Rational Distance

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Abstract: There are countable many rational distance squares, one square for each rational trigonometric Pythagorean pair $(s,c): s^2+c^2=1$ and a rational number r.

Problem: *Prove or disprove that there is an integer square ABCD and a point P in the plain of the square such that the segments AP BP, CP and DP are also integers. An equivalent problem set-up is in the set of the rational numbers.*

We place the square in the coordinate frame xOy with $OB=$ AB on the coordinate axes x, see the Picture 1 of the Figure 1. Orthogonal projections of the point P on the axes x and y are Q and R respectively. The edge of the square is an integer n and DP $=m$, CP $=k$, AP $=\mathrm{M}$, BP $=\mathrm{K}$, and BQ $=\xi$ and QP $=\eta$. We have to show that the segments $\{n;\mathrm{M},\mathrm{K},m,k\}$ may or may not be the integers/rational numbers.

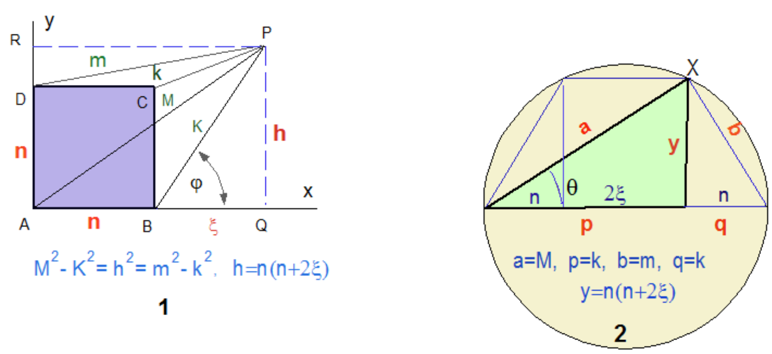

Figure 1: Rational Distance

Corollary 01. *For fixed segments n and ξ the differences $\mathrm{M}^2-\mathrm{K}^2$ and m^2-k^2 are identical and*

$$\mathrm{M}^2-\mathrm{K}^2=h^2=m^2-k^2,\quad h^2=n(n\pm 2\xi). \tag{1}$$

□ We use the cosine theorem on the triangles DCP and ABP, $\mathrm{K}\cos\varphi=\xi=k\cos\varphi'$. The sign of the segment ξ depends on the position of the P projection point Q relative to the square.Thus

$$\begin{aligned} \mathrm{M}^2 &= n^2+\mathrm{K}^2\pm 2n\xi \;\Leftrightarrow\; \mathrm{M}^2-\mathrm{K}^2 = n(n\pm 2\xi), = (\mathrm{M}-n)(\mathrm{M}+n)\\ m^2 &= n^2+k^2\pm 2n\xi \;\Leftrightarrow\; m^2-k^2 = n(n\pm 2\xi) = (m-n)(m+n)\\ \therefore\;\; \mathrm{M}^2-\mathrm{K}^2 &= m^2-k^2 = n(n\pm 2\xi). \end{aligned}$$

□

Corollary 02. *The rational distance problem is equivalent to the right rational triangle problem: Find if there is a right rational triangle $\mathcal{T} = \{\text{M}, m; d\}$ of the legs* M *and m and the hypotenuse $d = 2(n \pm \xi)$.*

□ A right triangle $\mathcal{T}$ of the hypotenuse $d = 2(n \pm \xi)$ inscribed in a circle of the radius $R = n \pm \xi$ must have the right angle vertex X on the circle Each such triangle, see the Picture 2 of the Figure 1, is specified by the position of the point X defined by the segments p and q cutup on the hypotenuse by the triangle hight $y : y^2 = pq$ from the point X. Once the segments p and q are fixed the triangle legs a and b are defined uniquely. The point X partitions the triangle $\mathcal{T}$ into right triangles $\triangle(p, a, y)$ and $\triangle(q, b, y)$, so that

$$a^2 - p^2 = y^2 = b^2 - q^2.$$

For given hypotenuse d there are infinitely many triangles $\mathcal{T}$ and their right triangle partition parts. Each partition is uniquely defined by the pair (p, q) or equivalently by the hight $y^2 = pq$ factorization.
Our particular triangle is set by the evaluation $y = h : h^2 = n(n \pm 2\xi)$ and identification

$$\triangle(a, b, d) = \triangle(M, m, K + k) \quad \therefore \quad a = \text{M},\ b = m,\ p = K, q = k,\ k + K = d$$

