Algunas integrales relacionadas con el número π

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Resumen

En esta nota mostramos algunas integrales relacionadas con la constante Pi:
$\pi = 3.1415926535...$

Integrales

Entry 1.

$$\pi = \int_{-\infty}^{\infty} \frac{x}{\cosh(x-1)} dx = \int_{-\infty}^{\infty} x \operatorname{sech}(x-1) dx \quad (1)$$

$$\pi = \int_{0}^{\infty} x\left(\operatorname{sech}(x-1) - \operatorname{sech}(x+1)\right) dx \quad (2)$$

$$\pi = 2\sinh 1 \int_{0}^{\infty} \frac{x \sinh x}{\cosh(x-1)\cosh(x+1)} dx \quad (3)$$

$$\pi = 4\sinh 1 \int_{0}^{\infty} \frac{x \sinh x}{\cosh 2 + \cosh(2x)} dx \quad (4)$$

$$\pi = 2\sinh 1 \int_{0}^{\infty} \frac{x \sinh x}{\cosh^2 x + \sinh^2 1} dx \quad (5)$$

$$\pi = 2\sinh 1 \int_{0}^{\infty} \frac{x \sinh x}{\sinh^2 x + \cosh^2 1} dx \quad (6)$$

$$\pi = \int_{-\infty}^{\infty} \frac{x \cosh(x+1)}{\cosh^2 x + \sinh^2 1} dx \quad (7)$$

$$\pi = \int_{-\infty}^{\infty} \frac{x\cosh(x+1)}{\cosh^2 1 + \sinh^2 x}\,dx \tag{8}$$

$$\pi = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2 \sinh(x-1)}{\cosh^2(x-1)}\,dx \tag{9}$$

$$\pi = 2\int_{1}^{\infty} \frac{1}{\sqrt{x^2-1}} \tan^{-1}\left(\frac{\sinh 1}{x}\right) dx \tag{10}$$

$$\pi = 2\int_{0}^{\infty} \tan^{-1}\left(\frac{\sinh 1}{\cosh x}\right) dx \tag{11}$$

$$\pi = 2\int_{0}^{\infty} \sin^{-1}\left(\frac{\sinh 1}{\sqrt{\cosh^2 x + \sinh^2 1}}\right) dx \tag{12}$$

$$\pi = 2\int_{0}^{\infty} \cos^{-1}\left(\frac{\cosh x}{\sqrt{\cosh^2 x + \sinh^2 1}}\right) dx \tag{13}$$

$$\pi = 2\int_{0}^{\tan^{-1}(\sinh 1)} \cosh^{-1}\left(\frac{\sinh 1}{\tan x}\right) dx \tag{14}$$

$$\pi = 2\int_{0}^{\sinh 1} \frac{1}{1+x^2} \cosh^{-1}\left(\frac{\sinh 1}{x}\right) dx \tag{15}$$

$$\pi = 2\sinh 1 \int_{1}^{\infty} \frac{\cosh^{-1} x}{x^2 + \sinh^2 1}\,dx = 2\sinh 1 \int_{1}^{\infty} \frac{\ln\left(x + \sqrt{x^2-1}\right)}{x^2 + \sinh^2 1}\,dx \tag{16}$$

$$\pi = 4\int_{0}^{\infty} \frac{x^2 \ln x}{\left(1+x^2\right)^2}\,dx \tag{17}$$

$$\pi = 2e\int_{0}^{\infty} \frac{\ln x}{e^2 + x^2}\,dx \tag{18}$$

Remark : $e = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.718281...$

Entry 2. Si $a \geq 0$, entonces

$$\pi + 2a\tan^{-1}\left(\frac{\sinh 1}{\cosh a}\right) = 2\int_0^a \tan^{-1}\left(\frac{\sinh 1}{\cosh x}\right)dx + 2\sinh 1\int_a^\infty \frac{x\sinh x}{\cosh^2 x + \sinh^2 1}dx \qquad (19)$$

$$\pi = 2a\tan^{-1}\left(\frac{\cosh a}{\sinh 1}\right) - 2\int_0^a \tan^{-1}\left(\frac{\cosh x}{\sinh 1}\right)dx + 2\sinh 1\int_a^\infty \frac{x\sinh x}{\cosh^2 x + \sinh^2 1}dx \qquad (20)$$

