On the smallest Volume Scale and Dark Energy

Paolo Bolzoni
Freie Waldorfschule Magdeburg, Kroatenwuhne 3, 39116, Magdeburg, Germany.
E-mail: pbolzoni@gmail.com

In this study we want to propose a heuristic model to compute and to interpret the dark energy content of our universe. To this purpose we include the mass-energy of the static gravitational field in Newtonian gravity, finding agreement with general relativity at large scales. We then compute its effect at very small distances also including quantum effects. From this analysis, we obtain an estimation of the smallest volume in empty space. Our result is compatible with loop quantum gravity and this enables the embedding in it. After that we show, how this can be used to compute a natural energy cutoff k_c for all quantum fields and study its utility in computing the dark energy density and its implications on the content of fermionic and bosonic elementary fields. Indeed for the vacuum equation of state $w = p_{vac}/\rho_{vac}$ we obtain an expression depending on $\Delta N = N_f - N_b$, which represents the difference between the number of species of fermions and bosons. Finally comparing our result with the measured value of w, we discuss general constraints on the field content beyond the Standard Model of the elementary particles.

1 Introduction

A common aspect of many different approaches to quantum gravity such as string theory (see *e.g.* [1]), causal sets [2, 3], spin foams [4], causal dynamical triangulation (CDT) [5, 6] and loop quantum gravity [7, 8] is the presence of a smallest geometrical scale. The experimental search of such a scale has gained in the last years a lot of importance and concrete projects have already been started [9, 10]. A first phenomenological review of these approaches to quantum gravity can be found for example in [11–14].

If one tries to quantize a non renormalizable theory like gravity, the presence of a smallest scale has usually the advantage to solve the problems associated with infinities. In particular quartic divergencies emerge when one interprets dark energy as being originated by quantum fluctuations and this is independent on the curvature. The problem is that, even if the final result is finite, it turns out to be anyway many orders of magnitudes above the observed value [15]. Interesting new approaches have been developed in the last years from the point of view of supersymmetry, the renormalization procedure [16, 17], the renormalization group flow [18], the holographic principle [19] and stability considerations concerning the Minkowski space-time [20, 21]. In all such approaches the results are all improved and some of them are also able to predict the dark energy with the correct order of magnitude.

The purpose of this work is to propose a simple phenomenological model to predict the smallest geometrical scale of loop quantum gravity and the dark energy density from quantum fluctuations. As we will show, our approach has implications also on the possible field content of dark matter, showing a strict connection between the two aspects.

An important point of our investigation is an idea introduced by Heim [22], which consists in considering how the field mass $\mu = E/c^2$ associated to the energy content E of the gravitational field generated by a central massive spherical body has an effect to the field itself in Newtonian gravity. Assuming this one arrives at two important consequences as discussed in [22]:

(i) A source (typically a central mass) and its associated gravitational field build a unity (called structure).

(ii) The Einsteins field equations,

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \qquad (1)$$

receive a more general interpretation: they are not only viewed as an equation with the "source" appearing on its right side and the "field" appearing on its left side, but they are, more generally, considered as an equivalence between the generalized phenomenological energy-density-tensor including the field mass of the structure and geometry including in the Einstein-tensor also the "geometrization" of the source.

In this way $T_{\mu\nu}$ represents the energy density of the full structure (mass and field mass) and $G_{\mu\nu}$ represents its geometrical side. Since matter has a quantum nature, its "geometrization" has to be intended as quantum geometry, which for example has been developed in loop quantum gravity. We note then that according to the principles (i) and (ii), Eqs.(1) are just a generalization of the field equations of general relativity: indeed if the "field mass" is sufficiently small to be neglected in the phenomenological energy-density-tensor, then $T_{\mu\nu}$ can be interpreted again as the source of the gravitational field described by the metric $g_{\mu\nu}$, which contributes to the Einstein tensor $G_{\mu\nu}$.

We shall show that including field mass effects in the energy-density-tensor, one reproduces, to a certain approximation, the result of general relativity without geometrizing. A further consequence of this model will also be the existence of a smallest geometrical scale, after that effects of usual quantum mechanics are considered. In [23–25] the smallest scale found with this method is called "metron", which is quite different from the result of our revisited analysis. However this will be in agreement with the result obtained by "geometrizing" matter, which is in its essence a quantum behavior. This is achieved by the attempts to quantize geometry, like loop quantum gravity and renormalization group quantum gravity . Indeed the advantage of our result will be the possible embedding in these theories.

In comparison to [26] this paper presents a revision of the derivation of the modified Newton potential and of its smallest scale. This is due to the following reasons:

- The natural context of our argument is the three dimensional space and not a surface.
- The quantum effects on the field mass were not properly included.
- The mass formula of Heim [24], which we reported in Eq.(1) of [26] should be consequently also be revisited and cannot be used as part of our argument anymore. This will eventually be done in a future publication [27].

