ON CERTAIN Π_q-IDENTITIES OF W. GOSPER

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ABSTRACT. In 2001 W. Gosper introduced the q-trigonometric functions and the constant Π_q and conjectured many intriguing identities on these q-trigonometric functions and Π_q. In this paper we employ some knowledge of modular equations with degree 5 to confirm several of Gosper's Π_q-identities. As a consequence, a q-identity involving Π_q and Lambert series, which was conjectured by Gosper, is proved. As an application, we confirm an interesting q-trigonometric identity of Gosper.

1. INTRODUCTION

Throughout this paper we assume that $|q| < 1$. W. Gosper [5] first introduced the q-constant Π_q :

$$\Pi_q = q^{1/4}\frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2}, \tag{1.1}$$

where $(a;q)_\infty$ is defined by

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n),$$

and then stated without proofs many identities involving Π_q based on the computer program MACSYMA. In particular, he [5, pp. 102–104] conjectured the following Π_q-identities:

$$\frac{\Pi_q^2}{\Pi_{q^2}\Pi_{q^4}} - \frac{\Pi_{q^2}^2}{\Pi_{q^4}^2} = 4, \tag{1.2}$$

$$\sqrt{\Pi_q\Pi_{q^9}}(\Pi_q + 3\Pi_{q^9}) = \Pi_{q^3}^2 + 3\Pi_q\Pi_{q^9}, \tag{1.3}$$

$$\Pi_{q^2}\Pi_{q^5}^4(16\Pi_{q^{10}}^4 - \Pi_{q^5}^4) = \Pi_{q^{10}}^3(5\Pi_{q^{10}} - \Pi_{q^2})(\Pi_{q^2} - \Pi_{q^{10}})^5,$$

$$\Pi_{q^{10}}\Pi_q^4(16\Pi_{q^2}^4 - \Pi_q^4) = \Pi_{q^2}^3(5\Pi_{q^{10}} - \Pi_{q^2})^5(\Pi_{q^2} - \Pi_{q^{10}}),$$

$$\Pi_q\Pi_{q^5}(16\Pi_{q^2}^4 - \Pi_q^4)^2 = \Pi_{q^2}^4(5\Pi_{q^5} - \Pi_q)^5(\Pi_{q^5} - \Pi_q),$$

$$\Pi_q\Pi_{q^5}(16\Pi_{q^{10}}^4 - \Pi_{q^5}^4)^2 = \Pi_{q^{10}}^4(5\Pi_{q^5} - \Pi_q)(\Pi_{q^5} - \Pi_q)^5,$$

$$(\Pi_q\Pi_{q^{10}} - \Pi_{q^2}\Pi_{q^5})^2 = \Pi_{q^2}\Pi_{q^{10}}(\Pi_{q^5} - \Pi_q)(5\Pi_{q^5} - \Pi_q)$$

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and

$$\frac{\sum_{n\geq 1}\frac{q^{2n-1}}{(1-q^{2n-1})^2}-5\sum_{n\geq 1}\frac{q^{10n-5}}{(1-q^{10n-5})^2}}{\Pi_{q^5}^2}=\sqrt{\frac{\Pi_q^3}{\Pi_{q^5}^3}-2\frac{\Pi_q^2}{\Pi_{q^5}^2}+5\frac{\Pi_q}{\Pi_{q^5}}}=\frac{\frac{\Pi_{q^5}^2}{\Pi_{q^{10}}^2}+16\frac{\Pi_{q^{10}}^2}{\Pi_{q^5}^2}}{\frac{\Pi_q}{\Pi_{q^5}}-4-\frac{\Pi_{q^5}}{\Pi_q}}.$$

The formula (1.2) was derived by Gosper [5, p. 93]. The Π_q-identity (1.3) was confirmed by the author and H.-C. Zhai [6] using an addition formula for the Jacobi theta function of Liu [8, Theorem 1] (for applications of Liu's addition formula, please see [7]). See M. El Bachraoui [3] for some other Π_q-identities.

In this paper we shall prove the following Π_q-identities by using some knowledge of modular equations with degree 5.

Theorem 1.1. *We have*

$$\Pi_{q^2}\Pi_{q^5}^4(16\Pi_{q^{10}}^4-\Pi_{q^5}^4)=\Pi_{q^{10}}^3(5\Pi_{q^{10}}-\Pi_{q^2})(\Pi_{q^2}-\Pi_{q^{10}})^5, \tag{1.4}$$

$$\Pi_{q^{10}}\Pi_q^4(16\Pi_{q^2}^4-\Pi_q^4)=\Pi_{q^2}^3(5\Pi_{q^{10}}-\Pi_{q^2})^5(\Pi_{q^2}-\Pi_{q^{10}}), \tag{1.5}$$

$$\Pi_q\Pi_{q^5}(16\Pi_{q^2}^4-\Pi_q^4)^2=\Pi_{q^2}^4(5\Pi_{q^5}-\Pi_q)^5(\Pi_{q^5}-\Pi_q), \tag{1.6}$$

$$\Pi_q\Pi_{q^5}(16\Pi_{q^{10}}^4-\Pi_{q^5}^4)^2=\Pi_{q^{10}}^4(5\Pi_{q^5}-\Pi_q)(\Pi_{q^5}-\Pi_q)^5, \tag{1.7}$$

$$(\Pi_q\Pi_{q^{10}}-\Pi_{q^2}\Pi_{q^5})^2=\Pi_{q^2}\Pi_{q^{10}}(\Pi_{q^5}-\Pi_q)(5\Pi_{q^5}-\Pi_q). \tag{1.8}$$

Theorem 1.2. *We have*

$$\frac{\sum_{n\geq 1}\frac{q^{2n-1}}{(1-q^{2n-1})^2}-5\sum_{n\geq 1}\frac{q^{10n-5}}{(1-q^{10n-5})^2}}{\Pi_{q^5}^2}=\sqrt{\frac{\Pi_q^3}{\Pi_{q^5}^3}-2\frac{\Pi_q^2}{\Pi_{q^5}^2}+5\frac{\Pi_q}{\Pi_{q^5}}}=\frac{\frac{\Pi_{q^5}^2}{\Pi_{q^{10}}^2}+16\frac{\Pi_{q^{10}}^2}{\Pi_{q^5}^2}}{\frac{\Pi_q}{\Pi_{q^5}}-4-\frac{\Pi_{q^5}}{\Pi_q}}. \tag{1.9}$$

