On Basic Tensor Concepts and the Taylor Expansion

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Abstract

In this article we derive an anomalous result that with curved space time transformations have to be linear. Technical difficulties with infinitesimal space time coordinates as tensors are exposed. Analysis with the Taylor series brings out a stupendous fact that it considers only such functions as are linear in the independent variables.

Introduction

Considering infinitesimal space time separations as tensors is a fact appreciated in General Relativity where space time transformations are ,in general, not linear in the rectangular Cartesian system. Consequently the Christoffel symbols and the Riemannian curvature tensor are not zero valued. But a simple investigation reveals that the transformation between the coordinates of space and time are linear or approximately linear. It has been clearly analyzed that considering infinitesimal space time separations as tensors is a flawed concept. Finally we derive that the Taylor series applies only to linear functions.

The Anomalous Result

We start with the result[1]

$$d\bar{x}^{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} dx^{\alpha} \quad (1)$$

From (1)

$$\bar{x}^{\mu} = \int_{0}^{x^{\alpha}} \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} dx^{\alpha} \quad (2)$$

$$\Rightarrow \bar{x}^{\mu} = x^{\alpha} \left[\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\right]_{mean} \quad (3)$$

Taking differentials on either side of (3) , we have,

$$d\bar{x}^{\mu} = dx^{\alpha}\left[\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\right]_{mean} + x^{\alpha} d\left[\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\right]_{mean} \quad (4)$$

With $x^{\alpha} \to \infty$, the product $x^{\alpha} d\left[\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\right]_{mean} \to \infty$ unless $d\left[\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\right]_{mean} \to 0$.

The right side of (4) and consequently $x^{\alpha} d\left[\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\right]_{mean}$ has to be an infinitesimal and not just a finite quantity ,let alone an infinitely large quantity.

But

$$d\left[\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\right]_{mean} \to 0 \Rightarrow \left[\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\right]_{mean} = constant$$

for large x^{α}. But at any point we can make x^{α} sufficiently large by shift of origin of the space –time coordinates. Interestingly,

$$\left[\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\right]_{mean} = constant \Rightarrow \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} = constant \quad (5)$$

that transformations should be linear like the Lorentz Transformations. That would mean that the Riemann curvature tensor and the Ricci tensors are null tensors and the Ricci scalar is zero.

[NB: $\Delta F = f(x)\Delta x$, ΔF will be an infinitesimal provided $f(x)$is not unbounded.]

We might consider the following option . Differentiating both sides of (3) with respect to proper time we haven

We now have ,

$$\frac{d\bar{x}^{\mu}}{d\tau} = \frac{dx^{\alpha}}{d\tau}\left[\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\right]_{mean} + x^{\alpha}\frac{d}{d\tau}\left[\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\right]_{mean} \quad (6)$$

With x^{α} tending to infinity the right side of (7) blows up. To save the situation we set $\frac{d}{d\tau}\left[\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\right]_{mean} = 0 \Rightarrow \left[\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\right]_{mean} = constant \Rightarrow \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} = constant$. Thus we have a linear transformation like the Lorentz transformations of flat space time. The curvature elements become zero. The Riemannian curvature tensor and the Ricci tensors become null tensors. The Ricci scalar becomes zero.

NB: Proper speed components can exceed the speed of light without violating modern relativity but with (6), $\bar{P}^{\mu} = m_0 \frac{d\bar{x}^{\mu}}{d\tau}$ is becoming dependent on position and time coordinates given by x^{α} and hence on $\bar{x}^{\mu} = \bar{x}^{\mu}(x^{\alpha})$

This type of an anomaly[as expressed in the discussion so far] is emerging from relation (1) which is purely a mathematical one. Therefore it would be important to investigate the validity/accuracy of (1)

The Technical Issue

We rewrite (1)

$$d\bar{x}^{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} dx^{\alpha}$$

By definition we take $dx^{\alpha} = \Delta x^{\alpha}$; $d\bar{x}^{\mu}$ is defined by (1).