Furthermore this will be exploited for to compute the dark energy content from vacuum quantum fluctuations. For the application to the actual universe we start from the usual Friedmann equations for a spacially flat universe reported in Eqs(33) and, similarly as in [20], compute the dark energy contribution to the stress tensor $T_{\mu\nu}$ from quantum fluctuations on this curved background. This point will be explained in detail in Section 4 together with the Appendix. To avoid the quartic divergence of the large cutoff k_c, we also assume, that the Minkowski space is stable, *i.e.* it is imposed as a general principle to have a vanishing vacuum energy, as it should be. This implies that the contribution of the flat space-time has always to be subtracted from the vacuum energy derived from the quantum fluctuations also in a general curved space-time. We will show that according to our interpretation of the result, we can compute the dark energy density in very good agreement with the actual measurements. In addition we will also obtain a prediction for the equation of state parameter of the dark energy $w = p_{vac}/\rho_{vac}$ from first principles using the computed cutoff k_c. The comparison with the current measurements will show, that even if it is not yet possible to discriminate between "quintessence" ($-1 \leq w < -1/3$) and "phantom-energy" ($w < -1$) , it is still generally possible to constrain the field content, fixing the difference between the number of species of fermions and bosons $\Delta N = N_f - N_b$. According to the Standard Model of elementary particles we have that $\Delta_{SM} = 60$ and hence our result provides also a way to determine the minimal amount of additional degrees of freedom, which can contribute to the dark matter.

The paper is organized as follows. In Section 2 we explain in detail the computation for the modified Newtonian potential due to the inclusion of field mass effects. The derivation of the smallest volume element is then shown in Section 3. The application of these results for the computation and explanation of the nature of the dark energy from vacuum fluctuations and the possible influence of the result on the field content of dark matter beyond the Standard Model follows in Section 4. Finally we write our conclusions in Section 5.

2 The inclusion of the field mass in the static potential

In this section we want to include the effects of the inclusion of the field mass on the Newtonian potential $\phi_n = -Gm_{(0)}/r$ of a central mass $m_{(0)}$, where G is the usual gravitational constant. Hereafter we also include quantum effects at small scales.

We denote the field mass by μ and we then assume it produces a modification of the Newtonian potential ϕ_n, leading to an effective function of the form $\phi = -Gm(r)/r$, where $m(r) = m_{(0)} + \mu(r)$. According to this definition we obtain for the Laplacian operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ that

$$\nabla^2\phi = -Gm(r)\nabla^2\left(\frac{1}{r}\right) - \frac{G}{r}\frac{d^2\mu(r)}{dr^2} \tag{2}$$
$$= 4\pi G\left(m(r)\delta(r) - 3V\frac{d^2\mu(r)}{dV^2} - 2\frac{d\mu(r)}{dV}\right),$$

where in the second line we have used the distributional relation $\nabla^2(1/r) = -4\pi\delta$ with δ the Dirac function and where we have performed the change of variables $V = 4/3\pi r^3$. A first natural condition for the field mass $\mu(r)$ is that for very large values of r it vanishes. As a second condition for $\mu(r)$ it seems to us reasonable to expect that at the Planck scale quantum gravity effects weaken the gravitational mass $m(r)$ till it reaches a maximal value before starting to fall rapidly down. This is because the quantum uncertainty acts against the localizing tendency of gravity. Under these circumstances and looking at the second part of Eq.(2) one can think there are three additional effective sources coming from the second derivative, which at the maximal point is negative. Accordingly the last term is expected to be negligible, because at the extremum $d\mu/dr = 0$. That the origin of this kind of behavior for gravity at very small distances has a purely quantum nature, has been shown in the context of the renormalization group approach in Ref. [28].

In this way we arrive at a modified field equation for the gravitational field $\vec{G}$ in a static system valid only at short distances nearly the extremum of the field mass:

$$\vec{\nabla}\vec{G} = \frac{\rho}{\alpha}, \tag{3}$$

with

$$\rho = \frac{1}{4}\left(m_{(0)}\delta(r) - 3V\frac{d^2\mu(r)}{dV^2}\right) \tag{4}$$
$$\alpha = (16\pi G)^{-1}.$$

At large distances the derivatives of μ can be considered small enough to be neglected and thus.

$$\rho_n = m_{(0)}\delta(r); \quad \alpha_n = (4\pi G)^{-1}, \tag{5}$$

recovering the usual Poisson equation of the Newtonian case. We have that in general α has not to be considered a constant. However its derivatives will be neglected in what follows limiting in this way the confidence of our approximation,which however, as we shall see, can be considered relatively good.

