A note on circle chains associated with the incircle of a triangle

Hiroshi Okumura
Maebashi Gunma 371-0123, Japan
e-mail: hokmr@yandex.com

Abstract. We generalize a problem in Wasan geometry involving the incircle of a triangle.

Keywords. circle chain, incircle of a triangle

Mathematics Subject Classification (2010). 01A27, 51M04

1. Introduction

We generalize the next problem [1] (see Figure 1).

Problem 1. For the incircle δ of a triangle ABC, let α be the incircle of the curvilinear triangle made by δ and CA and AB. Similarly we define the circles β and γ. If a, b, c, d are the radii of the circles α, β, γ, δ, respectively, show that the following relation holds.

$$d = \sqrt{ab} + \sqrt{bc} + \sqrt{ca}.$$

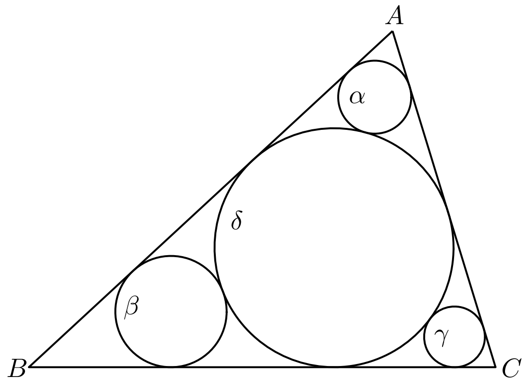

Figure 1.

2. Generalization

The fact stated in the problem is the case $n = 1$ in the next theorem (see Figure 2).

Theorem 1. *For the incircle δ of a triangle ABC, let $\alpha_0 = \delta$, and let α_n be the incircle of the curvilinear triangle made by α_{n-1} and the sides CA and AB if the circle α_{n-1} has been defined for a positive integer n. If a_n, b_n, c_n, d are the radii of α_n, β_n, γ_n, δ, respectively for a positive integer n, then we have*

$$d^{\frac{1}{n}} = (a_n b_n)^{\frac{1}{2n}} + (b_n c_n)^{\frac{1}{2n}} + (c_n a_n)^{\frac{1}{2n}}.$$

Proof. Let $a = a_1/d$, $b = b_1/d$, $c = c_1/d$. Then $a_n = da^n$, $b_n = db^n$, $c_n = dc^n$. While we have $d = \sqrt{a_1 b_1} + \sqrt{b_1 c_1} + \sqrt{c_1 a_1} = d(\sqrt{ab} + \sqrt{bc} + \sqrt{ca})$. Hence we get $\sqrt{ab} + \sqrt{bc} + \sqrt{ca} = 1$. Then

$$\begin{aligned}(a_n b_n)^{\frac{1}{2n}} + (b_n c_n)^{\frac{1}{2n}} + (c_n a_n)^{\frac{1}{2n}} &= (da^n db^n)^{\frac{1}{2n}} + (db^n dc^n)^{\frac{1}{2n}} + (dc^n da^n)^{\frac{1}{2n}} \\ &= d^{\frac{1}{n}}((ab)^{\frac{1}{2}} + (bc)^{\frac{1}{2}} + (ca)^{\frac{1}{2}}) = d^{\frac{1}{n}}.\end{aligned}$$

□

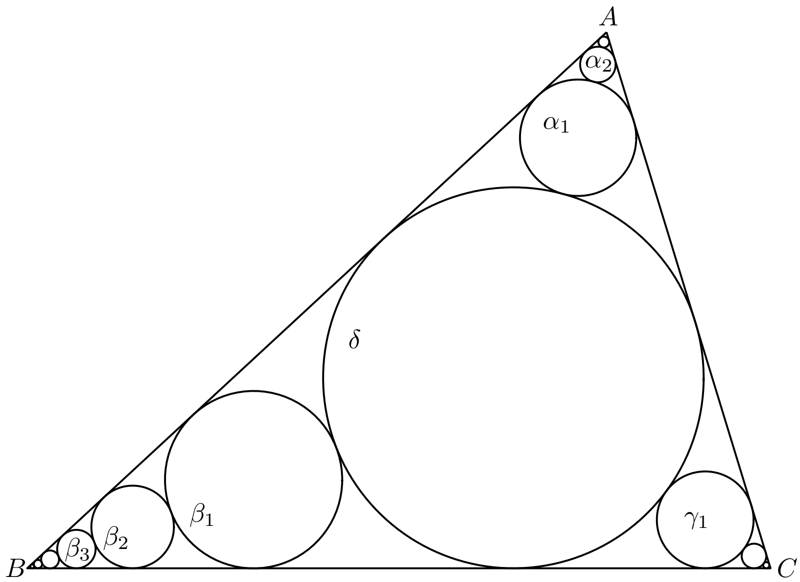

Figure 2.

References

[1] Fujita (藤田定資), Seiyō Sampō (精要算法) 1781 Tohoku University Digital Collection.