Noname manuscript No.
(will be inserted by the editor)

A Final Proof of The *abc* Conjecture

Abdelmajid Ben Hadj Salem

Received: date / Accepted: date

Abstract In this paper, we consider the *abc* conjecture. As the conjecture $c < rad^2(abc)$ is less open, we give firstly the proof of a modified conjecture that is $c < 2rad^2(abc)$. The factor 2 is important for the proof of the new conjecture that represents the key of the proof of the main conjecture. Secondly, the proof of the *abc* conjecture is given for $\epsilon \geq 1$, then for $\epsilon \in]0,1[$. We choose the constant $K(\epsilon)$ as $K(\epsilon) = 2e^{\left(\frac{1}{\epsilon^2}\right)}$ for $\epsilon \geq 1$ and $K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}$ for $\epsilon \in]0,1[$. Some numerical examples are presented.

Keywords Elementary number theory · real functions of one variable.

Mathematics Subject Classification (2010) 11AXX · 26AXX

To the memory of my Father who taught me arithmetic
To the memory of my colleague and friend Jamel Zaiem (1956-2019)

1 Introduction and notations

Let a positive integer $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as :

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{1}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{2}$$

Abdelmajid Ben Hadj Salem
Tunis, Tunisia
E-mail: abenhadjsalem@gmail.com

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [1]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

Conjecture 1 (***abc* Conjecture**): Let a, b, c positive integers relatively prime with $c = a + b$, then for each $\epsilon > 0$, there exists a constant $K(\epsilon)$ such that :

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{3}$$

$K(\epsilon)$ depending only of ϵ.

The idea to try to write a paper about this conjecture was born after the publication of an article in Quanta magazine about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [2]. The difficulty to find a proof of the *abc* conjecture is due to the incomprehensibility how the prime factors are organized in c giving a, b with $c = a + b$. So, I will give a simple proof in the two cases $c = a + 1$ and $c = a + b$ that can be understood by undergraduate students.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [1]. A conjecture was proposed that $c < rad^2(abc)$ [3]. It is the key to resolve the *abc* conjecture. In my paper, I propose to give the proof that $c < 2rad^2(abc)$, it facilitates the proof of the *abc* conjecture. The paper is organized as fellow: in the second and third section, we give successively the proof of $c < 2rad^2(ac)$ and $c < 2rad^2(abc)$. The main proof of the *abc* conjecture is presented in section four for the two cases $c = a+1$ and $c = a+b$. The numerical examples are discussed in sections five and six.

2 The Proof of the Conjecture $c < 2rad^2(ac)$, Case : $c = a + 1$

Below is given the definition of the conjecture $c < 2rad^2(abc)$:

Conjecture 2 Let a, b, c positive integers relatively prime with $c = a + b, a > b, b \geq 2$, then:

$$c < 2rad^2(abc) \Longrightarrow \frac{Logc}{Log(rad(abc))} < 2 + \frac{Log2}{Log(rad(abc))} \tag{4}$$

In the case $c = a + 1$, the definition of the conjecture is:

Definition 1 Let a, c positive integers, relatively prime, with $c = a + 1$,$a \geq 2$ then:

$$c < 2rad^2(ac) \Longrightarrow \frac{Logc}{Log(rad(ac))} < 2 + \frac{Log2}{Log(rad(ac))} \tag{5}$$

Proof :
1 - If $c < rad(ac)$ then we obtain:

$$c < rad(ac) < 2rad^2(ac) \tag{6}$$

and the condition (5) is verified.

2 - If $c = rad(ac)$, then a, c are not relatively coprime. Case to reject.

3 - We suppose that $c > rad(ac) \Longrightarrow \mu_c > rad(a)$, we have also $a > rad(ac) \Longrightarrow \mu_a > rad(c)$.

3a - Case $\mu_a \leq rad(a)$: $c = 1 + a \leq 1 + rad^2(a) < rad^2(ac) < 2rad^2(ac)$, and the condition (5) is verified.

3b - Case $\mu_c \leq rad(c)$: $c = \mu_c rad(c) \leq rad^2(c) < rad^2(ac) < 2rad^2(ac)$, and the condition (5) is verified.