Now the presence of a gravitational field $\vec{G}$ associated to central spherical source with radius r_0 and mass m_0 is from our point of view not a possibility but a necessity and this should be reflected in the fact that it is somehow produced by an "energetic convenience" *i.e.* a reduction of the system energy. Hence we can write for the energy ($E = m(r)c^2$) of the mass-field system up to a radial coordinate r from the center:

$$m(r)c^2 = m_0c^2 - \frac{\alpha}{2}\int_{V_0}^{V}\vec{G}^2\,dV. \tag{6}$$

The last term in this equation represents the field energy and is obtained from Eqs.(3,4) in complete analogy to the energy of the static electrical field. Now remembering that $\vec{G} = \vec{\nabla}\phi$ Eq.(6) becomes

$$m(r)c^2 = m_0c^2 - \frac{\alpha}{2}\int_{V_0}^{V}(\vec{\nabla}\phi)^2\,dV. \tag{7}$$

Now performing the first derivative, taking into account that $\phi = -Gm(r)/r$ and that for a spherical symmetric function $(\vec{\nabla}\phi)2 = (d\phi/dr)^2$, we obtain easily the following differential equation for the static potential ϕ:

$$\left(r\frac{d\phi}{dr}\right)^2 - \frac{c^2}{2\pi G\alpha}\left[\left(r\frac{d\phi}{dr}\right) + \phi\right] = 0. \tag{8}$$

This nonlinear differential equation can be easily solved viewing it as a quadratic equation in terms of $rd\phi/dr$, whose solutions are:

$$r\frac{d\phi}{dr} = \frac{c^2}{4\pi\gamma\alpha}\left(1 \pm \sqrt{1 + \frac{8\pi G\alpha}{c^2}\phi}\right). \tag{9}$$

With help of the following substitution,

$$q_\pm = 1 \pm \sqrt{1 + \frac{8\pi G\alpha}{c^2}\phi}, \tag{10}$$

one can straightforward rewrite Eq.(9) as:

$$r\frac{d(2q_\pm - q_\pm^2)}{dr} = -2q_\pm, \tag{11}$$

which according to $dx/x = d\ln(x)$ can be simplified to

$$d\ln(rq_\pm e^{-q_\pm}) = 0, \tag{12}$$

or equivalently to

$$rq_\pm e^{-q_\pm} = A, \tag{13}$$

where the integration constant A has been introduced.

We want now to fix the sign in Eq.(10) and the constant A in Eq.(13). Assuming that for $r \to \infty$, $\phi \to 0$, we obtain immediately that the negative sign in Eq.(10) is the only possibility. This follows from the fact that with this choice $q \to 0$ when $r \to \infty$ and only in this case remains our assumption consistent with the fact that in Eq.(13) A is a numerical constant. As far as the determination of the constant A is concerned, we can fix it requiring that the classical Newton potential $\phi_n = -Gm_{(0)}/r$ will be reproduced if $|\phi|/c^2 \ll 1$. Accordingly expanding Eq.(13) to the first order in ϕ/c^2, we obtain for the constant A:

$$A = rq_- e^{-q_-} = -r\frac{4\pi G\alpha}{c^2}\phi_n, \tag{14}$$

where according to Eq.(10) $0 \le q_- \le 1$. This equation defines implicitly the potential ϕ and, according to its dependence on $m(r)$, also the field mass $\mu(r)$.

We show now that for small q_-, Eq.(14) reproduces the result of general relativity up to corrections of order $q^3_{min}(\phi_n)$, if quantum corrections at small distances are not included, *i.e.* if $\alpha = (4\pi G)^{-1}$. First of all we notice that inverting Eq.(10) we have

$$\frac{\phi}{c^2} = -q_- + \frac{1}{2}q_-^2. \tag{15}$$

Then expanding Eq.(14) in q and using Eq.(15) we get:

$$\frac{\phi}{c^2} = \frac{\phi_n}{c^2} - \frac{1}{2}q_-^2(\phi) + O(q^3(\phi)) \tag{16}$$

$$= \frac{\phi_n}{c^2} - \frac{1}{2}q_-^2(\phi_n) + O(q^3(\phi_n)) \tag{17}$$

$$= -1 + \sqrt{1 + 8\pi G\,\alpha\frac{\phi_n}{c^2} + (1-4\pi G\,\alpha)\frac{\phi_n}{c^2}} + O(q^3(\phi_n)), \tag{18}$$

and putting $\alpha = (4\pi G)^{-1}$ we find

$$\frac{\phi}{c^2} = -1 + \sqrt{1 - \frac{2Gm_{(0)}}{rc^2}} + O(q^3(\phi_n)), \tag{19}$$

which agrees with the general relativity result (see e.g. Eq.(25.16) in [29]).