The identities (1.4)–(1.7) only contain three of the constants $\Pi_q, \Pi_{q^2}, \Pi_{q^5}$ and $\Pi_{q^{10}}$, but the formula (1.8) includes all of these four constants. The structures of these five identities are similar so that our proofs share the same pattern. The identity (1.9) involves and Π_q, which leads to its huge appearence and complicated proof. The key to our proof of the identity (1.9) is to deal with the denominator

$$\sum_{n\geq 1}\frac{q^{2n-1}}{(1-q^{2n-1})^2}-5\sum_{n\geq 1}\frac{q^{10n-5}}{(1-q^{10n-5})^2}$$

and the constant

$$\sqrt{\frac{\Pi_q^3}{\Pi_{q^5}^3}-2\frac{\Pi_q^2}{\Pi_{q^5}^2}+5\frac{\Pi_q}{\Pi_{q^5}}}.$$

In the next section we provide some auxiliary results, which are crucial in the derivation of Theorem 1.1. Section 3 is devoted to our proof of Theorem 1.1. We will show Theorem 1.2 in Section 4. As a result, we in Section 5 confirm a q-identity involving Π_q and Lambert series, which was also conjectured by Gosper:

$$\begin{aligned}&6\bigg(\sum_{n\geq 1}\frac{q^n}{(1-q^n)^2}-5\sum_{n\geq 1}\frac{q^{5n}}{(1-q^{5n})^2}\bigg)+1\\&=\bigg(\frac{\Pi_q}{\Pi_{q^5}}+2+5\frac{\Pi_{q^5}}{\Pi_q}\bigg)\bigg(\sum_{n\geq 1}\frac{q^{2n-1}}{(1-q^{2n-1})^2}-5\sum_{n\geq 1}\frac{q^{10n-5}}{(1-q^{10n-5})^2}\bigg).\end{aligned}$$

As an application, we employ Theorem 1.2 to confirm an interesting q-trigonometric identity of Gosper:

$$\begin{aligned}\sin_q 5z &= \frac{\Pi_q}{\Pi_{q^5}}(\cos_{q^5} z)^4 \sin_{q^5} z - \sqrt{\frac{\Pi_q^3}{\Pi_{q^5}^3}-2\frac{\Pi_q^2}{\Pi_{q^5}^2}+5\frac{\Pi_q}{\Pi_{q^5}}}(\cos_{q^5} z)^2(\sin_{q^5} z)^3\\&\quad+(\sin_{q^5} z)^5\end{aligned}$$

in the last section.

2. Auxiliary results

In this section and throughout this paper we will adopt the notations of [1, Chapters 5 and 6]. We begin this section with the definition of modular equations [1, (6.3.2)].

Let $0 < k,\ l < 1$ and let n be a positive integer. A relation between k and l induced by the formula

$$n\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-k^2)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;k^2)}=\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-l^2)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;l^2)}$$

is called a modular equation of degree n. Take $\alpha = k^2, \beta = l^2$, we say that β has degree n over α. The multiplier m is given by

$$m=\frac{z_1}{z_n},$$

where

$$z_n=\varphi^2(q^n)$$

and

$$\varphi(q)=\sum_{k=-\infty}^{\infty}q^{k^2}.$$

The value of m depends on n, but throughout this paper we only consider modular equations of degree 5, then it is always assumed that $n = 5$ and

$$m=\frac{z_1}{z_5}. \tag{2.1}$$

In order to prove Theorem 1.1 we need several auxiliary results.

Theorem 2.1. *If β has degree 5 over α, then*

$$256\frac{z_1}{z_5}\left(\frac{\alpha}{\beta}\right)^{1/2}\frac{1}{\beta}\left(1-\frac{1}{\beta}\right)=\left(5-\frac{z_1}{z_5}\left(\frac{\alpha}{\beta}\right)^{1/2}\right)\left(\frac{z_1}{z_5}\left(\frac{\alpha}{\beta}\right)^{1/2}-1\right)^5, \tag{2.2}$$

$$256\frac{z_5}{z_1}\left(\frac{\beta}{\alpha}\right)^{1/2}\frac{1}{\alpha}\left(1-\frac{1}{\alpha}\right)=\left(5\frac{z_5}{z_1}\left(\frac{\beta}{\alpha}\right)^{1/2}-1\right)^5\left(1-\frac{z_5}{z_1}\left(\frac{\beta}{\alpha}\right)^{1/2}\right), \tag{2.3}$$

$$\frac{z_5}{z_1}\left(\frac{\beta}{\alpha}\right)^{1/4}\left(\alpha-1\right)^2=\frac{\alpha}{16}\left(5\frac{z_5}{z_1}\left(\frac{\beta}{\alpha}\right)^{1/4}-1\right)^5\left(\frac{z_5}{z_1}\left(\frac{\beta}{\alpha}\right)^{1/4}-1\right), \tag{2.4}$$

$$\frac{z_1}{z_5}\left(\frac{\alpha}{\beta}\right)^{1/4}\left(\beta-1\right)^2=\frac{\beta}{16}\left(5-\frac{z_1}{z_5}\left(\frac{\alpha}{\beta}\right)^{1/4}\right)\left(1-\frac{z_1}{z_5}\left(\frac{\alpha}{\beta}\right)^{1/4}\right)^5, \tag{2.5}$$

$$\left(\frac{z_1}{z_5}\left(\frac{\alpha}{\beta}\right)^{1/4}-\frac{z_1}{z_5}\left(\frac{\alpha}{\beta}\right)^{1/2}\right)^2=\frac{z_1}{z_5}\left(\frac{\alpha}{\beta}\right)^{1/2}\left(1-\frac{z_1}{z_5}\left(\frac{\alpha}{\beta}\right)^{1/4}\right)\left(5-\frac{z_1}{z_5}\left(\frac{\alpha}{\beta}\right)^{1/4}\right). \tag{2.6}$$

Proof. We first prove (2.2) and (2.5). According to [2, Chapter 19, (13.12)–(13.15)] we have

$$\left(\frac{\alpha}{\beta}\right)^{1/4}=\frac{2m+\rho}{m(m-1)}, \tag{2.7}$$

$$\left(\frac{\beta}{\alpha}\right)^{1/4}=\frac{2m-\rho}{5-m}, \tag{2.8}$$

$$\left(\frac{1-\beta}{1-\alpha}\right)^{1/4}=\frac{2m+\rho}{5-m},$$

$$(\alpha\beta)^{1/2}=\frac{4m^3-16m^2+20m+\rho(m^2-5)}{16m^2}, \tag{2.9}$$

$$\{(1-\alpha)(1-\beta)\}^{1/2}=\frac{4m^3-16m^2+20m-\rho(m^2-5)}{16m^2}, \tag{2.10}$$

where

$$\rho=(m^3-2m^2+5m)^{1/2}.$$

Then

$$\beta=\left(\frac{2m-\rho}{5-m}\right)^2\frac{4m^3-16m^2+20m+\rho(m^2-5)}{16m^2}, \tag{2.11}$$

$$1-\beta=\left(\frac{2m+\rho}{5-m}\right)^2\frac{4m^3-16m^2+20m-\rho(m^2-5)}{16m^2}. \tag{2.12}$$