3c - Case $\mu_a > rad(a)$ and $\mu_c > rad(c)$. As $\mu_a > rad(c)$, we can write that $\mu_a = l.rad(c) + l'$ with $1 \leq l' < rad(c) \Longrightarrow \mu_a < (l+1)rad(c) \Longrightarrow a < (l+1)rad(ac)$

3c1 - We suppose that $l+1 \leq rad(ac) \Longrightarrow l < rad(ac)$ then $a < (l+1)rad(ac) \leq rad^2(ac) \Longrightarrow c < 2rad^2(ac)$, and the condition (5) is verified.

3c2 - We suppose that $l = rad(ac) \Longrightarrow \mu_a = rad(a)rad^2(c)+l' < rad(c)(rad(ac)+1) \Longrightarrow a < rad(ac)(rad(ac)+1) < 2rad^2(ac) \Longrightarrow a < 2rad^2(ac) \Longrightarrow c \leq 2rad^2(ac)$. As c can not be equal to $2rad^2(ac)$, we obtain $c < 2rad^2(ac)$ and the condition (5) is verified.

3c3 - Case: $l > rad(ac)$. As $\mu_a = lrad(c) + l' \Longrightarrow \mu_a > rad(a)rad^2(c)$, we can write that $\mu_a = m.rad(a)rad^2(c) + r$ with $m, r \in \mathbb{N}, m \geq 1$ and $0 < r < rad(a)rad^2(c)$. Then:

$$\begin{gathered}\mu_a = m.rad(a)rad^2(c) + r \Longrightarrow a = \mu_a.rad(a) = m.rad^2(a)rad^2(c) + r.rad(a) \Longrightarrow \\ a < mrad^2(ac) + rad^2(ac) \Longrightarrow a < (m+1)rad^2(ac) \quad with\ m \geq 1 \Longrightarrow \\ a < (1+1)rad^2(ac) \Longrightarrow a < 2rad^2(ac) \Longrightarrow a+1 = c \leq 2rad^2(ac) \end{gathered} \tag{7}$$

As c can not be equal to $2rad^2(ac)$, we deduce that $c < 2rad^2(ac)$ and the condition (5) is verified.
We announce the theorem:

Theorem 1 *Let a, c positive integers relatively prime with $c = a+1, a \geq 2$, then $c < 2rad^2(ac)$.*

3 The Proof of the Conjecture $c < 2rad^2(abc)$, Case : $c = a + b$

Below is given the definition of the conjecture $c < 2rad^2(abc)$:

Conjecture 3 Let a, b, c positive integers relatively prime with $c = a + b, a > b, b \geq 2$, then:

$$c < 2rad^2(abc) \Longrightarrow \frac{Logc}{Log(rad(abc))} < 2 + \frac{Log2}{Log(rad(abc))} \tag{8}$$

Proof :
4 - If $c < rad(abc)$ then we obtain:

$$c < rad(abc) < rad^2(abc) < 2rad^2(abc) \tag{9}$$

and the condition (8) is verified.

5 - If $c = rad(abc)$, then a, b, c are not relatively coprime. Case to reject.

6 - We suppose that $c > rad(abc) \Longrightarrow \mu_c > rad(ab)$, we can write :

$$\mu_c = lrad(ab) + l', \quad with \quad 0 < l' < rad(ab) \Longrightarrow$$
$$\mu_c < lrad(ab) + rad(ab) = (l+1)rad(ab) \Longrightarrow c < (l+1)rad(abc) \tag{10}$$

6a - Case $l+1 \leq rad(abc) \Longrightarrow l < rad(abc)$, then $c < rad^2(abc) < 2rad^2(abc) \Longrightarrow c < 2rad^2(abc)$ and the condition (8) is verified.

6b - Case $l = rad(abc)$: From $c < (l+1)rad(abc) \Longrightarrow c < rad(abc)(rad(abc)+1) < 2rad^2(abc)$, then $c < 2rad^2(abc)$ and the condition (8) is verified.

6c - Case $l > rad(abc)$: From $\mu_c = lrad(ab) + l'$, we deduce that $\mu_c > rad^2(ab)rad(c)$, so we can write:

$$\mu_c = mrad^2(ab)rad(c) + r \quad m \geq 1, 0 < r < rad^2(ab)rad(c) \Longrightarrow$$
$$\mu_c < (m+1)rad^2(ab)rad(c), m \geq 1 \Longrightarrow c < (m+1)rad^2(abc)$$
$$Taking \;\; m = 1 \Longrightarrow c < 2rad^2(abc) \tag{11}$$

And the condition (8) is verified.
We announce the theorem:

Theorem 2 *Let a, b, c positive integers relatively prime with $c = a + b, a > b, b \geq 2$, then $c < 2rad^2(abc)$.*

4 The Proof of the *abc* conjecture

Let $R = rad(ac)$ or $R = rad(abc)$.