$$d\bar{x}^{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \Delta x^{\alpha} \ (7)$$

Let us consider the inverse transformation

$$dx^{\alpha} = \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} d\bar{x}^{\mu} \ \ (8)$$

If with the right side of (6) we consider, by definition, $d\bar{x}^{\mu} = \Delta\bar{x}^{\mu}$ then a contradiction may be derived in the following manner.

$$dx^{\alpha} = \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} \Delta\bar{x}^{\mu} \ (9)$$

Since $d\bar{x}^{\mu} = \Delta\bar{x}^{\mu}$we may write (1) as

$$\Delta\bar{x}^{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \Delta x^{\alpha} \ \ (10)$$

Indeed by Taylor expansion[] we have,

$$\bar{x}^{\mu}(x^{\alpha} + \Delta x^{\alpha}) = \bar{x}^{\mu}(x^{\alpha}) + \frac{1}{1!}\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \Delta x^{\alpha} + \frac{1}{2!}\frac{\partial^2 \bar{x}^{\mu}}{\partial x^{\beta}\partial x^{\alpha}} \Delta x^{\alpha}\Delta x^{\beta} + \frac{1}{3!}\frac{\partial^2 \bar{x}^{\mu}}{\partial x^{\gamma}\partial x^{\beta}\partial x^{\alpha}} \Delta x^{\alpha}\Delta x^{\beta}\Delta x^{\gamma} + \cdots \frac{1}{r!}\frac{\partial^r \bar{x}^{\mu}}{\partial x^{\gamma}\partial x^{\beta}\partial x^{\alpha} \ldots.} \Delta x^{\alpha}\Delta x^{\beta}\Delta x^{\gamma} \ldots. + \cdots ..$$

$$\bar{x}^{\mu}(x^{\alpha} + \Delta x^{\alpha}) - \bar{x}^{\mu}(x^{\alpha}) = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \Delta x^{\alpha} + \frac{1}{2!}\frac{\partial^2 \bar{x}^{\mu}}{\partial x^{\beta}\partial x^{\alpha}} \Delta x^{\alpha}\Delta x^{\beta} + \frac{1}{3!}\frac{\partial^2 \bar{x}^{\mu}}{\partial x^{\gamma}\partial x^{\beta}\partial x^{\alpha}} \Delta x^{\alpha}\Delta x^{\beta}\Delta x^{\gamma} + \cdots \frac{1}{r!}\frac{\partial^r \bar{x}^{\mu}}{\partial x^{\gamma}\partial x^{\beta}\partial x^{\alpha} \ldots.} \Delta x^{\alpha}\Delta x^{\beta}\Delta x^{\gamma} \ldots. + \cdots.$$

$$\Delta\bar{x}^{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \Delta x^{\alpha} + \frac{1}{2!}\frac{\partial^2 \bar{x}^{\mu}}{\partial x^{\beta}\partial x^{\alpha}} \Delta x^{\alpha}\Delta x^{\beta} + \frac{1}{3!}\frac{\partial^2 \bar{x}^{\mu}}{\partial x^{\gamma}\partial x^{\beta}\partial x^{\alpha}} \Delta x^{\alpha}\Delta x^{\beta}\Delta x^{\gamma} + \cdots \frac{1}{r!}\frac{\partial^r \bar{x}^{\mu}}{\partial x^{\gamma}\partial x^{\beta}\partial x^{\alpha} \ldots.} \Delta x^{\alpha}\Delta x^{\beta}\Delta x^{\gamma} \ldots. + \cdots.$$

$$\Delta\bar{x}^{\mu} \neq \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \Delta x^{\alpha} \ \ (11)$$

Alternatively we consider (1) again

$$d\bar{x}^{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} dx^{\alpha}$$

where by definition is considered as $dx^{\alpha} \equiv \Delta x^{\alpha} \Rightarrow d\bar{x}^{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \Delta x^{\alpha}$

With the inverse transformation (8)

$$dx^{\alpha} = \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} d\bar{x}^{\mu}$$

$d\bar{x}^{\mu}$ on the right side of (8)is defined by $d\bar{x}^{\mu}$ on the left side of (1)[as opposed to what we did earlier].

