A weak extension of Complex structure on Hilbert spaces

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Abstract

The purpose of this paper is to try to replicate what happens in $\mathbb{C}$ on spaces where there are more then one of immaginary units. All these spaces, in our definition, will have the same Hilbert structure. At first we will introduce the sum and product operations on $\mathbb{C}(H) := \mathbb{R} \times H$ (where H is an Hilbert space), then we'll investigate on its algebraic properties. In our construction we lose only the associative of multiplication regardless of H, exept when $\dim H = 1$ (in this case $\mathbb{R} \times H \simeq \mathbb{C}$), and this is why we say *weak extension*. One of the most important result of this study is the Weak Integrity Theorem (th. 12) according to which in particular conditions there exist zero divisors. The next result is the Foundamental Theorem (th. 34) according to which for all $z \in \mathbb{C}(H)$ there exists $w \in \mathbb{C}(H)$ such that $z = w^2$. Afterwards we will study tranformations between these spaces which keep operation (that's why we will call them $\mathbb{C}$-morphisms). At the end we will look at the *commutative functions*, i.e. maps $\mathbb{C}(H) \to \mathbb{C}(H')$ which can be rapresented by complex transformations $\mathbb{C} \to \mathbb{C}$

1 The Pseudo-Complex Space

Definition 1. Let $(H, \langle \cdot \mid \cdot \rangle)$ be an Hilbert space on $\mathbb{R}$ and define two operation on $\mathbb{R} \times H$ as such

$$(x, f) + (y, g) := (x + y, f + g) \tag{1}$$

$$(x, f) \cdot (y, g) := (xy - \langle f \mid g \rangle, xg + yf) \tag{2}$$

for any choice of $x, y \in \mathbb{R}$ and $f, g \in H$. We call **pseudo-complex space** $\mathbb{C}(H)$ on H the triad $(\mathbb{R} \times H, +, \cdot)$. We will often use ab notation instead of $a \cdot b$ where $a, b \in \mathbb{C}(H)$ and $0 := (0, 0_H)$ and λ instead $(\lambda, 0)$ when $\lambda \in \mathbb{R}$. If $z = (x, f) \in \mathbb{C}(H)$ we denote $\Re(z) := x$ and $\Im(z) := f$ respectively the **real** and **immaginary part** of z

EXAMPLE 1. If $H = \mathbb{R}^n$ then for every $x, y \in \mathbb{R}$ and every $u, v \in H$ with $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ we have

$$(x, u) \cdot (y, v) = \left(xy - \sum_{i=1}^{n} u_i v_i, \begin{pmatrix} xv_1 + yu_1 \\ \vdots \\ xv_n + yu_n \end{pmatrix} \right)$$

EXAMPLE 2. If $H = L^2(\mathbb{R})$ then for every $x, y \in \mathbb{R}$ and every $f, g \in H$ we have

$$(x, f) \cdot (y, g) = \left(xy - \int_{\mathbb{R}} f g \, d\mu, xg + yf \right)$$

EXAMPLE 3. If $H = \mathbb{R}$ then there is a field isomorphism between $\mathbb{C}(H)$ and $\mathbb{C}$

$$(x, y) \leftrightarrow x + iy$$

EXAMPLE 4. If in H there is an orthonormal basis $\{e_j\}_{j \in I} \subset H$ then for all $f \in H$ there exist $\{a_j \in \mathbb{R}\}_{j \in I}$ such that

$$f = \sum_{j \in I} a_j e_j$$

So, for all $x \in \mathbb{R}$

$$(x, f) = \left(x, \sum_{j \in I} a_j e_j \right) = (x, 0) + \sum_{j \in I} a_j (0, e_j)$$

Calling $(0, e_j) := i_j$ for $j \in I$, we will have the algebraic notation

$$(x, f) = x + \sum_{j \in I} a_j i_j$$

By using definition, we can observe that $i_j i_k = -\delta_{jk}$ for $j, k \in I$

