Simplified research for the constellation of all roots of Dirichlet eta function in critical strip

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Abstract

The constellation of zeros of **Dirichlet eta function** is similar to constellation of zeros of important subclass of L-functions (like Dirichlet series etc.). The hereby proposed simplified research can help in researching this important subclass of L-functions.

Keywords:

Dirichlet eta function.

Roots of L-functions in the critical strip.

1 Introduction

The following are special cases of the certain class of L-function.

The **Lerch's Transcendent** for |z|<1 and Re(*s*)>1, |z|=1 from [5]:

$$\Phi(z,s,a) \equiv \sum_{n=0}^{\infty} \frac{z^n}{(a+n)^s} = \sum_{n=1}^{\infty} \frac{z^{(n-1)}}{(a+n-1)^s} = \sum_{n=1}^{\infty} \frac{|z|^{(n-1)} \cdot e^{i(n-1)\cdot ln(|z|)}}{(a+n-1)^s} \tag{1}$$

The **Dirichlet eta function** defined as following for Re(*s*)>0, $z = e^{i\cdot 2\pi/2}$:

$$\eta(s) \equiv \Phi(-1,s,1) \equiv \sum_{n=0}^{\infty} \frac{(-1)^n}{(1+n)^s} = \sum_{n=1}^{\infty} \frac{(-1)^{(n-1)}}{(n)^s} = \sum_{n=1}^{\infty} \frac{\left|e^{i\cdot 2\pi/2}\right|^{(n-1)} \cdot e^{i\cdot(n-1)\cdot 2\pi/2\cdot ln\left(\left|e^{i\cdot 2\pi/2}\right|\right)}}{(1+n-1)^s} \tag{2}$$

2 Let's define auxiliary function

Let's define following function for $s = r + i \cdot t$:

$$H(s) = \eta(s+1/2) = \eta(r+1/2+i\cdot t) = \sum_{n=1}^{\infty} \frac{(-1)^{(n-1)} \cdot e^{-(i\cdot t)\cdot ln(n)}}{n^{(r+1/2)}} =$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{(n-1)} \cdot \left(cos(-t\cdot ln(n)) + i\cdot sin(-t\cdot ln(n))\right)}{n^{(r+1/2)}} \tag{3}$$

$$Re\{H(s)\} = \sum_{n=1}^{\infty} \frac{(-1)^{(n-1)} \cdot \left(cos(-t\cdot ln(n))\right)}{n^{(r+1/2)}} \tag{3.RE}$$

$$Im\{H(s)\} = \sum_{n=1}^{\infty} \frac{(-1)^{(n-1)} \cdot \left(sin(-t\cdot ln(n))\right)}{n^{(r+1/2)}} \tag{3.IM}$$

For complex conjugate s^* we have:

$$H(s^*) = \eta(s^*+1/2) = \eta(r+1/2-i\cdot t) = \sum_{n=1}^{\infty} \frac{(-1)^{(n-1)} \cdot e^{(i\cdot t)\cdot ln(n)}}{n^{(r+1/2)}} =$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{(n-1)} \cdot \left(cos(t\cdot ln(n)) + i\cdot sin(t\cdot ln(n))\right)}{n^{(r+1/2)}} = \sum_{n=1}^{\infty} \frac{(-1)^{(n-1)} \cdot \left(cos(-t\cdot ln(n)) - i\cdot sin(-t\cdot ln(n))\right)}{n^{(r+1/2)}} \tag{4}$$

$$Re\{H(s^*)\} = \sum_{n=1}^{\infty} \frac{(-1)^{(n-1)} \cdot \left(cos(-t\cdot ln(n))\right)}{n^{(r+1/2)}} \tag{4.RE}$$

$$Im\{H(s^*)\} = -\sum_{n=1}^{\infty} \frac{(-1)^{(n-1)} \cdot \left(sin(-t\cdot ln(n))\right)}{n^{(r+1/2)}} \tag{4.IM}$$

I.e.:

$$Re\{H(s^*)\} = Re\{H(s)\} \tag{5.RE}$$

$$Im\{H(s^*)\} = -Im\{H(s)\} \tag{5.IM}$$

We know from [6], that in the critical strip zeros of $\eta(s)$ come in pairs as s1 = 1 - s2:

$$H\left(s2 - \tfrac{1}{2}\right) = \eta(s2) = 0 = \eta(1-s2) = \eta(s1) = H\left(s1 - \tfrac{1}{2}\right) = H\left(1 - s2 - \tfrac{1}{2}\right) = H\left(\tfrac{1}{2} - s2\right) \tag{6}$$

I.e. in the critical strip zeros of $H(s)$ come in pairs as s1 = -s2:

$$H(r2 + i\cdot t2) = H(s2) = 0 = H(-s2) = H(-r2 - i\cdot t2) = H(s1) = H(r1 + i\cdot t1) \tag{7}$$

For $s = |s| \cdot e^{i\cdot\theta}$ we have:

$$H(|s2| \cdot e^{i \cdot \theta 2}) = H(s2) = 0 = H(-s2) = H(|-s2| \cdot e^{i \cdot (\theta 2 + \pi)}) = H(s1) = H(|s1| \cdot e^{i \cdot \theta 1}) \quad (8)$$

From (108) we have:

$$Re\{H(|s2| \cdot e^{-i \cdot \theta 2})\} = Re\{H(s2^*)\} = Re\{H(s2)\} = 0 = Re\{H(-s2)\} = Re\{H(|-s2| \cdot e^{i \cdot (\theta 2 + \pi)})\} = Re\{H(|s2| \cdot e^{i \cdot (\theta 2 + \pi)})\} \quad (9.RE)$$

$$Im\{H(|s2| \cdot e^{-i \cdot \theta 2})\} = Im\{H(s2^*)\} = -Im\{H(s2)\} = 0 = -Im\{H(-s2)\} = -Im\{H(|-s2| \cdot e^{i \cdot (\theta 2 + \pi)})\} = -Im\{H(|s2| \cdot e^{i \cdot (\theta 2 + \pi)})\} \quad (9.IM)$$

From (9.RE) for every zero in the critical strip of $H(s)$ we have:

$$e^{-i \cdot \theta 2} = e^{i \cdot (\theta 2 + \pi)} \quad (10)$$

Consequently for every zero in the critical strip of $H(s)$ we have:

$$1 = e^{i \cdot (2 \cdot \theta 2 + \pi)} \quad (11)$$

Consequently for every zero in the critical strip of $H(s)$ we have:

$$2 \cdot \theta 2 + \pi = 2 \cdot \pi \cdot n, n \in Z \quad (12)$$

Consequently for every zero in the critical strip of $H(s)$ we have:

$$arg(s) = \theta = \pi \cdot n + \pi/2, n \in Z \quad (13)$$

3 Conclusion

This result means, that each zero of $H(s)$ in the critical strip lie on the line Re(s) = 0, and there are no zeros of $H(s)$ in the critical strip outside this line Re(s) = 0.

And this means, that each zero of $\eta(s)$ in the critical strip lie on the line Re(s) = 1/2, and there are no zeros of $\eta(s)$ in the critical strip outside this line Re(s) = 1/2.

The constellation of zeros of **Riemann Xi function** is similar to constellation of zeros of **Dirichlet eta function.** From this result we can also conclude for **Riemann Xi function** in the critical strip, “dass alle Wurzeln reell sind.”

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