From Eq.(14) we can also determine the smallest allowed r-value r_{min} from its reality condition. Indeed this is fulfilled by Eq.(10), when $|\phi| \le c^2/(8\pi G\alpha)$, which means that

$$r \ge r_{min} = \frac{8\pi G^2\alpha\, m(r_{min})}{c^2}. \tag{20}$$

This last result needs few words: First of all we notice that very often for a macroscopic collapsing system one can neglect field mass effects in the second of Eqs.(2) and one can so assume $\alpha = \alpha_n = (4\pi G)^{-1}$, according to Eq.(5). In this case we obtain for the smallest radius of the system $r_{min} = 2Gm/c^2$, which is equal to the well known Schwarzschield radius of the general relativity. Considering also quantum effects to the field mass typical of high density microscopic systems like elementary particles, we have for $r = r_{min}$ that $q = 1$ and that $\alpha = (16\pi G)^{-1}$, according to Eq.(4). In this case we get from Eqs.(13,14), that $r_{min} = eA = e4\pi G^2\alpha m_{(0)}/c^2$, showing by comparison with Eq.(20) also that

$$m(r_{min}) = \frac{m_{(0)}}{2}. \tag{21}$$

Notice that this result does not depend on α, although r_{min} does.

We notice that significant deviations accure down to distances around $10^{-17}m$ well below current experimental limits (see *e.g.* [11, 30, 31] and references therein).

3 The smallest (non vanishing) volume V_{min}

Following the same approach adopted in [23], we want now similarly to derive the smallest volume $Vmin$ for the system under consideration. Eq.(20) fixes a lower limit r_{min} for the radial coordinate due to relativistic and gravitational field mass effects and thus does not represent a smallest length for the empty space-time, because it vanishes with the mass m. However a microscopic mass system should also be characterized by its quantum behavior, which becomes important at the scale of the corresponding Compton wavelength $\lambda_c = h/\,m_{(0)}c$ (we take however $\lambda_c/2$, because for example in e^+e^- annihilation one has $2mc^2 = 2hc/\lambda$ for the two particle system). Conversely we have that in this case for a vanishing mass λ_c diverges. Hence if we are looking for a good definition of the smallest scale l_{min} for a spherical system with a vanishing mass m like the empty space, we can build the geometrical average between $\tilde{d}_{min} = 2r_{min}$ and $\lambda_c/2$ to find a well defined limit:

$$\begin{aligned} d_{min} &= \lim_{m_{(0)}\to 0} \sqrt{\tilde{d}_{min} \cdot \lambda_c/2} \\ &= \sqrt{\frac{16\pi\, G^2 \alpha\, m(r_{min})}{c^2} \cdot \frac{h}{2m_{(0)}c}} \\ &= \sqrt{\frac{\pi e \hbar G}{2c^3}} = \sqrt{\frac{\pi e}{2}}\, l_{Pl}, \end{aligned} \tag{22}$$

where we have used Eq.(20), Eq.(21) and the second of Eqs.(4) and where $l_{Pl} = \sqrt{\hbar G/c^3}$ is the well known Planck length. Using Eq.(22) we finally arrive at our approximated formula for the smallest non vanishing Volume of space:

$$\begin{aligned} V_{min} &= \frac{4}{3}\pi \left(\frac{d_{min}}{2}\right)^3 \\ &= \frac{\pi}{6}\left(\frac{\pi e \hbar G}{2c^3}\right)^{\frac{3}{2}}. \end{aligned} \tag{23}$$

We can here compare our result with the pure geometrical result from loop quantum gravity as given for example in the book by Rovelli and Vidotto [32] Eq.(1.65):

$$V_{min}^{LQG} = \frac{1}{\sqrt{6\sqrt{3}}}\left(\frac{8\pi\gamma\hbar G}{c^3}\right)^{\frac{3}{2}}, \tag{24}$$

where γ is the Barbero-Immirzi constant. Now equating Eq.(23) with Eq.(24) we find

$$\gamma = \frac{e}{16}\left(\frac{\pi^2\sqrt{3}}{6}\right)^{\frac{1}{3}} = 0,241. \tag{25}$$

The theoretical value of γ in loop quantum gravity is fixed by the Beckenstein-Hawking entropy interpreted statistically from quantum geometry and is given by (see Eq.(10.27) of [32]):

$$\gamma = 0,274, \tag{26}$$

showing that our result is compatible with loop quantum gravity. This is because our result has to be intended as an estimation due to the approximations assumed in the derivation.