1.1 An Hilbert space

Definition 2. Let's define the **conjugate** of $(x, f) \in H$ the element $\overline{(x, f)} := (x, -f)$

Observation 1. For any $z_1, z_2 \in \mathbb{C}(H)$ we have

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2 \quad ; \quad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$$

Proposition 1. *The application*

$$\mathbb{C}(H) \times \mathbb{C}(H) \to \mathbb{R}$$

$$(z_1, z_2) \to \frac{1}{2}(z_1 \bar{z}_2 + \bar{z}_1 z_2) \tag{3}$$

is a dot product

Proof. Let's note that if $z_1 = (x, f)$ and $z_2 = (y, g)$ then

$$(z_1 \mid z_2) = \frac{1}{2}(z_1 \bar{z}_2 + \bar{z}_1 z_2) = xy + \langle f \mid g \rangle$$

and so the proof ◻

Observation 2. $(z_1 \mid \bar{z}_2) = (\bar{z}_1 \mid z_2)$

Corollary 2. *If* $z_1 z_2 = 0$ *then* $(z_1 \mid \bar{z}_2) = 0$

Proof.

$$2(z_1 \mid \bar{z}_2) = \overline{z_1 z_2} + z_1 z_2 = 0$$

□

Henceforth we call $|z| := \sqrt{z\bar{z}} \quad \forall z \in \mathbb{C}(H)$. So we're ready to prove this:

Theorem 3. $\mathbb{C}(H)$ *is an Hilbert space*

Proof. $|z|$ is a norm on $\mathbb{C}(H)$ induced by (3) which complete the product space $\mathbb{R} \times H$. Indeed if $\{(x_n, f_n)\}_{n\in\mathbb{N}} \subset \mathbb{C}(H)$ is a Cauchy sequence so $|(x_n, f_n) - (x_m, f_m)| < \epsilon$, that is

$$(x_n - x_m)^2 + \|f_n - f_m\|_H^2 < \epsilon^2$$

i.e. $|x_n - x_m| < \epsilon$ and $\|f_n - f_m\|_H < \epsilon$. Since $\mathbb{R}$ and H are both complete there exist $\tilde{x} \in \mathbb{R}$ and $\tilde{f} \in H$ such that $x_n \to \tilde{x}$ and $f_n \xrightarrow{H} \tilde{f}$ i.e. $(x_n, f_n) \to (\tilde{x}, \tilde{f}) \in \mathbb{C}(H)$ □

Proposition 4. *Let* $z_1, z_2 \in \mathbb{C}(H)$. *Then*

$$|z_1 z_2| \leq |z_1|\,|z_2| \quad ; \quad \left|z_1^2\right| = |z_1|^2$$

Proof. Let's call $z_1 = (x, f)$ and $z_2 = (y, g)$. Then

$$|z_1 z_2|^2 = \left|\left(xy - \langle f \mid g\rangle, xg + yf\right)\right| = x^2y^2 + \langle f \mid g\rangle^2 + x^2\|g\|_H^2 + y^2\|f\|_H^2 \leq$$

$$\leq x^2y^2 + \|f\|_H^2\|g\|_H^2 + x^2\|g\|_H^2 + y^2\|f\|_H^2 = \left(x^2 + \|f\|_H^2\right)\left(y^2 + \|g\|_H^2\right) = |z_1|^2|z_2|^2$$

where we used the Cauchy-Schwarz inequality $|\langle f \mid g\rangle| \leq \|f\|_H\|g\|_H$. If $z_1 = z_2$ we got the second identity □

Corollary 5. *Since* $|z_1 z_2| \leq |z_1|\,|z_2|$, $\mathbb{C}(H)$ *is a Banach algebra*

1.2 $\mathbb{C}(H)$ algebra

Let's check some algebraic properties of these spaces:

Proposition 6. $(\mathbb{C}(H), +)$ *is an abelian group; multiplication satisfies commutativity property and is distributive with respect to addition. Furthermore there exists an identity element for multiplication and every non-null element admits a multiplicative inverse*