Substituting (2.1), (2.7), (2.11) and (2.12) into both sides of each of the identities (2.2) and (2.5), noticing that $\rho=(m^3-2m^2+5m)^{1/2}$ and then simplifying we find that both sides of each of the identities (2.2) and (2.5) are respectively equal to

$$\left(\frac{2}{m-1}\right)^2A(m)$$

and

$$\frac{m-5}{256m(m-1)}B(m),$$

where

$$A(m) = 4m^9 - 24m^8 + m^8\rho + 64m^7 - 2m^7\rho - 200m^6 + 6m^6\rho - 40m^5 - 98m^5\rho$$
$$- 40m^4 + 80m^4\rho - 1280m^3 - 470m^3\rho - 504m^2 - 470m^2\rho - 28m - 70m\rho - \rho$$

and

$$B(m) = 2m^6 - 10m^5 + m^5\rho - 5m^4\rho + 4m^4 + 10m^3\rho$$
$$- 4m^3 - 102m^2 - 42m^2\rho - 18m - 27m\rho - \rho.$$

These prove (2.2) and (2.5).

We now show (2.3) and (2.4). According to [2, Chapter 19, (13.12)] we get

$$\left(\frac{1-\alpha}{1-\beta}\right)^{1/4} = \frac{2m-\rho}{m(m-1)}. \tag{2.13}$$

It follows from (2.7), (2.9), (2.10) and (2.13) that

$$\alpha = \left(\frac{2m+\rho}{m(m-1)}\right)^2 \frac{4m^3 - 16m^2 + 20m + \rho(m^2-5)}{16m^2}, \tag{2.14}$$

$$1-\alpha = \left(\frac{2m-\rho}{m(m-1)}\right)^2 \frac{4m^3 - 16m^2 + 20m - \rho(m^2-5)}{16m^2}. \tag{2.15}$$

We subsitute (2.1), (2.8), (2.14) and (2.15) into both sides of each of the identities (2.3) and (2.4), note that $\rho = (m^3 - 2m^2 + 5m)^{1/2}$ and then simplify to deduce that both sides of each of the identities (2.3) and (2.4) equal

$$-\frac{2^{12}m^2}{(m-5)^{12}}C(m)$$

and

$$\frac{1-m}{256m^6(m-5)}D(m)$$

respectively, where

$$C(m) = 28m^9 + 2520m^8 - m^8\rho + 32000m^7 - 350m^7\rho + 5000m^6 - 11750m^6\rho$$
$$+ 25000m^5 - 58750m^5\rho + 625000m^4 + 50000m^4\rho - 1000000m^3 - 306250m^3\rho$$
$$+ 1875000m^2 + 93750m^2\rho - 1562500m - 156250m\rho + 390625\rho,$$

and

$$D(m) = 18m^6 + 510m^5 - m^5\rho + 100m^4 - 135m^4\rho - 1050m^3\rho$$
$$-500m^3 + 1250m^2\rho + 6250m^2 - 6250m - 3125m\rho + 3125\rho,$$

which prove (2.3) and (2.4).

We finally prove (2.6). We subsitute (2.1) and (2.7) into both sides of (2.6) and then simplify using the identity $\rho = (m^3 - 2m^2 + 5m)^{1/2}$ to derive that both sides of (2.6) are equal to

$$\frac{(m-5)^2(m^3 - m^2 + 7m + 2m\rho + 1 + 2\rho)}{(m-1)^4},$$

from which (2.6) follows readily. This finishes the proof of Theorem 2.1. □

3. Proof of Theorem 1.1

Let

$$\psi(q)=\sum_{n=0}^{\infty} q^{n(n+1)/2}.$$

It follows from (1.1) and [1, (1.3.14)] that

$$\Pi_q = q^{1/4}\psi^2(q). \tag{3.1}$$

Then

$$\Pi_{q^5} = q^{5/4}\psi^2(q^5). \tag{3.2}$$

Using these equations we see that the identities (1.4)–(1.8) are respectively equivalent to

(3.3)

$$\frac{\psi^2(q^2)}{\psi^2(q^{10})}\frac{\psi^8(q^5)}{\psi^8(q^{10})}\left(16q^5-\frac{\psi^8(q^5)}{\psi^8(q^{10})}\right)=\left(5q^2-\frac{\psi^2(q^2)}{\psi^2(q^{10})}\right)\left(\frac{\psi^2(q^2)}{\psi^2(q^{10})}-q^2\right)^5,$$

(3.4)

$$\frac{\psi^2(q^{10})}{\psi^2(q^2)}\frac{\psi^8(q)}{\psi^8(q^2)}\left(16q-\frac{\psi^8(q)}{\psi^8(q^2)}\right)=\left(5q^2\frac{\psi^2(q^{10})}{\psi^2(q^2)}-1\right)^5\left(1-q^2\frac{\psi^2(q^{10})}{\psi^2(q^2)}\right),$$

$$\frac{\psi^2(q^5)}{\psi^2(q)}\left(16q\frac{\psi^8(q^2)}{\psi^8(q)}-1\right)^2=\frac{\psi^8(q^2)}{\psi^8(q)}\left(5q\frac{\psi^2(q^5)}{\psi^2(q)}-1\right)^5\left(q\frac{\psi^2(q^5)}{\psi^2(q)}-1\right), \tag{3.5}$$

$$\frac{\psi^2(q)}{\psi^2(q^5)}\left(16q^5\frac{\psi^8(q^{10})}{\psi^8(q^5)}-1\right)^2=\frac{\psi^8(q^{10})}{\psi^8(q^5)}\left(5q-\frac{\psi^2(q)}{\psi^2(q^5)}\right)\left(q-\frac{\psi^2(q)}{\psi^2(q^5)}\right)^5, \tag{3.6}$$

$$\left(q\frac{\psi^2(q)}{\psi^2(q^5)}-\frac{\psi^2(q^2)}{\psi^2(q^{10})}\right)^2=\frac{\psi^2(q^2)}{\psi^2(q^{10})}\left(q-\frac{\psi^2(q)}{\psi^2(q^5)}\right)\left(5q-\frac{\psi^2(q)}{\psi^2(q^5)}\right). \tag{3.7}$$

We temporarily assume that $0<q<1$. Let β have 5 degree over α. According to [1, Theorem 5.4.2 (i) and (iii)] we have

$$\psi(q)=\sqrt{\frac{z_1}{2}}(\alpha/q)^{1/8}, \tag{3.8}$$

$$\psi(q^2)=\frac{1}{2}\sqrt{z_1}(\alpha/q)^{1/4}, \tag{3.9}$$

$$\psi(q^5)=\sqrt{\frac{z_5}{2}}(\beta/q^5)^{1/8}, \tag{3.10}$$

$$\psi(q^{10})=\frac{1}{2}\sqrt{z_5}(\beta/q^5)^{1/4}. \tag{3.11}$$

It follows from (3.9), (3.10) and (3.11) that

$$\frac{\psi(q^2)}{\psi(q^{10})}=\sqrt{\frac{z_1}{z_5}}\left(\frac{\alpha}{\beta}\right)^{1/4}q, \tag{3.12}$$

$$\frac{\psi(q^5)}{\psi(q^{10})}=\frac{\sqrt{2}}{(\beta/q^5)^{1/8}}. \tag{3.13}$$

Multiplying both sides of (2.2) by q^{12} and then using (3.12) and (3.13) in the resulting equation we can easily obtain the identity (3.3).

