Presentation of Finite Dimensions

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We present subsets of Euclidian spaces $\mathbb{R}^n$ in the ordinary plane $\mathbb{R}^2$. Naturally some informations are lost. We provide examples.

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1 Introduction

It is trivial that one can picture objects of the space $\mathbb{R}^n$ only if n is less than four. The best presentation is a picture in $\mathbb{R}^2$. Mathematicians often deal with objects in higher dimensional spaces, but since we live in a three dimensional space we have no real imagination of these objects. Here we show methods to represent something of the $\mathbb{R}^n$ in $\mathbb{R}^2$. The way is by dividing a vector of $\mathbb{R}^n$ into small parts consisting of some components. After this we take barycenters. The final point can be presented in a two dimensonal space.

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First we give names. We have methods $\mathsf{way}_2, \mathsf{way}_3, \mathsf{way}_4, \ldots$. To use way_2 we need points in $\mathbb{R}^n$ for $n \in \{4, 8, 16, \ldots\}$. To use way_3 we need points in $\mathbb{R}^n$ for $n \in \{9, 27, 81, \ldots\}$. Generally if we use way_k we need points in $\mathbb{R}^n$ for $n \in \{k^2, k^3, k^4, \ldots\}$ for $k > 1$. We calculate barycenters of k-polygons. Further we define method_k, which requires a vector from $\mathbb{R}^k$, and which is suitable for all integers $k > 1$ and which does not need barycenters.
First we show method_k. Let us take a vector $\vec{a} := (a_1, a_2, \ldots, a_{n-1}, a_n)$ of $\mathbb{R}^n$. We define

$$\mathsf{method}_n(\vec{a}) := \begin{cases} \Big(a_1 + a_2 + \ldots + a_{\frac{n}{2}-1} + a_{\frac{n}{2}},\ a_{\frac{n}{2}+1} + a_{\frac{n}{2}+2} + \ldots + a_{n-1} + a_n\Big) \\ \text{if } n \ \text{ is even}, \ \ n \text{ larger than } 4 \\ \Big(a_1 + a_2 + \ldots + a_{\frac{n-1}{2}-1} + a_{\frac{n-1}{2}} + \frac{1}{2} \cdot a_{\frac{n-1}{2}+1},\ \frac{1}{2} \cdot a_{\frac{n-1}{2}+1} + a_{\frac{n-1}{2}+2} + \ldots + a_{n-1} + a_n\Big) \\ \text{if } n \text{ is odd}, \ \ n \text{ larger than } 4 \end{cases}$$

We define $\mathsf{method}_2(a, b) := (a, b)$, $\mathsf{method}_3(a, b, c) := \Big(a + \frac{1}{2} \cdot b, \frac{1}{2} \cdot b + c\Big)$, $\mathsf{method}_4(a, b, c, d) := (a + b, c + d)$.

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Let us demonstrate way_2. If we have an element $(a,b,c,d) \in \mathbb{R}^4$ we take two vectors $(a,b),(c,d) \in \mathbb{R}^2$. Then we compute the barycenter and we get the image point

$$\mathsf{way}_2(a,b,c,d) := \left(\frac{1}{2}\cdot(a,b)+\frac{1}{2}\cdot(c,d)\right) = \left(\frac{1}{2}\cdot(a+c),\ \frac{1}{2}\cdot(b+d)\right), \tag{2.1}$$

which can be drawn in $\mathbb{R}^2$. In the case of a vector $\vec{y} := (a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8) \in \mathbb{R}^8$ we divide it in four parts $(a_1,a_2),(a_3,a_4),(a_5,a_6),(a_7,a_8) \in \mathbb{R}^2$. First we calculate two barycenters of the pairs $(a_1,a_2),(a_3,a_4)$ and $(a_5,a_6),(a_7,a_8)$, respectively. After this we take the barycenter of the two barycenters. We get

$$\mathsf{way}_2(\vec{y}) := \left(\frac{1}{2}\cdot\left[\frac{1}{2}\cdot(a_1+a_3),\ \frac{1}{2}\cdot(a_2+a_4)\right]+\frac{1}{2}\cdot\left[\frac{1}{2}\cdot(a_5+a_7),\ \frac{1}{2}\cdot(a_6+a_8)\right]\right)$$

hence

$$\mathsf{way}_2(\vec{y}) = \left(\frac{1}{4}\cdot(a_1+a_3+a_5+a_7),\ \frac{1}{4}\cdot(a_2+a_4+a_6+a_8)\right)$$

which is a point in $\mathbb{R}^2$.
If we have a vector $\vec{u} := (a_1,a_2,a_3,\ldots,a_{14},a_{15},a_{16}) \in \mathbb{R}^{16}$ we can use also way_2. We compute the barycenter of two barycenters of four barycenters of 8 points $(a_1,a_2),(a_3,a_4),(a_5,a_6),(a_7,a_8),(a_9,a_{10}),(a_{11},a_{12}),(a_{13},a_{14})$ and (a_{15},a_{16}). We get $\mathsf{way}_2(\vec{u}) =$

$$\left(\frac{1}{8}\cdot(a_1+a_3+a_5+a_7+a_9+a_{11}+a_{13}+a_{15}),\ \frac{1}{8}\cdot(a_2+a_4+a_6+a_8+a_{10}+a_{12}+a_{14}+a_{16})\right). \tag{2.2}$$