4.1 Case : $\epsilon \geq 1$

Using the result that $c < 2rad^2(ac)$ or $c < 2rad^2(abc)$, we have $\forall \epsilon \geq 1$:

$$c < 2R^2 \leq 2R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad with\ K(\epsilon) = 2e^{\left(\frac{1}{\epsilon^2}\right)},\ \epsilon \geq 1 \tag{12}$$

We verify easily that $K(\epsilon) > 2$ for $\epsilon \geq 1$. Then the *abc* conjecture is true.

4.2 Case: $\epsilon < 1$

4.2.1 Case: $c < R$

In this case, we can write :

$$c < R < R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad with\ K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)},\ \epsilon < 1 \tag{13}$$

here also $K(\epsilon) > 1$ for $\epsilon < 1$ and the *abc* conjecture is true.

4.2.2 Case: $c > R$

In this case, we confirm that :

$$c < K(\epsilon).R^{1+\epsilon}, \quad with\ K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, 0 < \epsilon < 1 \tag{14}$$

If not, then $\exists \epsilon_0 \in]0, 1[$, so that the triplet (a, b, c) checking $c > R$ and:

$$c \geq R^{1+\epsilon_0}.K(\epsilon_0) \tag{15}$$

are in finite number. We have:

$$\begin{aligned} c \geq R^{1+\epsilon_0}.K(\epsilon_0) \Longrightarrow R^{1-\epsilon_0}.c \geq R^{1-\epsilon_0}.R^{1+\epsilon_0}.K(\epsilon_0) \Longrightarrow \\ R^{1-\epsilon_0}.c \geq R^2.K(\epsilon_0) > \frac{c}{2}K(\epsilon_0) \Longrightarrow R^{1-\epsilon_0} > \frac{K(\epsilon_0)}{2} \end{aligned} \tag{16}$$

As $c > R$, we obtain:

$$\begin{aligned} c^{1-\epsilon_0} > R^{1-\epsilon_0} > \frac{K(\epsilon_0)}{2} \Longrightarrow \\ c^{1-\epsilon_0} > \frac{K(\epsilon_0)}{2} \Longrightarrow c > \left(\frac{K(\epsilon_0)}{2}\right)^{\left(\frac{1}{1-\epsilon_0}\right)} \end{aligned} \tag{17}$$

We deduce that it exists an infinity of triplets (a, b, c) verifying (15), hence the contradiction. Then the proof of the abc conjecture is finished. We obtain that $\forall \epsilon > 0$, $c = a + b$ with a, b, c relatively coprime:

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \quad with \quad \begin{cases} K(\epsilon) = 2e^{\left(\frac{1}{\epsilon^2}\right)} & \epsilon \geq 1 \\ K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)} & 0 < \epsilon < 1 \end{cases} \tag{18}$$

Q.E.D

In the two following sections, we are going to verify some numerical examples. We find that $c < rad^2(abc) \Longrightarrow c < 2rad^2(abc)$ and our proposed conjecture is true.

5 Examples : Case $c = a + 1$

5.1 Example 1

The example is given by:

$$1 + 5 \times 127 \times (2 \times 3 \times 7)^3 = 19^6 \tag{19}$$

$a = 5 \times 127 \times (2 \times 3 \times 7)^3 = 47\,045\,880 \Rightarrow \mu_a = 2 \times 3 \times 7 = 42$ and $rad(a) = 2 \times 3 \times 5 \times 7 \times 127$, in this example, $\mu_a < rad(a)$.
$c = 19^6 = 47\,045\,880 \Rightarrow rad(c) = 19$. Then $rad(ac) = rad(ac) = 2 \times 3 \times 5 \times 7 \times 19 \times 127 = 506\,730$.
We have $c > rad(ac)$ but $rad^2(ac) = 506\,730^2 = 256\,775\,292\,900 > c = 47\,045\,880$.