It is easily deduced from (3.8) and (3.9) that

$$\frac{\psi(q)}{\psi(q^2)} = \frac{\sqrt{2}}{(\alpha/q)^{1/8}}. \tag{3.14}$$

Then (3.4) follows by substituting (3.12) and (3.14) into (2.3).

It is easily seen from (3.8) and (3.10) that

$$\frac{\psi(q^5)}{\psi(q)} = \sqrt{\frac{z_5}{z_1}\left(\frac{\beta}{\alpha}\right)^{1/8}}/q^{1/2}. \tag{3.15}$$

Then (3.5) follows easily by dividing both sides of (2.4) by q and then using (3.14) and (3.15) in the resulting identity.

Multiplying both sides of (2.5) by q and then employing (3.13) and (3.15) in the resulting equation we can attain (3.6).

The identity (3.7) follows readily by multiplying both sides of (2.6) by q^4 and then using (3.12) and (3.15) in the resulting identity.

From these we see that (3.3)–(3.7) holds for $0 < q < 1$. By analytic continuation, these identities are also true for $|q| < 1$. This completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

We assume that $0 < q < 1$ temporarily. We first prove the first equality of (1.9). Let $P(q)$ be one of the Ramanujan Eisenstein series:

$$P(q) = 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}.$$

Since

$$\frac{x}{(1-x)^2} = \sum_{n\geq 1} nx^n, \ |x| < 1,$$

we see that

$$\sum_{m\geq 1}\frac{q^m}{(1-q^m)^2} = \sum_{m,n\geq 1} nq^{mn} = \sum_{n\geq 1}\frac{nq^n}{1-q^n} = \frac{1}{24}(1-P(q)). \tag{4.1}$$

Then

$$\begin{aligned}\sum_{n\geq 1}\frac{q^{2n-1}}{(1-q^{2n-1})^2} &= \sum_{n\geq 1}\frac{q^n}{(1-q^n)^2} - \sum_{n\geq 1}\frac{q^{2n}}{(1-q^{2n})^2}\\ &= \frac{1}{24}(1-P(q)) - \frac{1}{24}(1-P(q^2))\\ &= \frac{1}{24}(P(q^2)-P(q))\end{aligned}$$

and so

$$\sum_{n\geq 1}\frac{q^{10n-5}}{(1-q^{10n-5})^2} = \frac{1}{24}(P(q^{10})-P(q^5)).$$

Combining the above two identities we get

$$\begin{aligned}&\sum_{n\geq 1}\frac{q^{2n-1}}{(1-q^{2n-1})^2} - 5\sum_{n\geq 1}\frac{q^{10n-5}}{(1-q^{10n-5})^2}\\ &= \frac{1}{24}[(P(q^2)-5P(q^{10})) - (P(q)-5P(q^5))].\end{aligned} \tag{4.2}$$

Let β have degree 5 over α. According to [1, Theorem 5.4.9] we have

$$P(q) = (1-5\alpha)z_1^2 + 12\alpha(1-\alpha)z_1\frac{dz_1}{d\alpha}, \tag{4.3}$$

$$P(q^2) = (1-2\alpha)z_1^2 + 6\alpha(1-\alpha)z_1\frac{dz_1}{d\alpha}, \tag{4.4}$$

$$P(q^5) = (1-5\beta)z_5^2 + 12\beta(1-\beta)z_5\frac{dz_5}{d\beta}, \tag{4.5}$$

$$P(q^{10}) = (1-2\beta)z_5^2 + 6\beta(1-\beta)z_5\frac{dz_5}{d\beta}. \tag{4.6}$$

In view of [2, Chapter 18, Entry 24(vi)] we conclude that

$$\beta(1-\beta)z_5\frac{dz_5}{d\beta} = \frac{m\alpha(1-\alpha)}{5}z_1\frac{dz_5}{d\alpha}. \tag{4.7}$$

Since $z_1 = mz_5$, we know that

$$\frac{dz_1}{d\alpha} = m\frac{dz_5}{d\alpha} + z_5\frac{dm}{d\alpha}.$$

Substituting the identity

$$m\frac{dz_5}{d\alpha} = \frac{dz_1}{d\alpha} - z_5\frac{dm}{d\alpha}$$

into (4.7) we get

$$\beta(1-\beta)z_5\frac{dz_5}{d\beta} = \frac{\alpha(1-\alpha)}{5}z_1\frac{dz_1}{d\alpha} - \frac{\alpha(1-\alpha)}{5}z_1z_5\frac{dm}{d\alpha}.$$

Substituting this equation into (4.5) and (4.6) gives

$$P(q^5) = (1-5\beta)z_5^2 + \frac{12}{5}\alpha(1-\alpha)z_1\frac{dz_1}{d\alpha} - \frac{12}{5}\alpha(1-\alpha)z_1z_5\frac{dm}{d\alpha}, \tag{4.8}$$

$$P(q^{10}) = (1-2\beta)z_5^2 + \frac{6}{5}\alpha(1-\alpha)z_1\frac{dz_1}{d\alpha} - \frac{6}{5}\alpha(1-\alpha)z_1z_5\frac{dm}{d\alpha}. \tag{4.9}$$

Differentiating the identity [2, Chapter 19, (14.2)] with respect to m using the method of logarithmic differentiation and then simplifying yields

$$\frac{dm}{d\alpha} = \frac{1-2\alpha}{\alpha(1-\alpha)}\frac{m(m-1)(5-m)}{25-20m-m^2}. \tag{4.10}$$

We substitute (4.3), (4.4), (4.8) and (4.9) into (4.2) and then employ (4.10) in the resulting identity to get

$$\begin{aligned}
&\sum_{n\geq 1}\frac{q^{2n-1}}{(1-q^{2n-1})^2} - 5\sum_{n\geq 1}\frac{q^{10n-5}}{(1-q^{10n-5})^2}\\
&= \frac{1}{8}\left(\alpha z_1^2 - 5\beta z_5^2 - 2\alpha(1-\alpha)z_1z_5\frac{dm}{d\alpha}\right)\\
&= \frac{z_5^2}{8}\left(\alpha m^2 - 5\beta - 2\alpha(1-\alpha)m\frac{dm}{d\alpha}\right)\\
&= \frac{z_5^2}{8}\left(\alpha m^2 - 5\beta - 2(1-2\alpha)\frac{m^2(m-1)(5-m)}{25-20m-m^2}\right)
\end{aligned} \tag{4.11}$$

.

