THE RIEMANN HYPOTHESIS

Shekhar Suman

August 2019

0.1 Abstract

The Proof involves Functional Equation of the Riemann Zeta function defined on the whole of Complex plane except for a Pole at s=1.

Next we prove that the above functional equation is monotonically increasing in a domain.

Till date, 100000000000000 trivial zeroes of the Riemann Zeta function has been found, but for non trivial zeroes in the Critical Strip,

THE RIEMANN HYPOTHESIS (1859): The real part of every non trivial zero of Riemann Zeta Function is 1/2.

0.2 Proof

The analytic continuation (Ref.-[1]) of the Zeta Function is

$\zeta^*(s) = s\int_0^\infty([x]-x)/x^{s+1}\ dx\ ;0<Re(s)<1$

$let,\ s = \sigma + i\eta;\ \ 0<\sigma<1.$

Claim:- $\mid\zeta^*(\sigma+i\eta)\mid$ *is Monotonically increasing on* $1/2<\sigma<1.$

$Let,\ \ \sigma<\rho\ \ ;\ 1/2<\sigma,\rho<1$

$\mid\zeta^*(\sigma+i\eta)\mid^2 - \mid\zeta^*(\rho+i\eta)\mid^2 =$

$\mid(\sigma+i\eta)\int_0^\infty([x]-x)/x^{1+\sigma+i\eta}dx|^2 - \mid(\rho+i\eta)\int_0^\infty([x]-x)/x^{1+\rho+i\eta}dx|^2$

$\leq (\mid(\sigma^2+\eta^2)\mid\mid\int_0^\infty([x]-x)/x^{1+\sigma+i\eta}dx|^2 - \mid(\rho^2+\eta^2)\mid\mid\int_0^\infty([x]-x)/x^{1+\rho+i\eta}dx|^2\ )$

$\sigma<\rho,\ \ implies$

$\leq\mid(\rho^2+\eta^2)(\ \mid\mid\int_0^\infty([x]-x)/x^{1+\sigma+i\eta}dx|^2 - |\int_0^\infty([x]-x)/x^{1+\rho+i\eta}dx|^2)$

$\leq\mid(\rho^2+\eta^2)(\ \mid\mid\int_0^\infty([x]-x)/x^{1+\sigma}e^{-i\ \eta\ lnx}dx|^2 - |\int_0^\infty([x]-x)/x^{1+\rho}e^{-i\eta\ lnx}dx|^2\ )$

$\leq\mid(\rho^2+\eta^2)(\ \mid\mid\int_0^\infty([x]-x)/x^{1+\sigma}\mid e^{-i\ \eta\ lnx}\mid dx|^2 - |\int_0^\infty([x]-x)/x^{1+\rho}\mid e^{-i\eta\ lnx}\mid$

$dx|^2)$

$\leq\ (\rho^2+\eta^2)\ (\ (\ \int_0^\infty(x-[x])/x^{1+\sigma}dx)^2 - \mid(\int_0^\infty([x]-x)/x^{1+\rho}(e^{-\ i\eta\ lnx})dx|^2)\mid)$