We may rewrite (8) as

$$dx^{\alpha} = \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} d\bar{x}^{\mu} = \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \Delta x^{\alpha}$$

By contraction on μ in the term $\frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}$ we obtain a consistent result

$$dx^{\alpha} = \Delta x^{\alpha} \quad (12)$$

Now by Taylor expansion,

$$\Delta x^{\alpha} = \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} \Delta \bar{x}^{\mu} + \frac{1}{2!} \frac{\partial^2 x^{\alpha}}{\partial \bar{x}^{\nu} \partial \bar{x}^{\mu}} \partial \bar{x}^{\nu} \partial \bar{x}^{\mu} + \frac{1}{3!} \frac{\partial^2 x^{\alpha}}{\partial \bar{x}^{\rho} \partial \bar{x}^{\nu} \partial \bar{x}^{\mu}} \partial \bar{x}^{\rho} \partial \bar{x}^{\nu} \partial \bar{x}^{\mu} + \cdots$$

[Each index like μ, ν, ρ on the right run over all independent variables. Again on the right side α runsover all variables on the right side.]

Using the above result with $\bar{x}^{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \Delta x^{\alpha}$ we obtain

$$d\bar{x}^{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \left[\frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} \Delta \bar{x}^{\mu} + \frac{1}{2!} \frac{\partial^2 x^{\alpha}}{\partial \bar{x}^{\nu} \partial \bar{x}^{\mu}} \partial \bar{x}^{\nu} \partial \bar{x}^{\mu} + \frac{1}{3!} \frac{\partial^2 x^{\alpha}}{\partial \bar{x}^{\rho} \partial \bar{x}^{\nu} \partial \bar{x}^{\mu}} \partial \bar{x}^{\rho} \partial \bar{x}^{\nu} \partial \bar{x}^{\mu} + \cdots . \right]$$

$$d\bar{x}^{\mu} = \left[\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} \Delta \bar{x}^{\mu} + \frac{1}{2!} \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial^2 x^{\alpha}}{\partial \bar{x}^{\nu} \partial \bar{x}^{\mu}} \partial \bar{x}^{\nu} \partial \bar{x}^{\mu} + \frac{1}{3!} \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial^2 x^{\alpha}}{\partial \bar{x}^{\rho} \partial \bar{x}^{\nu} \partial \bar{x}^{\mu}} \partial \bar{x}^{\rho} \partial \bar{x}^{\nu} \partial \bar{x}^{\mu} + \cdots . \right]$$

$$d\bar{x}^{\mu} = \left[\Delta \bar{x}^{\mu} + \frac{1}{2!} \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial^2 x^{\alpha}}{\partial \bar{x}^{\nu} \partial \bar{x}^{\mu}} \partial \bar{x}^{\nu} \partial \bar{x}^{\mu} + \frac{1}{3!} \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial^2 x^{\alpha}}{\partial \bar{x}^{\rho} \partial \bar{x}^{\nu} \partial \bar{x}^{\mu}} \partial \bar{x}^{\rho} \partial \bar{x}^{\nu} \partial \bar{x}^{\mu} + \cdots . \right]$$

$$dx^{\alpha}=\frac{\partial x^{\alpha}}{\partial\bar{x}^{\mu}}d\bar{x}^{\mu}=\frac{\partial x^{\alpha}}{\partial\bar{x}^{\mu}}\left[\Delta\bar{x}^{\mu}+\frac{1}{2!}\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial^{2}x^{\alpha}}{\partial\bar{x}^{\nu}\partial\bar{x}^{\mu}}\partial\bar{x}^{\nu}\partial\bar{x}^{\mu}+\frac{1}{3!}\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial^{2}x^{\alpha}}{\partial\bar{x}^{\rho}\partial\bar{x}^{\nu}\partial\bar{x}^{\mu}}\partial\bar{x}^{\rho}\partial\bar{x}^{\nu}\partial\bar{x}^{\mu}+\cdots.\right]$$