Substituting (2.11) and (2.14) into (4.11) and then simplifying using the identity $\rho^2 = m^3 - 2m^2 + 5m$ we arrive at

$$\sum_{n\geq 1} \frac{q^{2n-1}}{(1-q^{2n-1})^2} - 5\sum_{n\geq 1} \frac{q^{10n-5}}{(1-q^{10n-5})^2} = \frac{z_5^2(m^2-5+2\rho)}{16}. \tag{4.12}$$

Squaring this identity and simplifying yields

$$\begin{aligned} &\left(\sum_{n\geq 1} \frac{q^{2n-1}}{(1-q^{2n-1})^2} - 5\sum_{n\geq 1} \frac{q^{10n-5}}{(1-q^{10n-5})^2}\right)^2 \\ &= \frac{z_5^4(m^4+4m^3-18m^2+4m^2\rho+20m+25-20\rho)}{2^8}. \end{aligned} \tag{4.13}$$

Using the identities (3.1) and (3.2) we know that

$$\begin{aligned} &\Pi_q^3\Pi_{q^5} - 2\Pi_q^2\Pi_{q^5}^2 + 5\Pi_q\Pi_{q^5}^3 \\ &= q^2\psi^6(q)\psi^2(q^5) - 2q^3\psi^4(q)\psi^4(q^5) + 5q^4\psi^2(q)\psi^6(q^5). \end{aligned} \tag{4.14}$$

Substituting the equations (3.8) and (3.10) into (4.14) yields

$$\begin{aligned} &\Pi_q^3\Pi_{q^5} - 2\Pi_q^2\Pi_{q^5}^2 + 5\Pi_q\Pi_{q^5}^3 \\ &= \frac{z_1z_5}{16}[z_1^2(\alpha^3\beta)^{1/4} - 2z_1z_5(\alpha\beta)^{1/2} + 5z_5^2(\alpha\beta^3)^{1/4}] \\ &= \frac{mz_5^4}{16}[m^2(\alpha^3\beta)^{1/4} - 2m(\alpha\beta)^{1/2} + 5(\alpha\beta^3)^{1/4}]. \end{aligned} \tag{4.15}$$

According to [2, Chapter 19, (13.10) and (13.11)] we get

$$(\alpha^3\beta)^{1/8} = \frac{\rho+3m-5}{4m},\ (\alpha\beta^3)^{1/8} = \frac{\rho+m^2-3m}{4m}.$$

We substitute these two equations and (2.9) into (4.15), note that $\rho^2 = m^3 - 2m^2 + 5m$ and then simplify to obtain

$$\begin{aligned} &\Pi_q^3\Pi_{q^5} - 2\Pi_q^2\Pi_{q^5}^2 + 5\Pi_q\Pi_{q^5}^3 \\ &= \frac{z_5^4(m^4+4m^3-18m^2+4m^2\rho+20m+25-20\rho)}{2^8}. \end{aligned} \tag{4.16}$$

Combining (4.13) and (4.16) we are led to

$$\left(\sum_{n\geq 1} \frac{q^{2n-1}}{(1-q^{2n-1})^2} - 5\sum_{n\geq 1} \frac{q^{10n-5}}{(1-q^{10n-5})^2}\right)^2 = \Pi_q^3\Pi_{q^5} - 2\Pi_q^2\Pi_{q^5}^2 + 5\Pi_q\Pi_{q^5}^3.$$

Comparing the coefficient of q in $\sum_{n\geq 1} \frac{q^{2n-1}}{(1-q^{2n-1})^2} - 5\sum_{n\geq 1} \frac{q^{10n-5}}{(1-q^{10n-5})^2}$ and that of q^2 in $\Pi_q^3\Pi_{q^5} - 2\Pi_q^2\Pi_{q^5}^2 + 5\Pi_q\Pi_{q^5}^3$ we deduce that

$$\sum_{n\geq 1} \frac{q^{2n-1}}{(1-q^{2n-1})^2} - 5\sum_{n\geq 1} \frac{q^{10n-5}}{(1-q^{10n-5})^2} = \sqrt{\Pi_q^3\Pi_{q^5} - 2\Pi_q^2\Pi_{q^5}^2 + 5\Pi_q\Pi_{q^5}^3}.$$

Dividing both sides of the above identity by $\Pi_{q^5}^2$ we see that (1.9) holds for $0 < q < 1$.

We now show the second equality. It follows from (3.1), (3.10) and (3.13) that

$$\begin{aligned}\Pi_{q^5}^2 &= q^{5/2}\psi^4(q^5) = \frac{z_5^2\beta^{1/2}}{4},\\ \frac{\Pi_{q^5}}{\Pi_{q^{10}}} &= \frac{\psi^2(q^5)}{q^{5/4}\psi^2(q^{10})} = \frac{2}{\beta^{1/4}}.\end{aligned} \tag{4.17}$$

Then

$$\frac{\Pi_{q^5}^2}{\Pi_{q^{10}}^2} + 16\frac{\Pi_{q^{10}}^2}{\Pi_{q^5}^2} = \frac{4}{\beta^{1/2}} + 4\beta^{1/2}.$$

Multiplying this identity by (4.17), using (2.3) in the resulting identity and simplifying we obtain

$$\Pi_{q^5}^2 \cdot \left(\frac{\Pi_{q^5}^2}{\Pi_{q^{10}}^2} + 16\frac{\Pi_{q^{10}}^2}{\Pi_{q^5}^2}\right) = \frac{\rho m^2 + 24m - 4\rho m - \rho}{16m} z_5^2. \tag{4.18}$$

It can be deduced from (3.1), (3.2) and (3.15) that

$$\frac{\Pi_q}{\Pi_{q^5}} - 4 - \frac{\Pi_{q^5}}{\Pi_q} = \frac{z_1}{z_5}\left(\frac{\alpha}{\beta}\right)^{1/4} - 4 - \frac{z_5}{z_1}\left(\frac{\beta}{\alpha}\right)^{1/4}.$$

Applying (2.1), (2.7) and (2.8) in this identity and simplifying yields

$$\frac{\Pi_q}{\Pi_{q^5}} - 4 - \frac{\Pi_{q^5}}{\Pi_q} = \frac{2m^3 - 16m^2 - \rho m^2 + 22m + 6\rho m - \rho}{(m-1)m(5-m)}.$$