Proof. We'll not prove these items. We just want to highlight the fact that

$$|z|^2 = z\bar{z} \quad \Longrightarrow \quad z^{-1} = \frac{\bar{z}}{|z|^2}$$

when $z \neq 0$ □

Proposition 7 (Weak Associativity)**.** *For any choice of* $A = (a, \alpha), B = (b, \beta), C = (c, \gamma) \in \mathbb{C}(H)$

$$A(BC) = (AB)C \iff \alpha\langle\beta \mid \gamma\rangle = \langle\alpha \mid \beta\rangle\gamma \tag{4}$$

Observation 3. From (4) we can observe that the associativity property is satisfied when $\alpha \in \operatorname{Span}\gamma$ or when $\langle\alpha \mid \beta\rangle = 0 = \langle\beta \mid \gamma\rangle$

Corollary 8. *$\mathbb{C}(H)$ is a field iff* $\dim H \in \{0,1\}$

Proof. If $\dim H \in \{0,1\}$ then $\mathbb{C}(H)$ is $\mathbb{R}$ or $\mathbb{C}$. If $\mathbb{C}(H)$ is a field then $A(BC) = (AB)C$ for any $A, B, C \in \mathbb{C}(H)$. In particular $\alpha \in \operatorname{Span}\gamma$ for any $\alpha \in H$ so $\dim H \in \{0,1\}$ □

Corollary 9. *For any $z \in \mathbb{C}(H)$*

$$z^n z^m = z^{n+m} \quad \forall n, m \in \mathbb{N}$$

Corollary 10. *For any $z, w \in \mathbb{C}(H)$*

$$z(wz^{-1}) = (zw)z^{-1}$$

Proof. It is sufficient to apply (4) □

Observation 4. If $z, w \in \mathbb{C}(H)$ with $z = (x, f)$ and $w = (y, g)$ then

$$zwz^{-1} = w \quad \Leftrightarrow \quad g = \lambda f \ \ \forall \lambda \in \mathbb{R}$$

Proposition 11. *$\mathbb{C}(H)$ is a $\mathbb{R}-$algebra*

Theorem 12 (Weak Integrity)**.** *Fixed $z_1, z_2 \in \mathbb{C}(H)$ such that $\Re(z_1) \neq 0$ and $z_1 z_2 = 0$ then $z_2 = 0$*

Proof. Let's take $z_1 = (x, f)$ and $z_2 = (y, g)$ so

$$0 = (xy - \langle f \mid g \rangle, xg + yf) \tag{5}$$

Since $x \neq 0$ we have

$$g = -\frac{y}{x} f \tag{6}$$

therefore replacing it in the real part of (5)

$$0 = xy + \frac{y}{x} \|f\|_H^2 \quad \Rightarrow \quad y|z_1|^2 = 0$$

that is $y = 0$. Replacing it in (6) we have $g = 0$ indeed $z_2 = 0$ □

Observation 5. If there exist $f, g \in H$ such that $\langle f \mid g \rangle = 0$ then

$$(0, f) \cdot (0, g) = 0$$

Observation 6. If $\Re(z) \neq 0$ there is only one inverse moltiplicative for z. Otherwise, fixed $f \in H$ for all $g \in H$ such that $\langle f \mid g \rangle = 0$ we have

$$(0, f) \cdot \left(0, g - \frac{f}{\|f\|_H^2}\right) = 1$$

Proposition 13. *If $A, B \in \mathbb{C}(H)$ with $\Re(A) \neq 0$ there is a unique $z \in \mathbb{C}(H)$ such that*

$$Az + B = 0 \tag{7}$$

Proof. The uniqueness comes from Theorem 12: if z_1, z_2 are the solutions of (7) then $A(z_1 - z_2) = 0$. Since $\Re(A) \neq 0$ follows that $z_1 - z_2 = 0$.
Calling $A = (a, \alpha)$, $B = (b, \beta)$, we define

$$x = -\frac{1}{|A|^2}(ab + \langle \alpha \mid \beta \rangle) \quad ; \quad f = -\frac{1}{a}(\beta + x\alpha)$$

so (x, f) is a solution of (7) □

Observation 7. In general $\mathbb{C}(H)$ is not algebraically closed. Indeed if $H = \mathbb{R}^2$, calling $A = (1, 0)$ and $B = (0, 1)$ the equation $(0, A) \cdot z + (0, B) = 0$ has got no solution

