Numbers: Part 3, "Ramanujan's Integral"

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ABSTRACT. We recall a Ramanujan's integral: $\int_0^1 f(x)\,dx = \frac{\pi^2}{15}$.

I. Ramanujan's Integral

Entry 1. (Ramanujan ~ 1915)

$$\frac{\pi^2}{15} = \int_0^1 \frac{1}{x} \ln\left(\frac{1+\sqrt{1+4x}}{2}\right) dx \tag{1}$$

Formula (1) appears in : B.C. Berndt , Ramanujan' s Notebooks Part IV , p .325 , Entry 40 .

II. Related Integrals and Series

Entry 2.

$$\frac{\pi^2}{15} = -2 \int_0^1 \frac{\ln(x)}{1+4x+\sqrt{4x+1}} dx \tag{2}$$

$$\frac{\pi^2}{15} = \int_1^{\phi} \frac{(2x-1)\ln(x)}{x(x-1)} dx \tag{3}$$

$$\frac{\pi^2}{15} = 2 \int_0^{\infty} \frac{x e^{-x}}{\sqrt{4e^{-x}+1}+4e^{-x}+1} dx \tag{4}$$

$$\frac{\pi^2}{15} = 2\ln(2)\ln(\phi) - \frac{1}{2}\int_0^4 \frac{\ln(x)}{1+x+\sqrt{x+1}} dx \tag{5}$$

$$\frac{\pi^2}{15} = -2 \int_0^u \tanh\left(\frac{x}{2}\right) \ln\left(\frac{\sinh(x)}{2}\right) dx \tag{6}$$

$$\frac{\pi^2}{15} = 4 \int_0^{u} \coth(x) \ln\left(\cosh\left(\frac{x}{2}\right)\right) dx \tag{7}$$

$$\frac{\pi^2}{15} = 4 \int_0^{v} \frac{x\left(2 e^{2x} - 1\right)}{e^{2x} - 1} dx \tag{8}$$

$$\frac{\pi^2}{15} = -4 \int_0^{u/2} \tanh(x) \ln\left(\frac{\sinh(2x)}{2}\right) dx \tag{9}$$

$$\frac{\pi^2}{15} = -4 \int_0^{u/2} \tanh(x) \ln(\sinh(x)\cosh(x))\, dx \tag{10}$$

$$\frac{\pi^2}{15} = -4 \int_1^{e^v} \frac{1}{x} \ln\left(x\sqrt{x^2 - 1}\right) dx \tag{11}$$

$$\frac{\pi^2}{15} = -4 \int_0^{r} \frac{x}{1 + x^2} \ln\left(x\sqrt{1 + x^2}\right) dx \tag{12}$$

$$\frac{\pi^2}{15} = 2 \int_{\ln(\phi)}^{\infty} \coth(x) \ln\left(\frac{1 + \coth(x)}{2}\right) dx \tag{13}$$

$$\frac{\pi^2}{15} = 4 \int_0^{\tan^{-1}(2)} \csc(2x) \ln\left(\frac{1 + \sec(x)}{2}\right) dx \tag{14}$$

$$\frac{\pi^2}{15} = 4 \int_{\tan^{-1}(1/2)}^{\pi/2} \csc(2x) \ln\left(\frac{1 + \csc(x)}{2}\right) dx \tag{15}$$

$$\frac{\pi^2}{15} = 2 \int_{\ln(\phi)}^{\infty} \coth(x)\,(x - \ln(2\sinh(x)))\, dx \tag{16}$$

$$\frac{\pi^2}{15} = 2 \int_{1/2}^{\infty} \frac{1}{x} \left(\sinh^{-1}(x) - \ln(2x)\right) dx \tag{17}$$

$$\frac{\pi^2}{15} = 2 \int_0^{1} \frac{1}{x} \left(\ln(x) + \sinh^{-1}\left(\frac{1}{2x}\right)\right) dx \tag{18}$$

$$\frac{\pi^2}{15} = -4 \int_0^{1/\phi} \frac{x}{1 - x^2} \ln\left(\frac{x}{1 - x^2}\right) dx \tag{19}$$

$$\frac{\pi^2}{15} = 4 \int_0^{u/2} (\tanh(x) + \coth(x)) \ln(\cosh(x))\, dx \tag{20}$$

$$\frac{\pi^2}{15} = 2\left(\ln\left(\frac{\sqrt{2 + \sqrt{5}}}{2} + \frac{1}{2\sqrt{2 + \sqrt{5}}}\right)\right)^2 + 4 \int_0^{u/2} \coth(x) \ln(\cosh(x))\, dx \tag{21}$$

$$\frac{\pi^2}{15} = \int_0^{1/\phi} \frac{(1 + 2x)}{x(1 + x)} \ln(1 + x)\, dx \tag{22}$$

$$\frac{\pi^2}{15} = \frac{1}{2} (\ln(\phi))^2 + \frac{1}{2} \int_{1/\phi}^{1} \frac{\ln\left(1 + \sqrt{1 - x}\right)}{1 - x} dx \tag{23}$$

$$\frac{\pi^2}{15} = \frac{1}{2}(\ln(\phi))^2 + 4\int_0^t \coth(x)\ln(\cosh(x))\,dx \tag{24}$$

$$\frac{\pi^2}{15} = 2\int_0^1\int_0^1 \frac{1}{1+4xy+\sqrt{1+4xy}}\,dx\,dy \tag{25}$$

$$\frac{\pi^2}{15} = \int_0^1\int_0^1 \frac{\phi+2y}{(\phi+y)(\phi+xy)}\,dx\,dy \tag{26}$$

where

$$\phi = \frac{1+\sqrt{5}}{2},\ u = \ln\left(2+\sqrt{5}\right),\ v = \ln\left(\frac{1}{2}\left(\sqrt{5}-1\right)\sqrt{2+\sqrt{5}}\right) \tag{27}$$

$$r = \frac{1}{2}\left(3-\sqrt{5}\right)\sqrt{2+\sqrt{5}},\ t = \ln\left(\sqrt{\phi}+\frac{1}{\sqrt{\phi}}\right) \tag{28}$$