We also notice that a similar result can be obtained in the contest of renormalization group quantum gravity. Indeed in [28] the improved Schwarzschild metric obtained in the case of a static, spherically symmetric spacetime with a point mass $m_{(0)}$ situated at the origin is given by:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2(\theta)d\phi^2\right), \tag{27}$$

where $f(r)$ is the improved radial function

$$f(r) = 1 - \frac{4Gm_{(0)}r^2/c^2}{2r^3 + \tilde{\omega} l_{Pl}^2 (2r + 9Gm_{(0)}/c^2)}, \tag{28}$$

with $\tilde{\omega} = 118/15\pi$. Below to a critical mass $m_{(0)c} = 3,5027 m_{Pl}$, where m_{Pl} is the Planck mass, there is no event horizon anymore and the function $f(r)$ exhibits in general an extremal point due to the weakening effect of the quantum corrections proportional to the Planck length l_{Pl} in Eq(28). This is actually the kind of behavior we have anticipated in Section 2. Imposing now $f'(r) = 0$ we obtain for the extremal points the following condition:

$$r^3 - \tilde{\omega} l_{Pl}^2 \left(\frac{r}{2} - \frac{9Gm_{(0)}}{2c^2}\right) = 0. \tag{29}$$

In the limit $m_{(0)} \to 0$ we get for the non vanishing minimal radius $r_{min} = \sqrt{\tilde{\omega}/2}\, l_{Pl}$ and thus the following formula for the minimal volume element:

$$V_{min} = \frac{4\pi}{3}\left(\frac{\tilde{\omega}\hbar G}{2c^3}\right)^{\frac{3}{2}}. \tag{30}$$

Comparing this result again with Eq.(24), we obtain for the Barbero-Immirzi constant the result:

$$\gamma = \frac{\tilde{\omega}}{4}\left(\frac{\sqrt{3}}{6\pi}\right)^{\frac{1}{3}} = 0,282, \tag{31}$$

which is in good agreement with Eq.(26).

After this discussion we will from now on assume for the smallest geometrical scale Eq.(24) together with Eq.(26). This result can be used to estimate the natural cutoff for the quantum fluctuations. Indeed to this purpose we propose to substitute in the usual uncertainty principle $\Delta x \cdot \Delta p \geq \hbar/2$ the position uncertainty Δx with $\sqrt[3]{V_{min}^{LQG}}$ of Eq.(24). This is possible, because in loop quantum gravity V_{min} is coordinates independent. Moreover the momentum uncertainty Δp is identified with the UV momentum cutoff k_c/c. In this way and assuming a minimal uncertainty for the higher energy fluctuations, we obtain

$$k_c = \frac{\hbar c}{2\sqrt[3]{V_{min}^{LQG}}} = \frac{\left(6\sqrt{3}\right)^{\frac{1}{6}}}{\sqrt{8\pi\gamma}}\frac{E_{Pl}}{2}, \quad (32)$$

where $E_{Pl} = \sqrt{\hbar c^5/G}$ is the Planck energy.

4 A possible description of the dark sector

In this Section we want to investigate the cosmological consequences of our result obtained in Eq.(32). A few years ago a new approach in considering the zero-point energy fluctuations of the quantum fields has been proposed in [20]. According to their method it was possible to obtain a consistent formula for the computation of the cosmological dark energy density ρ_{vac} entirely from vacuum energy quantum fluctuations. The basic additional principles of the authors in [20] are that the empty Minkowski space should be gravitational stable ($\rho_{vac} = 0$), that our universe is spatially flat and that the vacuum stress energy tensor should have the form $< T_{\mu\nu} >= -\rho_{vac}\, g_{\mu\nu}$ with $\dot{\rho}_{vac} = 0$. These are the usual properties assumed in the Standard Model of Cosmology, the ΛCDM-model, for the cosmological constant. Following [20], we remind the reader that according to the Friedmann equations,

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3\,c^2}\,\rho \\ \left(\frac{\dot{a}}{a}\right)^2 &+ 2\,\frac{\ddot{a}}{a} = -\frac{8\pi G}{c^2}\,p, \end{aligned} \quad (33)$$

one can easily check that

$$\dot{\rho}_{vac} = -3\left(\frac{\dot{a}}{a}\right)(\rho_{vac} + p_{vac}), \quad (34)$$

This implies that putting $p_{vac} = w\rho_{vac}$ with $w = -1$, for the vacuum energy equation of state one satisfies simultaneously the constraints $\dot{\rho}_{vac} = 0$ and $< T_{\mu\nu} >= -\rho_{vac}\, g_{\mu\nu}$ and hence also $\nabla^{\mu} < T_{\mu\nu} >= 0$ with ∇_{μ} the usual covariant derivative. The result computed in [20] is:

$$\rho_{vac} = \frac{g\,c^2}{8\pi\gamma}\left(\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\ddot{a}}{a}\right) \quad (35)$$

with

$$g = \frac{3\gamma}{8\pi\hbar c^5}\,\Delta N k_c^2, \quad (36)$$

where $\Delta N = N_f - N_b$ is the difference between the number of species of fermions and bosons and k_c is an UV-cutoff. The reader can find in the Appendix a detailed derivation of Eq.(35). Now substituting $p_{vac} = w\rho_{vac}$ with ρ_{vac} given by Eq.(35) into the second of Eqs.(33), remembering that for non relativistic matter $p_m = 0$ and neglecting the relativistic radiation density, which is a factor $\sim 10^{-5}$ smaller than the total energy density, one can very easily check that

$$w = -\frac{1}{g}. \quad (37)$$

Clearly this result is consistent with the assumptions of [20] outlined at the beginning of this section only if $g = 1$.