Let's see one last thing about the algebra on $\mathbb{C}(H)$

Proposition 14. *$\mathbb{C}(H)$ is a Lie algebra and $[z, w] := zw - wz$ is the Lie bracket*

Proof. The bilinearity and nilpotency of $[\cdot, \cdot]$ are immediately. Let's check Jacobi's identity: $\forall A, B, C \in \mathbb{C}(H)$ we should have

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$$

But

$$[A, [B, C]] = A(BC) - (AB)C = \Big(0, \big(\gamma \langle \beta \mid \alpha \rangle - \alpha \langle \beta \mid \gamma \rangle\big)\Big)$$
$$[B, [C, A]] = B(CA) - (BC)A = \Big(0, \big(\alpha \langle \beta \mid \gamma \rangle - \beta \langle \alpha \mid \gamma \rangle\big)\Big)$$
$$[C, [A, B]] = C(AB) - (CA)B = \Big(0, \big(\beta \langle \gamma \mid \alpha \rangle - \gamma \langle \beta \mid \alpha \rangle\big)\Big)$$

Summing, it follows the proof □

1.3 $\mathbb{C}-$morphisms

From these observations we can now define maps among pseudo-complex spaces:

Definition 3. Let H and H' be Hilbert spaces. A $\mathbb{C}-$**morphism** $T : \mathbb{C}(H) \to \mathbb{C}(H')$ is a continuos map such that $\forall z_1, z_2 \in \mathbb{C}(H)$

C1: $T(z_1 + z_2) = T(z_1) + T(z_2)$

C2: $T(z_1 z_2) = T(z_1)T(z_2)$

The set of $\mathbb{C}-$morphism form H to H' is named $\mathscr{C}(H, H')$. If $H = H'$ we denote $\mathscr{C}(H, H')$ simply $\mathscr{C}(H)$

EXAMPLE 5. If $H = \mathbb{R}^n$ and $H = L^2(\mathbb{R})$ let's take the operator

$$\Lambda : \mathbb{R}^n \to L^2(\mathbb{R})$$

$$(u_1, \ldots, u_n) \to \sum_{k=1}^{n} u_k \chi_{[k,k+1)}$$

Then the map $T : \mathbb{C}(\mathbb{R}^n) \to \mathbb{C}(L^2(\mathbb{R}))$ such that

$$T\big((x, u)\big) = \big(x, \Lambda(u)\big) \qquad \forall x \in \mathbb{R}\ \forall u \in \mathbb{R}^n$$

is a $\mathbb{C}-$morphism

Proposition 15. *For every Hilbert space H, the map $z \mapsto \bar{z}$ is a $\mathbb{C}-$morphism in $\mathscr{C}(H)$*

Corollary 16. *The constant map $z \mapsto 0$ is a $\mathbb{C}-$morphism in $\mathscr{C}(H, H')$ for every H, H' Hilbert spaces*

Proposition 17. *Let H_1, H_2 and H_3 be Hilbert spaces and $T_1 \in \mathscr{C}(H_1, H_2)$ and $T_2 \in \mathscr{C}(H_2, H_3)$ $\mathbb{C}-$morphisms. Then $T_2 \circ T_1 \in \mathscr{C}(H_1, H_3)$*