We multiply this equation by (4.12) and simplify using the identity $\rho^2 = m^3 - 2m^2 + 5m$ to arrive at

$$\begin{aligned}&\left(\frac{\Pi_q}{\Pi_{q^5}} - 4 - \frac{\Pi_{q^5}}{\Pi_q}\right)\left(\sum_{n\geq 1}\frac{q^{2n-1}}{(1-q^{2n-1})^2} - 5\sum_{n\geq 1}\frac{q^{10n-5}}{(1-q^{10n-5})^2}\right)\\ &= \frac{\rho m^2 + 24m - 4\rho m - \rho}{16m} z_5^2.\end{aligned} \tag{4.19}$$

Combining (4.18) and (4.19) we deduce that

$$\begin{aligned}&\left(\frac{\Pi_q}{\Pi_{q^5}} - 4 - \frac{\Pi_{q^5}}{\Pi_q}\right)\left(\sum_{n\geq 1}\frac{q^{2n-1}}{(1-q^{2n-1})^2} - 5\sum_{n\geq 1}\frac{q^{10n-5}}{(1-q^{10n-5})^2}\right)\\ &= \Pi_{q^5}^2 \cdot \left(\frac{\Pi_{q^5}^2}{\Pi_{q^{10}}^2} + 16\frac{\Pi_{q^{10}}^2}{\Pi_{q^5}^2}\right).\end{aligned}$$

Dividing both sides of this identity by

$$\Pi_{q^5}^2\left(\frac{\Pi_q}{\Pi_{q^5}} - 4 - \frac{\Pi_{q^5}}{\Pi_q}\right)$$

gives that the second equality of (1.9) holds for $0 < q < 1$.

From the above we see that the two equalities of (1.9) are ture for $0 < q < 1$. By analytic continuation, these two equalities hold for $|q| < 1$. This finishes the proof of Theorem 1.2.

5. A q-identity involving Π_q and Lambert series

Gosper [5, p. 104] conjectured the following q-identity:

$$\begin{aligned}&6\left(\sum_{n\geq 1}\frac{q^n}{(1-q^n)^2}-5\sum_{n\geq 1}\frac{q^{5n}}{(1-q^{5n})^2}\right)+1\\ &=\left(\frac{\Pi_q}{\Pi_{q^5}}+2+5\frac{\Pi_{q^5}}{\Pi_q}\right)\left(\sum_{n\geq 1}\frac{q^{2n-1}}{(1-q^{2n-1})^2}-5\sum_{n\geq 1}\frac{q^{10n-5}}{(1-q^{10n-5})^2}\right).\end{aligned} \tag{5.1}$$

In this section we will use (4.12) to confirm this identity. The key to our proof of (5.1) is to handle the Lambert series

$$\sum_{n\geq 1}\frac{q^n}{(1-q^n)^2}-5\sum_{n\geq 1}\frac{q^{5n}}{(1-q^{5n})^2}.$$

Theorem 5.1. *The identity* (5.1) *is true.*

From (5.1) and the second equality of (1.9) we deduce

$$\begin{aligned}&\sum_{n\geq 1}\frac{q^n}{(1-q^n)^2}-5\sum_{n\geq 1}\frac{q^{5n}}{(1-q^{5n})^2}\\ &=\frac{\Pi_{q^5}^2\left(\frac{\Pi_q}{\Pi_{q^5}}+2+5\frac{\Pi_{q^5}}{\Pi_q}\right)\left(\frac{\Pi_{q^5}^2}{\Pi_{q^{10}}^2}+16\frac{\Pi_{q^{10}}^2}{\Pi_{q^5}^2}\right)-\left(\frac{\Pi_q}{\Pi_{q^5}}-4-\frac{\Pi_{q^5}}{\Pi_q}\right)}{6\left(\frac{\Pi_q}{\Pi_{q^5}}-4-\frac{\Pi_{q^5}}{\Pi_q}\right)}.\end{aligned}$$

This indicates that the Lambert series

$$\sum_{n\geq 1}\frac{q^n}{(1-q^n)^2}-5\sum_{n\geq 1}\frac{q^{5n}}{(1-q^{5n})^2}$$

can be represented as a rational function of Π_q, Π_{q^5} and $\Pi_{q^{10}}$.

Proof of Theorem 5.1. We first assume that $0<q<1$. Let β have degree 5 over α. It follows from (4.1) that

$$\begin{aligned}\sum_{n\geq 1}\frac{q^n}{(1-q^n)^2}-5\sum_{n\geq 1}\frac{q^{5n}}{(1-q^{5n})^2}&=\frac{1}{24}(1-P(q))-\frac{5}{24}(1-P(q^5))\\ &=\frac{1}{24}(5P(q^5)-P(q))-\frac{1}{6}.\end{aligned}$$

Using (4.3), (4.8) and (4.10) in the above identity gives

$$\begin{aligned}&\sum_{n\geq 1}\frac{q^n}{(1-q^n)^2}-5\sum_{n\geq 1}\frac{q^{5n}}{(1-q^{5n})^2}\\ &=\frac{1}{24}\left(5(1-5\beta)z_5^2-(1-5\alpha)z_1^2-12\alpha(1-\alpha)z_1z_5\frac{dm}{d\alpha}\right)-\frac{1}{6}\\ &=\frac{z_5^2}{24}\left(5(1-5\beta)-(1-5\alpha)m^2-12(1-2\alpha)\frac{m^2(m-1)(5-m)}{25-20m-m^2}\right)-\frac{1}{6}.\end{aligned}$$

Substituting (2.11) and (2.14) into this identity and simplifying using the equality $\rho^2 = m^3 - 2m^2 + 5m$ we get

$$\sum_{n\geq 1} \frac{q^n}{(1-q^n)^2} - 5\sum_{n\geq 1} \frac{q^{5n}}{(1-q^{5n})^2} = \frac{6m^3 + m^2\rho + 14m\rho - 30m + 5\rho}{96m} z_5^2 - \frac{1}{6}.$$

Then

$$6\left(\sum_{n\geq 1} \frac{q^n}{(1-q^n)^2} - 5\sum_{n\geq 1} \frac{q^{5n}}{(1-q^{5n})^2}\right) + 1 = \frac{6m^3 + m^2\rho + 14m\rho - 30m + 5\rho}{16m} z_5^2. \tag{5.2}$$