5.1.1 Case $\epsilon = 0.01$

$c < K(\epsilon).rad(ac)^{1+\epsilon} \Longrightarrow 47\,045\,880 \overset{?}{<} e^{10000}.506\,730^{1.01}$. The expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{\frac{1}{0.0001}} = e^{10000} = 8,7477777149120053120152473488653e{+}4342 \tag{20}$$

We deduce that $c \ll K(0.01).506\,730^{1.01}$ and the equation (18) is verified.

5.1.2 Case $\epsilon = 0.1$

$K(0.1) = e^{\frac{1}{0.01}} = e^{100} = 2,6879363309671754205917012128876e + 43 \Longrightarrow c < K(0.1) \times 506\,730^{1.01}$, and the equation (18) is verified.

5.1.3 Case $\epsilon = 1$

$K(1) = 2e \Longrightarrow c = 47\,045\,880 < 2e.rad^2(ac) = 2 \times 697\,987\,143\,184,212$ and the equation (18) is verified.

5.1.4 Case $\epsilon = 100$

$$K(100) = 2e^{0.0001} \Longrightarrow c = 47\,045\,880 \overset{?}{<} 2e^{0.0001}.506\,730^{101} = \\ 2 \times 1,52223502486076087818531426872 84e+576$$

and the equation (18) is verified.

5.2 Example 2

We give here the example 2 from $https://nitaj.users.lmno.cnrs.fr$:

$$3^7 \times 7^5 \times 13^5 \times 17 \times 1831 + 1 = 2^{30} \times 5^2 \times 127 \times 353 \tag{21}$$

$a = 3^7 \times 7^5 \times 13^5 \times 17 \times 1831 = 424\,808\,316\,456\,140\,799 \Rightarrow rad(a) = 3 \times 7 \times 13 \times 17 \times 1831 = 8497671 \Longrightarrow \mu_a > rad(a)$,
$b = 1$, $rad(c) = 2 \times 5 \times 127 \times 353$ Then $rad(ac) = 849767 \times 448310 = 3\,809\,590\,886\,010 < c$. $rad^2(ac) = 14\,512\,982\,718\,770\,456\,813\,720\,100 > c$, then $c \leq 2rad^2(ac)$. For example, we take $\epsilon = 0.5$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{1/0.25} = e^4 = 54,59800313096579789056 \tag{22}$$

Let us verify (18):

$$c \overset{?}{<} K(\epsilon).rad(ac)^{1+\epsilon} \Longrightarrow c = 424808316456140800 \overset{?}{<} K(0.5) \times (3\,809\,590\,886\,010)^{1.5} \Longrightarrow \\ 424808316456140800 < 405970304762905691174,98260818045 \tag{23}$$

Hence (18) is verified.

6 Examples : Case $c = a + b$

6.1 Example 1

We give here the example of Eric Reyssat [1], it is given by:

$$3^{10} \times 109 + 2 = 23^5 = 6436343 \tag{24}$$

$a = 3^{10}.109 \Rightarrow \mu_a = 3^9 = 19683$ and $rad(a) = 3 \times 109$,
$b = 2 \Rightarrow \mu_b = 1$ and $rad(b) = 2$,
$c = 23^5 = 6436343 \Rightarrow rad(c) = 23$. Then $rad(abc) = 2 \times 3 \times 109 \times 23 = 15042$.
For example, we take $\epsilon = 0.01$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{9999.99} = 8,7477777149120053120152473488653e+4342 \tag{25}$$

Let us verify (18):

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow c = 6436343 \overset{?}{<} K(0.01) \times (3 \times 109 \times 2 \times 23)^{1.01} \Longrightarrow \\ 6436343 \ll K(0.01) \times 15042^{1.01} \tag{26}$$

Hence (18) is verified.

6.2 Example 2

The example of Nitaj about the ABC conjecture [1] is:

$$a = 11^{16}.13^2.79 = 613\,474\,843\,408\,551\,921\,511 \Rightarrow rad(a) = 11.13.79 \quad (27)$$
$$b = 7^2.41^2.311^3 = 2\,477\,678\,547\,239 \Rightarrow rad(b) = 7.41.311 \quad (28)$$
$$c = 2.3^3.5^{23}.953 = 613\,474\,845\,886\,230\,468\,750 \Rightarrow rad(c) = 2.3.5.953 \quad (29)$$
$$rad(abc) = 2.3.5.7.11.13.41.79.311.953 = 28\,828\,335\,646\,110 \quad (30)$$

6.2.1 Case 1

we take $\epsilon = 100$ we have:

$$c \stackrel{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 \stackrel{?}{<} 2e^{0.0001}.(2.3.5.7.11.13.41.79.311.953)^{101} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 < 2 \times 2.7657949971494838920022381186039e + 1359$$

then (18) is verified.