Proof. Since T_1 and T_2 are continuos, $T_2 \circ T_1$ is continuos. Moreover, calling $T := T_2 \circ T_1$ for all $z, w \in \mathbb{C}(H)$ we have

$$T(z+w) = T_2(T_1(z+w)) = T_2(T_1(z) + T_1(w)) = T_2(T_1(z)) + T_2(T_1(w)) = T(z) + T(w)$$

$$T(zw) = T_2(T_1(zw)) = T_2(T_1(z)T_1(w)) = T_2(T_1(z))T_2(T_1(w)) = T(z)T(w)$$

and so the proof □

Proposition 18. *Let $T \in \mathscr{C}(H, H')$ be a $\mathbb{C}-$morphism. If there exists $z_0 \in \mathbb{C}(H) \setminus \{0\}$ such that $T(z_0) = 0$ then $T \equiv 0$*

Proof. Suppose $\Re\mathrm{e}(z_0) \neq 0$. From Proposition 13, for all $z \in \mathbb{C}(H)$ there exists $w \in \mathbb{C}(H)$ such that $z_0 w = z$. That is

$$T(z) = T(z_0 w) = T(z_0)T(w) = 0$$

Without restrictions on $\Re\mathrm{e}(z_0)$, if $z_0 \neq 0$ such that $T(z_0) = 0$ then

$$T(|z_0|^2) = T(z_0)T(\bar{z}_0) = 0$$

But $|z_0|^2 \in \mathbb{R} \setminus \{0\}$ and the result is proved □

Corollary 19. *Let $T \in \mathscr{C}(H, H')$ be a non-null $\mathbb{C}-$morphism, then T is injective*

Proof. Suppose T isn't injective. Then there exist $z_1, z_2 \in \mathbb{C}(H)$ such that $z_1 \neq z_2$ and $T(z_1) = T(z_2)$. That is $T(z_1 - z_2) = 0$ i.e. $T \equiv 0$ by Prop. 18 □

Proposition 20. *Let $T \in \mathscr{C}(H, H')$ be an invertible $\mathbb{C}-$morphism. Then $T^{-1} \in \mathscr{C}(H', H)$*

Proof. if $w_1, w_2 \in \mathbb{C}(H')$, there exist $z_1, z_2 \in \mathbb{C}(H)$ such that $w_1 = T(z_1)$ and $w_2 = T(z_2)$. So

$$T^{-1}(w_1 + w_2) = T^{-1}(T(z_1) + T(z_2)) = T^{-1}(T(z_1 + z_2)) = z_1 + z_2 = T^{-1}(w_1) + T^{-1}(w_2)$$

$$T^{-1}(w_1 w_2) = T^{-1}(T(z_1)T(z_2)) = T^{-1}(T(z_1 z_2)) = z_1 z_2 = T^{-1}(w_1)T^{-1}(w_2)$$

The continuity of T^{-1} follows from the continuity and invertibility of T □

Proposition 21. *$T \in \mathscr{C}(H, H')$ is a Lie homomorphism*

Proof. For all $z_1, z_2 \in \mathbb{C}(H)$

$$[T(z_1), T(z_2)] = T(z_1)T(z_2) - T(z_2)T(z_1) = T(z_1 z_2) - T(z_2 z_1) = T(z_1 z_2 - z_2 z_1) = T([z_1, z_2])$$

□

Proposition 22. *If $T \in \mathscr{C}(H, H')$ a non-null $\mathbb{C}-$morphism. Then for all $z \in \mathbb{C}(H)$*

1. $T(0) = 0$
2. $T(1) = 1$
3. $T(-z) = -T(z)$
4. $T(z^{-1}) = T(z)^{-1}$ *when* $z \neq 0$
5. $T(\lambda z) = \lambda T(z) \quad \forall \lambda \in \mathbb{R}$
6. $T(x) = x \ \forall x \in \mathbb{R}$