It is deduced from (3.1), (3.2) and (3.15) that

$$\begin{aligned}\frac{\Pi_q}{\Pi_{q^5}} + 2 + 5\frac{\Pi_{q^5}}{\Pi_q} &= \frac{\psi^2(q)}{q\psi^2(q^5)} + 2 + \frac{5q\psi^2(q^5)}{\psi^2(q)} \\ &= m\left(\frac{\alpha}{\beta}\right)^{1/4} + 2 + \frac{5}{m}\left(\frac{\beta}{\alpha}\right)^{1/4}.\end{aligned}$$

Then, by (2.7) and (2.8),

$$\frac{\Pi_q}{\Pi_{q^5}} + 2 + 5\frac{\Pi_{q^5}}{\Pi_q} = \frac{2m+\rho}{m-1} + 2 + \frac{5(2m-\rho)}{m(5-m)}.$$

Multiplying this equation by (4.12) and then simplifying by employing the identity $\rho^2 = m^3 - 2m^2 + 5m$ we have

$$\left(\frac{\Pi_q}{\Pi_{q^5}} + 2 + 5\frac{\Pi_{q^5}}{\Pi_q}\right)\left(\sum_{n\geq 1} \frac{q^{2n-1}}{(1-q^{2n-1})^2} - 5\sum_{n\geq 1} \frac{q^{10n-5}}{(1-q^{10n-5})^2}\right) = \frac{6m^3 + m^2\rho + 14m\rho - 30m + 5\rho}{16m} z_5^2. \tag{5.3}$$

Combining (5.2) and (5.3) produces that (5.1) holds for $0 < q < 1$. By analytic continuation, we see that (5.1) holds for $|q| < 1$. This ends the proof of Theorem 5.1. □

6. An application

The Jacobi theta functions $\theta_j(z|\tau)$ for $j = 1, 2$ are defined by [10] [12, p. 464]:

$$\theta_1(z|\tau) = -\mathrm{i}q^{\frac{1}{4}} \sum_{k=-\infty}^{\infty} (-1)^k q^{k(k+1)} e^{(2k+1)z\mathrm{i}},$$

$$\theta_2(z|\tau) = q^{\frac{1}{4}} \sum_{k=-\infty}^{\infty} q^{k(k+1)} e^{(2k+1)z\mathrm{i}},$$

where $q = \exp(\pi\mathrm{i}\tau)$ and τ is a complex number with $\operatorname{Im}\tau > 0$. The notations $\vartheta_1'(\tau) = \theta_1'(0|\tau)$ and $\vartheta_2(\tau) = \theta_2(0|\tau)$ will be used in this section. We have the following relations:

$$\theta_1\left(z + \frac{\pi}{2}|\tau\right) = \theta_2(z|\tau),\ \theta_2\left(z + \frac{\pi}{2}|\tau\right) = -\theta_1(z|\tau).$$

In [5] Gosper introduced q-analogues of $\sin z$ and $\cos z$:

$$\sin_q(\pi z) := q^{(z-1/2)^2}\prod_{n=1}^{\infty}\frac{(1-q^{2n-2z})(1-q^{2n+2z-2})}{(1-q^{2n-1})^2},$$
$$\cos_q(\pi z) := q^{z^2}\prod_{n=1}^{\infty}\frac{(1-q^{2n-2z-1})(1-q^{2n+2z-1})}{(1-q^{2n-1})^2}. \tag{6.1}$$

Gosper also gave two identities between $\sin_q$, $\cos_q$ and the functions θ_1 and θ_2, which are equivalent to the following:

$$\sin_q z = \frac{\theta_1(z|\tau')}{\vartheta_2(\tau')},\ \cos_q z = \frac{\theta_2(z|\tau')}{\vartheta_2(\tau')}, \tag{6.2}$$

where $\tau' = -\frac{1}{\tau}$. He conjectured various identities involving $\sin_q z$ and $\cos_q z$. In particular, he stated [5, pp. 99–100]

$$\begin{aligned}\sin_q 2z &= \frac{\Pi_q}{\Pi_{q^2}}\sin_{q^2} z\cos_{q^2} z\\ &= \frac{1}{2}\frac{\Pi_q}{\Pi_{q^4}}\sqrt{(\sin_{q^4} z)^2-(\sin_{q^2} z)^4},\end{aligned} \tag{6.3}$$

$$\begin{aligned}\sin_q 3z &= \frac{\Pi_q}{\Pi_{q^3}}(\cos_{q^3} z)^2\sin_{q^3} z-(\sin_{q^3} z)^3\\ &= \frac{1}{3}\frac{\Pi_q}{\Pi_{q^9}}\sin_{q^9} z-\left(1+\frac{1}{3}\frac{\Pi_q}{\Pi_{q^9}}\right)(\sin_{q^3} z)^3\end{aligned} \tag{6.4}$$

and

$$\begin{aligned}\sin_q 5z = \frac{\Pi_q}{\Pi_{q^5}}(\cos_{q^5} z)^4\sin_{q^5} z-\sqrt{\frac{\Pi_q^3}{\Pi_{q^5}^3}-2\frac{\Pi_q^2}{\Pi_{q^5}^2}+5\frac{\Pi_q}{\Pi_{q^5}}}(\cos_{q^5} z)^2(\sin_{q^5} z)^3\\ +(\sin_{q^5} z)^5.\end{aligned} \tag{6.5}$$

The first equality in (6.3) was confirmed by Mező [11] by using the method of logarithmic derivatives. The identity (6.4) and the second equality in (6.3) were proved by M. El Bachraoui [4] by employing the theory of elliptic functions. The identity (6.5) is a theta function (or q-)analogue for the well-known trigonometric identity:

$$\sin 5z = 5(\cos z)^4\sin z-10(\cos z)^2(\sin z)^3+(\sin z)^5.$$

In this section we will use Theorem 1.2 to prove (6.5).

Theorem 6.1. *The identity* (6.5) *holds for any complex number* z.

Our proof of the identity (6.5) is different from those of (6.3) and (6.4) and the proof is more complicated. The key to proving the identity (6.5) is to determine the constant

$$-\sqrt{\frac{\Pi_q^3}{\Pi_{q^5}^3}-2\frac{\Pi_q^2}{\Pi_{q^5}^2}+5\frac{\Pi_q}{\Pi_{q^5}}}$$

in front of $(\cos_{q^5} z)^2(\sin_{q^5} z)^3$. Theorem 1.2 plays an important role in determining this constant.

In order to show Theorem 6.1 we also need the following interesting result.

Theorem 6.2. [9, Theorem 2.2] *Suppose that $f_1(z),\ f_2(z), \cdots f_r(z)$ are r linearly independent nonzero entire functions of z and satisfy the functional equations:*

$$f(z) = (-1)^r f(z+\pi) = (-1)^r q^r e^{2riz} f(z+\pi\tau). \tag{6.6}$$

Let $f(z)$ be any nonzero entire function satisfying (6.6). Then $f(z)$ is a linear combination of the functions $f_1(z),\ f_2(z), \cdots f_r(z)$.