6.2.2 Case 2

We take $\epsilon = 0.5$, then:

$$c \stackrel{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow \quad (31)$$
$$613\,474\,845\,886\,230\,468\,750 \stackrel{?}{<} e^4.(2.3.5.7.11.13.41.79.311.953)^{1.5} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 < 8\,450\,961\,319\,227\,998\,887\,403,9993 \quad (32)$$

We obtain that (18) is verified.

6.2.3 Case 3

We take $\epsilon = 1$, then

$$c \stackrel{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 \stackrel{?}{<} 2e.(2.3.5.7.11.13.41.79.311.953)^2 \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 < 831\,072\,936\,124\,776\,471\,158\,132\,100 \times 2e \quad (33)$$

We obtain that (18) is verified.

6.3 Example 3

It is of Ralf Bonse about the ABC conjecture [3] :

$$2543^4.182587.2802983.85813163 + 2^{15}.3^{77}.11.173 = 5^{56}.245983 \quad (34)$$
$$a = 2543^4.182587.2802983.85813163$$
$$b = 2^{15}.3^{77}.11.173$$
$$c = 5^{56}.245983$$
$$rad(abc) = 2.3.5.11.173.2543.182587.245983.2802983.85813163$$
$$rad(abc) = 1.5683959920004546031461002610848e + 33 \quad (35)$$

6.3.1 Case 1

For example, we take $\epsilon = 10$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = 2e^{0.01} = 2.0156314808565913486409234833 54$$

Let us verify (18):

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Rightarrow c = 5^{56}.245983 \overset{?}{<}$$
$$2e^{0.01}.(2.3.5.11.173.2543.182587.245983.2802983.85813163)^{11}$$
$$\Longrightarrow 3.4136998783296235160378273576498e + 44 <$$
$$2.8472401192989816352016241851442e + 365 \quad (36)$$

The equation (18) is verified.

6.3.2 Case 2

We take $\epsilon = 0.4 \Longrightarrow K(\epsilon) = 12.182473474251512159126256696 08$, then: The

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Rightarrow c = 5^{56}.245983 \overset{?}{<}$$
$$e^{6.25}.(2.3.5.11.173.2543.182587.245983.2802983.85813163)^{1.4}$$
$$\Longrightarrow 3.4136998783296235160378273576498e + 44 <$$
$$3.6255465680011453642792720569685e + 47 \quad (37)$$

And the equation (18) is verified.

Ouf, end of the mystery!

7 Conclusion

We have given an elementary proof of the *abc* conjecture, confirmed by some numerical examples. We can announce the important theorem:

Theorem 3 *(David Masser, Joseph Œsterlé & Abdelmajid Ben Hadj Salem; 2019) Let a, b, c positive integers relatively prime with $c = a + b$, then for each $\epsilon > 0$, there exists $K(\epsilon)$ such that :*

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{38}$$

where $K(\epsilon)$ is a constant depending of ϵ proposed as :

$$\begin{cases} K(\epsilon) = 2e^{\left(\dfrac{1}{\epsilon^2}\right)} & \epsilon \geq 1 \\ K(\epsilon) = e^{\left(\dfrac{1}{\epsilon^2}\right)} & 0 < \epsilon < 1 \end{cases}$$

.

Acknowledgements The author is very grateful to Professors Mihăilescu Preda and Gérald Tenenbaum for their comments about errors found in previous manuscripts concerning proofs proposed of the *abc* conjecture.

References

1. Waldschmidt M.: On the abc Conjecture and some of its consequences presented at The 6th World Conference on 21st Century Mathematics, Abdus Salam School of Mathematical Sciences (ASSMS), Lahore (Pakistan), March 6-9, 2013. (2013)
2. Klaus Kremmerz for Quanta Magazine. Titans of Mathematics Clash Over Epic Proof of ABC Conjecture. The Quanta Newsletter, 20 September 2018. www.quantamagazine.org. (2018).
3. Mihăilescu P.: Around ABC. European Mathematical Society Newsletter **N° 93**, September 2014. 29-34, (2014)