$Since 1/2 < \sigma < 1.$

$so,$

$\leq (\rho^2+\eta^2)\ (\ (\ \int_0^\infty 1/x^{3/2}dx)^2 -$

$(\ \int_0^\infty 1/x^2 cos(\ \eta\ lnx)dx)^2$

$+ ((\ \int_0^\infty 1/x^{3/2} sin(\eta lnx)\ \ )^2)\ dx$

$cos\theta, sin\theta \leq 1;$

$\leq (\rho^2+\eta^2)\ (\ (\ \int_0^\infty 1/x^{3/2}dx)^2 -$

$(\ \int_0^\infty 1/x^2 dx)^2$

$+ ((\ \int_0^\infty 1/x^{3/2}\ \ )^2)\ dx$

$\leq (\rho^2+\eta^2)(\ \ (\ 2(\ \int_0^\infty 1/x^{3/2}dx)^2 -$

$(\ \int_0^\infty 1/x^2 dx)^2)\ \ )$

$\leq (\rho^2+\eta^2)\ ((-1/x^{1/2})^2 - (1/x)^2$

$\leq (\rho^2+\eta^2)\ (1/x^2 - 1/x^2)$

$\leq (\rho^2+\eta^2)\ 0$

≤ 0

$|\ \zeta^*(\sigma+i\eta)\ |^2 - |\ \zeta^*(\rho+i\eta)\ |^2 \leq\ 0.$

$|\ \zeta^*(\sigma+i\eta)\ |^2 \leq |\ \zeta^*(\rho+i\eta)\ |^2\ .$

$So,$

$\sigma < \rho, implies$

$| \zeta^*(\sigma + i\eta) |^2 \leq | \zeta^*(\rho + i\eta) |^2$.

CLAIM: $\zeta^*(\sigma + i\eta) is\ Monotonically\ Decreasing\ on\ (0, 1/2]$.

$Observe,$

$| \zeta^*(s) |=| \zeta^*(1 - s) |$

$1/2 \leq \eta, \sigma \leq 1,$

$imply$

$| \zeta^*(\sigma + i\eta) |^2 \leq | \zeta^*(\rho + i\eta) |^2$.

$0< \ 1 - \sigma< \ 1 - \eta \leq 1/2$

$Implies$

$| \zeta^*(1 - \sigma - i\eta) |^2\geq| \zeta^*(1 - \rho - i\eta) |^2\ Put, 1 - \sigma = \sigma^1 and$

$1 - \rho = \rho^1$

$So,$

$0< \sigma^1 < \eta^1 \leq 1/2 implies,$

$| \zeta^*(\sigma^1 + i\eta) |^2 \geq | \zeta^*(\rho^1 + i\eta) |^2$.

So,$| \zeta^*(\sigma + i\eta) |\ \ is\ Monotonically\ Decreasing$

$Thus,$

$0<\sigma<\rho<`1/2 \ \ implies$

$| \zeta^*(1/2 + i\rho) | \geq | \zeta^*(\sigma + i\rho) |$

$Also, 1/2 \leq \sigma < \rho \leq 1, implies$

$| \zeta^*(\sigma + i\eta) | \leq | \zeta^*(\rho + i\eta) | .$

$| \zeta^*(1/2 + i\eta) | \leq | \zeta^*(\sigma + i\eta) | .$

$Combining \ the \ above \ ineualities,$

$| \zeta^*(\sigma + i\eta) | \leq | \zeta^*(1/2 + i\eta) | \leq | \zeta^*(\sigma + i\eta) | .$

$By \ \ hypothesis, | \zeta^*(\sigma + i\eta) = 0, 0< \sigma<1$

$0 \leq | \zeta^*(1/2 + i\eta) | \leq \ 0.$

$$| \zeta^*(1/2 + i\eta) | = \ 0$$

Hence, all non trivial zeroes lie on the line with real part 1/2.

This proves the Riemann hypothesis.

0.3 References:-

[1]. T.I.F.R. Lectures Pg-94 on Riemann Zeta function by K. Chandrasekharan

[2] Peter Roquette - Lecture Notes in Mathematics 2222 The Riemann Hypothesis in Characteristic p in Historical Perspective [1st ed.]- Springer International Publishing 2018.

Ariel Jacobs - The Proof of the Riemann Hypothesis.

[3] Kevin Broughan - Equivalents of the Riemann Hypothesis : Arithmetic Equivalents.

[4] Lars Ahlfors - Complex analysis [3 ed.] McGraw -Hill 1979.

[5] Tom M. Apostol - Introduction to Analytical Number Theory 1976.

[6] Complex Analysis - Walter and Rudin 3rd Edition 1970.

[7] https://www.claymath.org/millennium-problems/riemann-hypothesis.

[8] A note on S(t) and the zeros of the Riemann zeta-function - DA Goldston, SM Gonek.

Ramanujan Summation- Springer, Bernard Candelgrapher[10] Riemann Rearrangement Theorem- V.M Cadets, American Mathematical Society.

Contact:

Shekhar Suman.

N.B.H.M. M.Sc. Scholar-2015-17.

I.I.T. Delhi M.Sc. (2015-17).

Email - shekharsuman068@gmail.com