We now in the position to prove Theorem 6.1.

Proof of Theorem 6.1. It is clear that all of the five entire functions

$$\frac{\theta_1(5z|5\tau)}{\theta_1(z|\tau)},\ \theta_2^4(z|\tau),\ \theta_1^2(z|\tau)\theta_2^2(z|\tau),\ \theta_1^4(z|\tau), \theta_1(2z|\tau)$$

satisfy the functional equations:

$$f(z) = f(z+\pi) = q^4 e^{8iz} f(z+\pi\tau).$$

We now prove that the four functions

$$\theta_2^4(z|\tau),\ \theta_1^2(z|\tau)\theta_2^2(z|\tau),\ \theta_1^4(z|\tau),\ \theta_1(2z|\tau)$$

are linearly independent over $\mathbb{C}$. Assume that

$$C_1\theta_2^4(z|\tau) + C_2\theta_1^2(z|\tau)\theta_2^2(z|\tau) + C_3\theta_1^4(z|\tau) + C_4\theta_1(2z|\tau) = 0 \tag{6.7}$$

for some complex numbers $C_1,\ C_2,\ C_3,\ C_4$. Setting $z = 0$ in (6.7) gives $C_1 = 0$. Replacing z by $-z$ in (6.7) we have $C_4 = 0$. Substituting $C_1 = C_4 = 0$ into (6.7), dividing both sides of the resulting identity by $\theta_1^2(z|\tau)$ and then setting $z = 0$ we obtain $C_2 = 0$ and so $C_3 = 0$. Hence, these four functions are linearly independent over $\mathbb{C}$.

In view of Theorem 6.2 we get

$$\frac{\theta_1(5z|5\tau)}{\theta_1(z|\tau)} = D_1\theta_2^4(z|\tau) + D_2\theta_1^2(z|\tau)\theta_2^2(z|\tau) + D_3\theta_1^4(z|\tau) + D_4\theta_1(2z|\tau) \tag{6.8}$$

for some complex numbers $D_1,\ D_2,\ D_3,\ D_4$. These four constants are independent of z but depend on τ, and we sometimes denote D_i as $D_i(\tau)$ in the sequel. Putting $z = 0$ in (6.8) we are led to

$$D_1 = \frac{1}{\vartheta_2^4(\tau)} \lim_{z\to 0} \frac{\theta_1(5z|5\tau)}{\theta_1(z|\tau)} = \frac{5\vartheta_1'(5\tau)}{\vartheta_2^4(\tau)\vartheta_1'(\tau)}. \tag{6.9}$$

Replacing z by $-z$ in (6.8) gives

$$D_4 = 0. \tag{6.10}$$

We set $z = \frac{\pi}{2}$ in (6.8) to obtain

$$D_3 = \frac{\vartheta_2(5\tau)}{\vartheta_2^5(\tau)}. \tag{6.11}$$

Multiplying both sides of (6.8) by $\theta_1(z|\tau)$, replacing z by $z+\pi/2$ and substituting (6.9), (6.10) and (6.11) into the resulting identity we get

(6.12)

$$\theta_2(5z|5\tau) = \frac{5\vartheta_1'(5\tau)}{\vartheta_2^4(\tau)\vartheta_1'(\tau)}\theta_1^4(z|\tau)\theta_2(z|\tau) + D_2(\tau)\theta_1^2(z|\tau)\theta_2^3(z|\tau) + \frac{\vartheta_2(5\tau)}{\vartheta_2^5(\tau)}\theta_2^5(z|\tau).$$

It follows from (6.2) that

$$\sin_{q^5} z = \frac{\theta_1(z|\tau'/5)}{\vartheta_2(\tau'/5)},\ \cos_{q^5} z = \frac{\theta_2(z|\tau'/5)}{\vartheta_2(\tau'/5)}. \tag{6.13}$$

According to [6, Lemma 3.1] we have

$$\frac{\Pi_q}{\Pi_{q^5}} = \frac{5\vartheta_1'(\tau')\vartheta_2(\tau'/5)}{\vartheta_1'(\tau'/5)\vartheta_2(\tau')}. \tag{6.14}$$

Dividing both sides of (6.12) by $\vartheta_2(5\tau)$, replacing τ by $\tau'/5$ and then applying (6.13) and (6.14) in the resulting identity we find that

$$\cos_q 5z = \frac{\Pi_q}{\Pi_{q^5}}(\sin_{q^5} z)^4 \cos_{q^5} z + E(q)(\sin_{q^5} z)^2(\cos_{q^5} z)^3 + (\cos_{q^5} z)^5, \tag{6.15}$$

where

$$E(q) = \frac{D_2(\tau'/5)\vartheta_2^5(\tau'/5)}{\vartheta_2(\tau')}.$$

Noticing the difference between the identities (6.5) and (6.15), we only need to determine the constant $E(q)$ not D_2. We now calculate the constant $E(q)$.

Subtracting $(\cos_{q^5} z)^5$ from both sides of (6.15), dividing the resulting identity by $(\sin_{q^5} z)^2$, setting $z \to 0$ and then using L'Hôpital's rule two times we deduce that

$$E(q) = \frac{25\cos_q'' 0 - 5\cos_{q^5}'' 0}{2(\sin_{q^5}' 0)^2}. \tag{6.16}$$

From the definition of Π_q [5, p. 85] we see that

$$\sin_q' 0 = -\frac{2\ln q}{\pi}\Pi_q,$$

and so

$$\sin_{q^5}' 0 = -\frac{10\ln q}{\pi}\Pi_{q^5}. \tag{6.17}$$

Differentiating both sides of (6.1) with respect to z using the method of logarithmic differentiation and then setting $z = 0$ we have

$$\cos_q'' 0 = \frac{2\ln q}{\pi^2}\left(1 - 4\ln q \sum_{n\geq 1} \frac{q^{2n-1}}{(1-q^{2n-1})^2}\right). \tag{6.18}$$

Then

$$\cos_{q^5}'' 0 = \frac{10\ln q}{\pi^2}\left(1 - 20\ln q \sum_{n\geq 1} \frac{q^{10n-5}}{(1-q^{10n-5})^2}\right). \tag{6.19}$$

We substituting (6.17), (6.18) and (6.19) into (6.16) and then employ the first equality of (1.9) to get

$$E(q) = -\sqrt{\frac{\Pi_q^3}{\Pi_{q^5}^3} - 2\frac{\Pi_q^2}{\Pi_{q^5}^2} + 5\frac{\Pi_q}{\Pi_{q^5}}}.$$

Then (6.5) follows readily by substituting this equation into (6.15) and then replacing z by $z + \pi/2$ in the resulting identity. This concludes the proof of Theorem 6.1. □

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