Entry 3. If $0 < a \le 1 \le b \le 2$, then

$$\frac{\pi^2}{15} = 2\ln(2)\ln(a) - (\ln(2))^2 - (\ln(b))^2 + \sum_{n=1}^{\infty}\binom{2n}{n}\frac{(-1)^{n-1}2^{-2n-1}a^{2n}}{n^2} + \sum_{n=0}^{\infty}\binom{2n}{n}\frac{(-1)^n 2^{-2n+1}\left(b^{-2n-1}-2^{-2n-1}\right)}{(2n+1)^2} + 2\int_a^b \frac{1}{x}\ln\left(1+\sqrt{1+x^2}\right)dx \tag{29}$$

Entry 4.

$$\frac{\pi^2}{15} = 2\left(\ln\left(\frac{\sqrt{2+\sqrt{5}}}{2}+\frac{1}{2\sqrt{2+\sqrt{5}}}\right)\right)^2 + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}\left(\frac{\sqrt{2+\sqrt{5}}}{2}-\frac{1}{2\sqrt{2+\sqrt{5}}}\right)^{2n} \tag{30}$$

Entry 5. If $\phi = \frac{1+\sqrt{5}}{2}$, then

$$\frac{\pi^2}{15} = \ln(\phi) - \sum_{n=1}^{\infty}\frac{n+2}{n(n+1)}\left\{\ln(\phi) + \sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^k\left(\phi^k-1\right)}{k}\right\} \tag{31}$$

$$\frac{\pi^2}{15} = \frac{1}{2}(\ln(\phi))^2 + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}\phi^{-n}}{n^2} \tag{32}$$

$$\frac{\pi^2}{15} = 2\ln(\phi) + \frac{1}{\phi} - 1 + \sum_{n=1}^{\infty}\frac{(-1)^n}{n+1}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\left(\frac{2\left(\phi^{k-n}-1\right)}{k-n} - \frac{\phi^{k-n-1}-1}{k-n-1}\right) \tag{33}$$

Entry 6. If $\phi = \frac{1+\sqrt{5}}{2}$ and $H_n = \sum_{k=1}^{n}\frac{1}{k}$, then

$$\frac{\pi^2}{15}=\sum_{n=1}^{\infty}(-1)^{n-1}H_n\left(\frac{2\,\phi^{-n-1}}{n+1}+\frac{\phi^{-n}}{n}\right) \tag{34}$$

$$\frac{\pi^2}{15}=\frac{1}{\phi}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\,\phi^{-n-1}}{n+1}\left(H_n-\frac{1}{n+1}\right) \tag{35}$$

Entry 7. If $\phi=\frac{1+\sqrt{5}}{2}$, then

$$\frac{\pi^2}{15}=\ln(\phi)+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\,(n+2)\,\phi^{-n-2}}{n\,(n+1)^2}F\left(1,\,1;\,n+2;\,\phi^{-2}\right) \tag{36}$$

$$\frac{\pi^2}{15}=\ln(\phi)+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\,(n+2)\,\phi^{-n-1}}{n\,(n+1)^2}F\left(1,\,n+1;\,n+2;\,-\phi^{-1}\right) \tag{37}$$

$$\frac{\pi^2}{15}=\frac{1}{2}(\ln(\phi))^2+\sum_{n=1}^{\infty}\frac{\phi^{-n}}{n^2}F\left(n,\,n;\,n+1;\,-\phi^{-1}\right) \tag{38}$$

$$\frac{\pi^2}{15}=\frac{1}{2}(\ln(\phi))^2+\sum_{n=1}^{\infty}\frac{\phi^{-2n}}{n^2}F\left(1,\,n;\,n+1;\,\phi^{-2}\right) \tag{39}$$

$$\frac{\pi^2}{15}=\frac{1}{2}(\ln(\phi))^2+\sum_{n=1}^{\infty}\frac{\phi^{-2n+1}}{n^2}F\left(1,\,1;\,n+1;\,-\phi^{-1}\right) \tag{40}$$

$$\frac{\pi^2}{15}=\frac{1}{2}(\ln(\phi))^2+\sum_{n=0}^{\infty}\frac{2^{-2n}\,\phi^{-2n-1}}{(2n+1)^2}F\left(2n+1,\,2n+1;\,2n+2;\,-\frac{1}{2\,\phi}\right) \tag{41}$$

$$\frac{\pi^2}{15}=3\sum_{n=0}^{\infty}\frac{\phi^{-6n-6}}{n+1}\sum_{k=0}^{n}\frac{1}{2k+1}+2\sum_{n=0}^{\infty}\frac{\phi^{-6n-3}}{(2n+1)^2}F\left(1,\,n+\frac{1}{2};\,n+\frac{3}{2};\,\phi^{-6}\right) \tag{42}$$

$F(a,\,b;\,c;\,x)$ is the Gauss hypergeometric function.

III. References

1. Berndt, B.C., Ramanujan's Notebooks, Part IV ,Springer-Verlag,New York, 1994.
2. Boros, G., and Moll, V.H., Irresistible Integrals,Cambridge University Press,2004.
3. Gradshteyn, I.S., and Ryzhik, I.M., Table of Integrals, Series and Products.7th ed., edited by Alan Jeffrey and Daniel Zwillinger, Academic Press,2007.