$$dx^{\alpha}=\Delta\bar{x}^{\mu}+\frac{1}{2!}\frac{\partial x^{\alpha}}{\partial\bar{x}^{\mu}}\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial^{2}x^{\alpha}}{\partial\bar{x}^{\nu}\partial\bar{x}^{\mu}}\partial\bar{x}^{\nu}\partial\bar{x}^{\mu}+\frac{1}{3!}\frac{\partial x^{\alpha}}{\partial\bar{x}^{\mu}}\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial^{2}x^{\alpha}}{\partial\bar{x}^{\rho}\partial\bar{x}^{\nu}\partial\bar{x}^{\mu}}\partial\bar{x}^{\rho}\partial\bar{x}^{\nu}\partial\bar{x}^{\mu}+\cdots.. \quad (13)$$

In the above we cannot contract on α and write $\frac{\partial x^{\alpha}}{\partial\bar{x}^{\mu}}\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}=1$ since the third factor [for example$\frac{\partial^{2}x^{\alpha}}{\partial\bar{x}^{\nu}\partial\bar{x}^{\mu}}$in $\partial\bar{x}^{\nu}\partial\bar{x}^{\mu}\frac{\partial x^{\alpha}}{\partial\bar{x}^{\mu}}\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial^{2}x^{\alpha}}{\partial\bar{x}^{\nu}\partial\bar{x}^{\mu}}$] contains α as an index[dummy index]. Therefore from (13).[We cannot also contract on μ]

$$dx^{\alpha}\neq\Delta\bar{x}^{\mu} \ (14)$$

Relations (14) and (12) stand in contradiction to each other.

With the first alternative we have taken $dx^{i}=\Delta x^{i}$ for all frames of reference. All frames of reference have been considered equivalent as demanded by modern relativity. And we have obtained a contradiction.

With the second alternative we have considered $dx^{i}=\Delta x^{i}$ with one particular frame of reference . With other frames of reference $d\bar{x}^{\mu}\neq\Delta\bar{x}^{\mu}$. Thus here we do have a preferred frame of reference which is of a suspicious nature. And again, we have arrived at a contradiction.

Since relation (1) is not an accurate one unless we are in flat space time where the higher order derivatives vanish we did obtain an anomalous result.

The definition of space time infinitesimals as tensors as expressed through (1) and (8) has technically issues of difficulty as indicated.

Taylor Series Issues

First we consider the Taylor series in a single variable

$$f(x+\Delta x)=f(x)+f'(x)\Delta x+\frac{1}{2!}F''(x)(\Delta x)^{2}+\frac{1}{3!}F'''(x)(\Delta x)^{3}+\cdots.+\frac{1}{r!}F^{(r)}(x)(\Delta x)^{r}+\cdots. \ (15)$$

Δx is chosen so that the right side is convergent

$$f(x+\Delta x)-f(x)=f'(x)\Delta x+\frac{1}{2!}F''(x)(\Delta x)^{2}+\frac{1}{3!}F'''(x)(\Delta x)^{3}+\cdots.+\frac{1}{r!}F^{(r)}(x)(\Delta x)^{r}$$

$$df=f'(x)\Delta x+\frac{1}{2!}F''(x)(\Delta x)^{2}+\frac{1}{3!}F'''(x)(\Delta x)^{3}+\cdots.+\frac{1}{r!}F^{(r)}(x)(\Delta x)^{r}+\cdots \ (16)$$