Consequently, the h^2 has the following representation

$$\text{M}^2 - \text{K}^2 = n(n \pm 2\xi) = m^2 - k^2,$$

and the rational square problem is equivalent to the problem of the rational triangle $\mathbf{T} = \{\text{M}, m, \text{K}+k\}$ with hypotenuse $d = \text{K} + k$ and hight $h = \sqrt{n(n \pm 2\xi)}$. □

Corollary 03. *The rational distance problem is equivalent to the rational square problem of the triangle $\boldsymbol{T}$ in the polar representation, and*

$$\begin{aligned}
&\text{M} = 2R\cos\theta, \quad m = 2R\sin\theta,\\
&\text{K} = 2R\cos^2\theta, \quad k = 2R\sin^2\theta,\\
&h = R\sin 2\theta, \quad n = 2R\sin^2\theta,\\
&\xi = \pm R(1 - 2\sin^2\theta).
\end{aligned}$$

□ We introduce the angle θ between rays a and p, see the Picture 2, and the polar relations follow from the triangle $\mathbf{T}$. The segment ξ is calculated from the $2R = (n \pm 2\xi) + n = 2(n \pm \xi)$. Hence, all the rational square segments are dependent only on the circle radius R and angle θ.

Thus, the problem is *to find if there is an integer R and an angle θ such that all rational square segments are integers/rational numbers.* □

<u>Definition</u>: *The collection of all integer triples $(\alpha, \beta; \gamma)$, $\alpha^2 + \beta^2 = \gamma^2$, are the Pythagorean numbers. The integer $|\alpha, \beta| = \gamma$ is the norm of the Pythagorean number. The collection of the rational pairs (s, c), $s^2 + c^2 = 1$ are trigonometric or the unit Pythagorean numbers.*

Corollary 04. *Pythagorean and trigonometric Pythagorean numbers are equivalent.*

□ For, the number γ of the Pythagorean triple $(\alpha, \beta; \gamma)$ is its norm so that

$$|\alpha, \beta|^2 = \alpha^2 + \beta^2 \equiv \gamma^2 \quad \Leftrightarrow \quad 1 = \frac{\alpha^2}{|\alpha, \beta|^2} + \frac{\beta^2}{|\alpha, \beta|^2} = s^2 + c^2.$$

Conclusion

There are countable many rational distance squares. For each rational trigonometric Pythagorean pair $(s,c)\colon s^2+c^2=1$ *and each rational number* R *there is one rational distance square.*

$\square$ For each Pythagorean trigonometric pair (s,c) there is an angle θ

$$s=\sin\theta,\ c=\cos\theta,\ \ \sin 2\theta=2sc,\ \cos 2\theta=1-2s^2$$

so that all segments

$$\textsc{m}=2Rc,\quad m=2Rs,\ \ \textsc{k}=2Rc^2,\quad k\ =2Rs^2,\ \ h=\frac{Rcs}{2},$$

are rational numbers whenever R is a rational number. Further $n=k=2Rs^2$ is a rational number. Since

$$2R=2(n\pm\xi)\ \ \Rightarrow\ \ \xi=\pm R(1-2s^2)$$

all of $\{n;m,k,h,\textsc{m},\textsc{k},\xi\}$ are rational numbers. Hence, the collection of the segments $\{n;m,k,h,\textsc{m},\textsc{k},\xi\}$ corresponds to each rational point $(s,c;R)$, and there are countable many rational distance squares. The integer segments are guaranteed by the choice of $R=2\textsc{n}\,|\alpha,\beta|^2$ where $\textsc{n}$ is an integer. $\square$

References

[1] W. E. Deskins, *Abstract Algebra,* The MacMilan Company, New York,

[2] George E. Andrews, *Number Theory,* Dower Publications, Inc. New York.