In our study we relax the constraint $g = 1$ of [20] allowing more general and exotic possibilities, which deviates from the usual cosmological constant vacuum energy scenario with $w = -1$. Obviously, according to the actual experimental observations, a realistic description of the dark energy imposes that the deviations of g from the unity are expected to be small. Indeed taking recent fits from the observations of Type Ia supernovae dynamics [33, 34] of the HZSN and the the SCP collaborations we can estimate that

$$\left(\frac{\ddot{a}}{a}\right)_{t=t_0} \approx 0.58\,H_0^2, \quad (38)$$

where t_0 is the actual time and H_0 the actual Hubble constant. One can check this result for example computing the time derivatives of the fitting function for the scale factor $a(t)$ in Eq.(26.82a) in the book of Thomas Müller [35]. Although this is only a qualitative argument, because the specific fitted function of [35] is model dependent, it provides anyway a plausible estimation of the correct value. Substituting this result in Eq.(35), and using the usual definition for the critical density $\rho_{c\,0} = 3c^2H_0^2/(8\pi\gamma)$, we obtain

$$\Omega_{vac\,0} = \frac{\rho_{vac\,0}}{\rho_{c\,0}} \approx g\cdot\frac{1 + 2\cdot 0.58}{3} \approx g\cdot 0.7, \quad (39)$$

which with $g \approx 1$ is in quite good agreement with recent analysis from CMB measurements [36, 37], considering the experimental uncertainties in Eq.(38).

We have already shown that Eq.(35) satisfies the second of Eqs.(33) with the identification $g = -1/w$. We want now to discuss the solution of the first of Eqs.(33)

for the scale parameter $a(t)$ in the more general case $g \neq 1$. To this purpose we put the expression of the vacuum energy given in Eq.(35) into the first of of Eqs.(33) and thus we get the following differential equation for the scale factor $a(t)$:

$$\left(1 - \frac{g}{3}\right)\left(\frac{\dot{a}}{a}\right)^2 - \frac{2g}{3}\frac{\ddot{a}}{a} = H_0^2 \frac{\Omega_{m\,0}}{a^3}, \qquad (40)$$

where as usual $\Omega_{m\,0} = \rho_{m\,0}/\rho_{c\,0}, \rho_m = \rho_{m\,0}/a^3$ and where $\Omega_{rad\,0}$ has been again neglected. Performing now the change of variables,

$$w(a) = a\dot{a}^2, \qquad (41)$$

one has that Eq.(40) becomes a first order linear differential equation in w, whose solution is given by

$$w(a) = \Omega_{m\,0}H_0^2 + (1 - \Omega_{m\,0})a^{3/g}H_0^2, \qquad (42)$$

where the usual initial conditions $a_0 = a(t_0) = 1$ and $w_0 = w(a_0) = H_0^2$ have been imposed. According to this result and treating Eq.(41) as a separable variables differential equation, we can rewrite and integrate it as follows:

$$\int_{a_0}^{a} \frac{da}{\sqrt{\Omega_{m0}a^{-1} + (1 - \Omega_{m0})a^{-(1-3/g)}}} = H_0(t - t_0), \quad (43)$$

again with the initial condition $a_0 = a(t_0) = 1$. This integral represents the general solution to the Friedmann equations with the presence of matter with $w_m = 0$ and dark energy with $w_\Lambda = -1/g$ as expected by consistency. To our knowledge there is not a simple general analytic expression that solves the integral in Eq.(43) for $g \neq 1$. However a very simple solution can be obtained at early times ($a \ll a_0$), when the universe was matter dominated, and at later times ($a \gg a_0$), when the universe will be dark energy dominated:

$$a_g(t) \;\propto\; t^{2/3}; \qquad a \ll a_0; \qquad (44)$$

$$a_g(t) \;\propto\; \left[1 + \frac{3(g-1)}{2g}\sqrt{1 - \Omega_{m0}}H_0(t - t_0)\right]^{\frac{2g}{3(g-1)}}; \quad a \gg a_0. \qquad (45)$$

Consistently in the limit $g \to 1$ we obtain the later times behavior of the ΛCDM model according to which $a(t) \propto \exp\left[\sqrt{1 - \Omega_{m0}}H_0(t - t_0)\right]$. For the case $0 < g < 1$ we have that the scale factor $a_g(t)$ rapidly expands and diverges in the finite time $t = 2g/[3(1 - g)\sqrt{1 - \Omega_{m0}}H_0] + t_0$, producing a "Big Rip" as it is well known in the case that dark energy is phantom energy [38].