Proof.
1. using C1 we have $T(0) = T(0+0) = T(0) + T(0)$
2. using C2 we have $T(1) = T(1 \cdot 1) = T(1)^2$
3. $0 = T(0) = T(z - z) = T(z) + T(-z)$
4. $1 = T(1) = T(zz^{-1}) = T(z)T(z^{-1})$
5. if $n \in \mathbb{N}$

$$T(nz) = T\left(\sum_{k=1}^{n} z\right) = \sum_{k=1}^{n} T(z) = nT(z)$$

Using 3. it follows the property for $n \in \mathbb{Z}$ and $T(n) = n$. If $a, b \in \mathbb{Z}$ with $b \neq 0$ then

$$T(ab^{-1}) = T(a)T(b)^{-1} = \frac{a}{b}$$

in according with 4. That is $T(qz) = qT(z)$ for all $q \in \mathbb{Q}$. Let $\lambda \in \mathbb{R}$ and $\{q_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Q}$ such that $q_n \to \lambda$, that is

$$T(q_n z) = q_n T(z) \to \lambda T(z)$$

Since T is continous

$$T(q_n z) \to T(\lambda z)$$

6. $T(x) = T(x \cdot 1) = xT(1) = x$

□

Corollary 23. *Let $T \in \mathscr{C}(H, H')$ be a non-null $\mathbb{C}-$morphism. Then $\Re e\, T(0, f) = 0$ $\forall f \in H$*

Proof. Let's suppose $T(0, f) = (c, g)$ where $g \in H'$. Then

$$\|f\|_H^2 = T(\|f\|_H^2, 0) = T((0, f) \cdot (0, -f)) = T(0, f) \cdot T(0, -f) =$$

$$= -T(0, f)^2 = -(c, g)^2 = -(c^2 - \|g\|_{H'}^2, 2cg)$$

that is $cg = 0$. If $c = 0$ the result is proved; otherwise if $g = 0$ then $\|f\|_H^2 = -c^2$ which is possible iff $c = \|f\|_H = 0$ and so the proof □

Corollary 24. *Let $T \in \mathscr{C}(H, H')$ be a non-null $\mathbb{C}-$morphism. Then*

$$T(\bar{z}) = \overline{T(z)} \quad \forall z \in \mathbb{C}(H)$$

Proof. By the previous Corollary, there exists $g \in H'$ such that $T(0, f) = (0, g)$. That is, for all $x \in \mathbb{R}$

$$T(x, f) = T(x, 0) + T(0, f) = (x, 0) + (0, g) = (x, g)$$
$$T(x, -f) = T(x, 0) + T(0, -f) = (x, 0) - T(0, f) = (x, 0) - (0, g) = (x, -g)$$

which proves the result □

Corollary 25. *Let $T \in \mathscr{C}(H, H')$ be a non-null $\mathbb{C}-$morphism. Then $\forall z, w \in \mathbb{C}(H)$*

$$\Big(T(z) \mid T(w)\Big)_{H'} = (z \mid w)_H$$

Proof.

$$\Big(T(z) \mid T(w)\Big)_{H'} = \frac{1}{2}\big(T(z)\overline{T(w)} + \overline{T(z)}T(w)\big) = T\left(\frac{z\bar{w} + \bar{z}w}{2}\right) = T((z \mid w)_H)$$

But $(z \mid w)_H \in \mathbb{R}$ and so the proof □

Corollary 26. *Let $T \in \mathscr{C}(H, H')$ be a non-null $\mathbb{C}-$morphism. Then T is an isometry*

Theorem 27 (Representation)**.** *Let $T \in \mathscr{C}(H, H')$ be a $\mathbb{C}-$isomorphism. Then there exist a unique unitary operator $\Lambda \in \mathscr{L}(H, H')$ such that*

$$T(x, f) = \big(x, \Lambda f\big) \quad \forall (x, f) \in \mathbb{C}(H)$$

Proof. We can write

$$T(x, f) = T(x, 0_H) + T(0, f) = (x, 0_H) + (0, k(f)) = (x, k(f))$$

where $k : H \to H'$. From hypothesis, exists $\Lambda \in \mathscr{L}(H, H')$ such that $k(f) = \Lambda f$. From hypothesis, using C2 with $(x_1, f_1), (x_2, f_2) \in \mathbb{C}(H)$, we have now