To demonstrate way_3 for $\mathbb{R}^9$ we use a point $\vec{v} := (a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9)$. First we take the barycenter of the triangle of three points $(a_1,a_2,a_3),(a_4,a_5,a_6),(a_7,a_8,a_9)$ of $\mathbb{R}^3$, then we use method_3. We define

$$\mathsf{way}_3(\vec{v}) := \left(\frac{1}{3}\cdot(a_1+a_4+a_7)+\frac{1}{6}\cdot(a_2+a_5+a_8),\ \frac{1}{6}\cdot(a_2+a_5+a_8)+\frac{1}{3}\cdot(a_3+a_6+a_9)\right). \tag{2.3}$$

In the case of a vector $\vec{w} := (a_1,a_2,a_3,\ldots,a_{26},a_{27})$ from $\mathbb{R}^{27}$ we use method_3 for the barycenter of three barycenters of three triangles, which are generated by nine points $(a_1,a_2,a_3),(a_4,a_5,a_6),\ldots,(a_{25},a_{26},a_{27})$ of $\mathbb{R}^3$. This means

$$\mathsf{way}_3(\vec{w}) := \left(\frac{1}{9}\cdot a+\frac{1}{18}\cdot b,\ \frac{1}{18}\cdot b+\frac{1}{9}\cdot c\right) \tag{2.4}$$

where

$$a := a_1+a_4+a_7+a_{10}+a_{13}+a_{16}+a_{19}+a_{22}+a_{25}\ , \tag{2.5}$$
$$b := a_2+a_5+a_8+a_{11}+a_{14}+a_{17}+a_{20}+a_{23}+a_{26}\ , \tag{2.6}$$
$$c := a_3+a_6+a_9+a_{12}+a_{15}+a_{18}+a_{21}+a_{24}+a_{27}\ . \tag{2.7}$$

We omit to show a general formula of way_k.

Remark 2.1. *If $n < k$ it holds $\mathbb{R}^n \subset \mathbb{R}^k$. In place of the vector $(a_1,a_2,a_3,\ldots,a_{n-1},a_n)$ of $\mathbb{R}^n$ we can use $(a_1,a_2,a_3,\ldots,a_{n-1},a_n,0,0,0,\ldots,0,0) \in \mathbb{R}^k$.*

Remark 2.2. *To avoid fractions we may multiply $\mathsf{method}_k(\vec{v})$ or $\mathsf{way}_k(\vec{v})$ with a suitable factor to ensure an integer at the same position in $\vec{v}$ if $\vec{v}$ has an integer there.*

3 Examples

As an example we take the four dimensional cube, which is the convex hull of four dimensional vectors $(a, b, c, d) \in \mathbb{R}^4$, where the variables a, b, c, d either are 0 or 1. Each such point is called a *vertex* of the cube. Hence a four dimensional cube has 16 vertices.

By method_4 we get 9 vertices (x, y), where x and y is 0 or 1 or 2.
We get the same result if we use way_2 and Remark 2.2. By line (2.1) and Remark 2.2 the presentation in $\mathbb{R}^2$ has 9 vertices $(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2)$.

We repeat the presentation by Remark 2.1 since $\mathbb{R}^4 \subset \mathbb{R}^5$. Instead of points (a, b, c, d) we use points $(a, b, c, d, 0)$ with variables a, b, c, d either 0 or 1. It holds $\mathsf{method}_5(a, b, c, d, 0) = \big(a + b + \frac{1}{2} \cdot c, \frac{1}{2} \cdot c + d\big)$. By Remark 2.2 we multiply all points with 2 to avoid fractions. We get with method_5 12 points (x, y), where x is 0,1,2,3,4,5, and y is from the set $\{0, 1, 2, 3\}$. Note that c occurs both in x and y.

With way_3 we repeat the presentation by Remark 2.1, since $\mathbb{R}^4 \subset \mathbb{R}^9$. Instead of points $(a, b, c, d) \in \mathbb{R}^4$ we take points $(a, b, c, d, 0, 0, 0, 0, 0) \in \mathbb{R}^9$. By Remark 2.2 we multiply the 12 resulting points with 6. We get by line (2.3): $(0,0),(2,0),(4,0)$,$(0,2),(2,2),(4,2),(1,1)$, $(3,1),(5,1)$, $(1,3),(3,3),(5,3)$.

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