The prime gaps between successive primes to ensure that there is atleast one prime between their squares assuming the truth of the Riemann Hypothesis

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Abstract: Based on Dudek's proof that assumed the truth of the Riemann's hypothesis, that there exists a prime between $\{x - (4/\pi) x^{1/2} \log x\}$ and x, we determine the size of prime gaps that must exist between successive primes, so that we can be sure that there is atleast one prime number between their squares.

Results:

Let "l" and "m" represent two successive primes. Based on Dudek's proof [1] where he assumed the truth of the Riemann Hypothesis, the prime gap in this case is a= (m-l) and it must be definitely smaller than $(4/\pi) m^{1/2} \log m$, since there exists no prime between them.

$(4/\pi) m^{1/2} \log m > a$

Consider l^2 and m^2, what must their values be, so that there is atleast one prime definitely between them as suggested by Dudek[1], assuming Riemann's hypothesis.

l^2 must be equal to or smaller than $\{m^2 - (8/\pi) m \log m\}$, which is obtained by replacing "$x = m^2$" in the expression

"$x - (4/\pi) x^{1/2} \log x$"

Therefore the interval $(m^2 - (8/\pi) m \log m, m^2]$ must contain atleast one prime. Note that m^2 is composite, so the prime will be located within the interval.

Another way to write the gap between the squares is $(m^2 - l^2) = m^2 - (m-a)^2 = 2ma - a^2$

If this gap is greater than or equal to the minimum gap needed based on Dudek's results, then we can expect atleast one prime in between them.

$2ma - a^2 \geq (8/\pi) m \log m$

$2ma - (8/\pi) m \log m \geq a^2$ ……………………….. inequality (A)

$m \{2a - (8/\pi) \log m\} \geq a^2$

Since m= l+a, we can be sure that m>a. However, ensuring

$2a - (8/\pi) \log m \geq a$ will guarantee the left hand side of inequality (A) will be greater than the right side.

$2a - (8/\pi) \log m \geq a$

$a \geq (8/\pi) \log m$

So when two primes "l" and "m", (and m> l), are separated by a prime gap "a" where,

$(4/\pi)\, m^{1/2} \log m > a \geq (8/\pi) \log m$

the two primes, l and m must be successive primes and there must be atleast a single prime between their squares assuming the Riemann Hypothesis to be true.

References:

1. Dudek, Adrian W. (2014-08-21), "On the Riemann hypothesis and the difference between primes", *International Journal of Number Theory*, **11** (3): 771–778