We sum up each term of the above series on the interval (a, b)

$$\sum df - \sum f'(x)\Delta x = \sum \frac{1}{2!}F''(x)(\Delta x)^2 + \sum \frac{1}{3!}F'''(x)(\Delta x)^3 + \cdots. + \sum \frac{1}{r!}F^{(r)}(x)\,(\Delta x)^r + \cdots$$

If Δx is the same for each partition on the interval , then,

$$\sum df - \sum f'(x)\Delta x$$
$$= \frac{1}{2!}\Delta x \sum F''(x)\Delta x + \Delta x \sum \frac{1}{3!}F'''(x)(\Delta x)^2 + \cdots. + \Delta x \sum \frac{1}{r!}F^{(r)}(x)\,(\Delta x)^{r-1} + \cdots$$

$$\frac{\sum df - \sum f'(x)\Delta x}{\Delta x} = \frac{1}{2!}\sum F''(x)\Delta x + \frac{1}{3!}\Delta x \sum F'''(x)(\Delta x)^2 + \cdots. + \frac{1}{r!}\Delta x \sum F^{(r)}(x)\,(\Delta x)^{r-1} + \cdots$$

$$lim_{\Delta x \to 0}\frac{\sum \Delta f - \sum f'(x)\Delta x}{\Delta x}$$
$$= \frac{1}{2!}lim_{\Delta x \to 0}\sum F''(x)\Delta x + lim_{\Delta x \to 0}\Delta x$$
$$\times\, lim_{\Delta x \to 0}\left[\sum \frac{1}{3!}F'''(x)\Delta x + \cdots. + \frac{1}{r!}\sum F^{(r)}(x)\,(\Delta x)^{r-2} + \cdots\right]$$

$$lim_{\Delta x \to 0}\frac{\sum \Delta f - \sum f'(x)\Delta x}{\Delta x} = \frac{1}{2!}lim_{\Delta x \to 0}\sum F''(x)\Delta x$$

$$lim_{\Delta x \to 0}\sum \frac{\Delta f}{\Delta x} - lim_{\Delta x \to 0}\sum f'(x) = \frac{1}{2!}lim_{\Delta x \to 0}\sum F''(x)\Delta x \quad (17)$$

$$\sum f'(x) - \sum f'(x) = \frac{1}{2!}\int_a^b F''(x)dx$$

$$\int_a^b F''(x)dx = 0 \ (18)$$

Since a and b are arbitrary real numbers [a<b], we have

$$F''(x) = 0 \quad (19)$$

$$F'(x) = C[onstant]$$

$$F(x) = Cx + D \quad (20)$$

[In the above, $F(x) \leftrightarrow x$,represents a linear transformation in one variable]

The above in general is not true since F(x) could be any arbitrary function [continuous and differentiable]

If we consider a many variable Taylor expansion in 'n' variables we might hold n-1variables constant and arrive at the same conclusion.

Integrating along curves along which t, y and z are constant we obtain

$$F(t,x,y,z) = Ax + D_1(t,y,z) + C_1 \; (21)$$

Integrating along curves along which t, x and z are constant we obtain

$$F(t,x,y,z) = By + D_2(t,x,z) + C_2 \; (22)$$

Integrating along curves along which t, x and y are constant we obtain

$$F(t,x,y,z) = Cz + D_3(t,x,y) + C_3 \; (23)$$

Integrating along curves along which t, x and y are constant we obtain

$$F(t,x,y,z) = Dt + D_4(x,y,z) + C_4 \; (24)$$

The last four relations imply

$$F(t,x,y,z) = Ax + By + Cz + Dt + K \; (25)$$

In place of $F(t,x,y,z)$ we may consider $F^{\mu}(t,x,y,z); \mu = 0,1,2,3$

We are left with linear relations as concluded in the first section, "The Anomalous Result".

Conclusion

The strange difficulties with infinitesimal separations as tensors leading to the fact that transformations in the most general context have to be linear deserve attention. It is quite stupendous that the Taylor series considers only such functions as are linear in the independent variables. It is important to take stock of such a situation in view of mathematics and its applications like physics.

References

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