After that we come back to the physical interpretation of Eq.(36). First of all we substitute the computed result for the cutoff k_c of Eq.(32) into Eq.(36) and with Eq.(37) we obtain that

$$w = -\frac{8\pi\hbar c^5}{3Gk_c^2\Delta N} = -\frac{256\,\pi^2\gamma}{3(6\sqrt{3})^{\frac{1}{3}}\Delta N}. \qquad (46)$$

This is the main result of this paper. We can fix ΔN trying to satisfy the constraint $w = -1$ as accurately as possible. According to this point of view, we would find that,

$$w = -0.9976 \qquad \text{for} \quad \Delta N = 106, \qquad (47)$$

expecting for this case at least 46 additional fermionic degrees of freedom. Indeed in [20] the authors have shown that for the Standard Model of elementary particles $\Delta N_{SM} = 60$ ($N_f = 2 \times 3 + 4 \times 3 + 3 \times 4 \times 6$ including neutrinos, leptons and quarks and $N_b = 4 + 2 + 2 \times 3 + 2 \times 8 + 2$ including the Higgs, before symmetry breaking is performed, the photon, weak bosons, gluons and the graviton).

Finally we notice, that both the result for the equation of state $p_{vac} = w\rho_{vac}$ predicted in Eq.(47) and the dark energy density obtained by Eq.(39) are in agreement with the experimental measurements. However with the actual uncertainties it is not yet possible to discriminate all the possibilities above $\Delta N \approx 97$ and below $\Delta N \approx 108$. To come to this last statement we have compared with the central values of w reported in chapter 27 of [40]. The additional degrees of freedom could come from dark matter, whose nature has not yet been cleared.

5 Conclusions

Summarizing, we have firstly reviewed the computation of the modified Newtonian potential coming from the inclusion of the field mass and quantum effects in the source. Without any additional assumption and limiting ourselves to the small distance effects, we find for the smallest volume element $V_{min} = 4\pi/3\,(d_{min}/2)^3$ with $d_{min} = \sqrt{\pi e\hbar G/2c^3}$ of the empty space-time, a result which is compatible with the loop quantum gravity computation and also with the renormalization group approach (asymptotic safety). Hence imposing the loop quantum gravity result to the scale length of the quantum fluctuations we compute a natural UV-cutoff for the modes of the zero point energy finding the following expression: $k_c = (6\sqrt{3})^{\frac{1}{6}}/(2\sqrt{8\pi\gamma})E_{Pl}$, where $E_{Pl} = \sqrt{\hbar c^5/\gamma}$ is the Planck energy. Substituting this result into the formula of Bernard and LeClair for the cosmological constant given in [20], we obtain for the dark energy equation of state $p_{vac} = w\rho_{vac}$ the result $w = -256\pi^2\gamma/(3(6\sqrt{3})^{1/3}\Delta N)$, where γ is the usually called Barbero-Immirzi constant and where ΔN is the difference between the number of species of fermions and bosons. We find so that $w = -0.9976$ with $\Delta N = 106$. More generally comparing with the recent experimental determinations [40], we found that the number of additional fields beyond the Standard Model should at least include 37 fermionic degrees of freedom (implying $\Delta N = 97$), which could account for dark matter. Furthermore we also estimate that ΔN, according to the actual experimental constraints, should be bounded from above by 108.

Appendix

In this Appendix we compute the result for the vacuum energy from quantum fluctuations given in Eq.(35). We start with the action of a single bosonic field on a curved background:

$$S^b = \int dtd^3x\sqrt{-g}\frac{1}{2}\left(-\partial^\mu\phi\,\partial_\mu\phi - m^2\phi^2\right), \quad (48)$$

where g is the determinant of metric $g_{\mu\nu}$. We take as backgroung the FLRW-metric in the case of a spacially flat universe

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \quad (49)$$

thus implying that $g_{\mu\nu} = \text{diag}(-1, a^2, a^2, a^2)$ and that that $g = -a^6$. Before proceeding with the canonical quantization of the field, we first perform the change of variable $\phi = \chi/a^{3/2}$, in oder to remove the time dependence appearing in the measure of the integral action coming from g. Indeed in this way one obtains after some algebra that the action in Eq.(48) becomes

$$S^b = \int dtd^3x\frac{1}{2}\left((\partial_t\chi)^2 - \frac{1}{a^2}(\vec{\nabla}\chi)^2 - (m^2 - \mathcal{A})\chi^2\right), \quad (50)$$

where

$$\mathcal{A} = \frac{3}{4}\left(\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\ddot{a}}{a}\right). \quad (51)$$