Entry 3.

$$\int_0^1 \tan^{-1}\left(\frac{\cosh x}{\sinh 1}\right)dx = \int_1^\infty \tan^{-1}\left(\frac{\sinh 1}{\cosh x}\right)dx \qquad (21)$$

$$\frac{\pi}{4} = \int_0^1 \tan^{-1}\left(\frac{\sinh 1 - \cosh x}{\sinh 1 + \cosh x}\right)dx + \int_1^\infty \tan^{-1}\left(\frac{\sinh 1}{\cosh x}\right)dx \qquad (22)$$

Entry 4. Si $a \geq 0$, entonces

$$\pi(1-a) = -2\int_0^a \tan^{-1}\left(\frac{\cosh x}{\sinh 1}\right)dx + 2\int_a^\infty \tan^{-1}\left(\frac{\sinh 1}{\cosh x}\right)dx \qquad (23)$$

Entry 5. Si $k \in \mathbb{R}$, entonces

$$k\pi = \int_{-\infty}^\infty \frac{x}{\cosh(x-k)}dx \qquad (24)$$

$$k\pi = \int_0^\infty x\left(\operatorname{sech}(x-k) + \operatorname{sech}(x+k)\right)dx \qquad (25)$$

Entry 6. Si $G = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^2}$,(Constante de Catalan), entonces

$$\pi = 2\int_0^{\sinh 1} \frac{\ln\left(\sinh 1 + \sqrt{\sinh^2 1 - x^2}\right)}{1+x^2}dx - 2\int_0^{\sinh 1} \frac{\ln x}{1+x^2}dx \qquad (26)$$

$$\pi - 2G = 2\int_0^{\sinh 1} \frac{\ln\left(\sinh 1 + \sqrt{\sinh^2 1 - x^2}\right)}{1+x^2}dx - 2\int_1^{\sinh 1} \frac{\ln x}{1+x^2}dx \qquad (27)$$

$$\pi - 2G = 2\int_0^{\sinh 1} \frac{\ln\left(\sinh 1 + \sqrt{\sinh^2 1 - x^2}\right)}{1+x^2}dx - 2\int_0^{\sinh 1 - 1} \frac{\ln(1+x)}{2+2x+x^2}dx \qquad (28)$$

Entry 7.

$$\frac{\pi}{4}\ln\sinh 1+\tan^{-1}\left(\frac{\sinh 1-1}{\sinh 1+1}\right)\ln\sinh 1= \\ =\int_0^{\sinh 1}\frac{\ln\left(\sinh 1+\sqrt{\sinh^2 1-x^2}\right)}{1+x^2}dx-\sinh 1\int_0^1\frac{\ln\left(1+\sqrt{1-x^2}\right)}{1+x^2\sinh^2 1}dx \tag{29}$$

Entry 8.

$$\pi+2\tan^{-1}(\sinh 1)\ln(\sinh 1)+i\left(Li_2(1-i\sinh 1)-Li_2(1+i\sinh 1)\right)= \\ =2\int_0^{\tan^{-1}\sinh 1}\ln\left(\sinh 1+\sqrt{\sinh^2 1-\tan^2 x}\right)dx \tag{30}$$

Remark: $Li_2(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^2}\quad ,|z|\le 1\,;\, Li_2(z)=-\int_0^z t^{-1}\ln(1-t)\,dt\quad ,z\in\mathbb{C}-(1,\infty)$

Remark: $i=\sqrt{-1}$.

Referencias

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