The corresponding equation of motion for the field χ is then

$$\partial_t^2\chi - \frac{1}{a^2}\nabla^2\chi + (m^2 - \mathcal{A})\chi = 0. \quad (52)$$

We rewrite now the field χ as a Fourier integral with a relativistic invariant measure

$$\chi = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_{k/a}}}\left(a_{\vec{k}}u_{\vec{k}}(t)e^{i\vec{k}\cdot\vec{x}} + a^\dagger_{\vec{k}}u^*_{\vec{k}}(t)e^{-i\vec{k}\cdot\vec{x}}\right), \quad (53)$$

where

$$\omega^2_{k/a} = \frac{k^2}{a^2} + m^2 - \mathcal{A}, \quad (54)$$

where $a^\dagger_{\vec{k}}, a_{\vec{k}}$ are the usual creation and annihilation operators of a particle state with momentum $\vec{k}$ and where $u_{\vec{k}}$ is a time dependent function. Substituting the Fourier integral into Eq.(52) one obtains for $u_{\vec{k}}$ the following equation:

$$(\partial_t^2 + \omega^2_{k/a})u_{\vec{k}}(t) = 0. \quad (55)$$

In the so called “adiabatic limit” one assumes that the time dependence of $\omega_{k/a}$ can be neglected in our actual universe and one can easily find the solution to Eq.(55), which is

$$u_{\vec{k}}(t) = u_{\vec{k}}\,e^{-i\omega_{k/a}t}. \quad (56)$$

After that performing a Legendre transformation of the Lagrangian in the action for χ Eq.(50) and substituting Eq.(53) together with Eq.(56) into it, one finds for the Hamiltonian

$$\begin{aligned} H^b &= \int d^3x\frac{1}{2}\left((\partial_t\chi)^2 + \frac{1}{a^2}(\vec{\nabla}\chi)^2 + (m^2 - \mathcal{A})\chi^2\right) \\ &= \frac{1}{2}\int d^3k\,\omega_{k/a}(a^\dagger_{\vec{k}}a_{\vec{k}} + a_{\vec{k}}a^\dagger_{\vec{k}}). \end{aligned} \quad (57)$$

Now repeating a similar computation for a fermionic field one obtains

$$H^f = \frac{1}{2}\int d^3k\,\omega_{k/a}(b^\dagger_{\vec{k}}b_{\vec{k}} - b_{\vec{k}}b^\dagger_{\vec{k}}). \quad (58)$$

We introduce now the usual commutation (anticommutation) relations for bosons (fermions):

$$[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = \left\{b_{\vec{k}}, b^\dagger_{\vec{k}'}\right\} = \delta_3(\vec{k} - \vec{k}') = \int \frac{d^3x}{(2\pi)^3}e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}, \quad (59)$$

where we have also added the integral representation of the Dirac function. Remembering that $a_{\vec{k}}|vac >=$

$b_{\vec{k}}|vac>= 0$ for the vacuum state $|vac>$ and using the relations in Eq.(59) one gets

$$\rho^{b(f)}_{vac,0} \equiv \frac{1}{a^3 V_0} < vac|H^{b(f)}|vac >= \pm\frac{\delta_3(\vec{0})}{2V_0}\int d^3k\,\omega_k, \tag{60}$$

where the change of variables $\vec{k} \to \vec{k}/a$ has been performed and where + has to be chosen for bosons and − has to be chosen for fermions. According to the last equality in Eq.(59) one has that $\delta_3(\vec{0}) = V_0/(2\pi)^3$ and Eq.(60) becomes

$$\rho^{b(f)}_{vac,0} = \pm\frac{1}{16\pi^3}\int d^3k\,\omega_k. \tag{61}$$

Now as mentioned in the Introduction, one has to subtract from it the contribution from the flat space-time ($\mathcal{A} = 0$) and the vacuum energy contributions to dark energy becomes:

$$\rho^{b(f)}_{vac} = \pm\frac{1}{16\pi^3}\int d^3k\left(\sqrt{k^2+m^2-\mathcal{A}} - \sqrt{k^2+m^2}\right), \tag{62}$$

where Eq.(54) has been used. Finally remembering that $d^3k = k^2dk\sin(\theta)d\theta d\phi$, introducing the large cutoff k_c to regulate the integral and considering N_f fermionic and N_b bosonic fields, one obtains at the leading order

$$\rho_{vac} = N_f\rho^f_{vac} + N_b\rho^b_{vac} = \frac{\Delta N k_c^2\mathcal{A}}{16\pi^2} + \ldots, \tag{63}$$

where $\Delta N = N_f - N_b$ and where the additional terms are all suppresed by powers of $1/k_c^2$. As a last step one can easily check that Eq.(63) coincides with Eq.(35).

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