$$T\Big((x_1, f_1) \cdot (x_2, f_2)\Big) = \Big(x_1 x_2 - \langle f_1 \mid f_2\rangle_H \,, \Lambda(x_1 f_2 + x_2 f_1)\Big)$$

$$T(x_1, f_1) \cdot T(x_2, f_2) = \Big(x_1 x_2 - \langle \Lambda f_1 \mid \Lambda f_2\rangle_{H'} \,, \Lambda(x_1 f_2 + x_2 f_1)\Big)$$

which implies that

$$\Big(x_1 x_2 - \langle f_1 \mid f_2\rangle_H \,, \Lambda(x_1 f_1 + x_2 f_2)\Big) = \Big(x_1 x_2 - \langle \Lambda f_1 \mid \Lambda f_2\rangle_{H'} \,, \Lambda(x_1 f_2 + x_2 f_1)\Big)$$

that is

$$\langle f_1 \mid f_2\rangle_H = \langle \Lambda f_1 \mid \Lambda f_2\rangle_{H'}$$

for all $f_1, f_2 \in H$. So Λ must be unitary.
Suppose T is represented by two operators Λ_1 and Λ_2 such that $\forall x \in \mathbb{R}$ and $\forall f \in H$

$$T(x, f) = (x, \Lambda_1 f) \quad ; \quad T(x, f) = (x, \Lambda_2 f)$$

Using linearity of T we have

$$0 = T(x, f) - T(x, f) = (x, \Lambda_1 f) - (x, \Lambda_2 f) = \big(0, (\Lambda_1 - \Lambda_2) f\big)$$

and so the uniqueness. □

EXAMPLE 6. $\mathscr{C}(\mathbb{R}^{n+1}, \mathbb{R}^n) = \{0\}$

Proof. $T \in \mathscr{C}(\mathbb{R}^{n+1}, \mathbb{R}^n)$ is a linear map on $\mathbb{R}$ so there exists $A \in \mathcal{M}_{n+1,n}(\mathbb{R})$ such that $T(x, u) = (x, Au)$ for all $(x, u) \in \mathbb{C}(\mathbb{R}^{n+1})$. It follows that $\mathrm{rg}(A) \leq n$ but $\mathrm{rg}(A) = n + 1 - \dim\ker A$ so $\dim\ker A \geq 1$. If $u, v \in \mathbb{R}^{n+1}$ such that $u \neq v$ and $u - v \in \ker A$ then

$$T(0, u - v) = (0, A(u - v)) = 0$$

that is $T \equiv 0$ by Proposition 18 □

1.4 Subspaces

Definition 4. A subset $M \subseteq \mathbb{C}(H)$ is called **subspace of** $\mathbb{C}(H)$ if is complete and for all $z, w \in M$ and $\forall\lambda \in \mathbb{R}$ we have $z + w \in M$, $\lambda z \in M$ e $zw \in M$. A subspace M is **autonomous** if for all $z \in M$ there is at least one $w \in M$ such that $w^2 = z$

EXAMPLE 7. $\{(0, 0)\}$ is an autonomous subspace of every $\mathbb{C}(H)$; $\mathbb{R} \times \{0\}$ is a non autonomous subspace of $\mathbb{C}(\mathbb{R})$

Theorem 28. *For every pair of morphisms* $F, G : \mathbb{C}(H) \to \mathbb{C}(H')$ *the set*

$$\mathrm{EQ}_{F,G} := \{z \in \mathbb{C}(H) : F(z) = G(z)\}$$

is a subspace of $\mathbb{C}(H)$

Proof. Since F and G are linear, $\mathrm{EQ}_{F,G}$ is a vectorial space on $\mathbb{R}$ with the dot product induced by $\mathbb{C}(H)$. If $z_1, z_2 \in \mathrm{EQ}_{F,G}$ then

$$F(z_1 z_2) = F(z_1)F(z_2) = G(z_1)G(z_2) = G(z_1 z_2)$$

i.e. $z_1 z_2 \in \mathrm{EQ}_{F,G}$. Let's take a Cauchy sequence $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathrm{EQ}_{F,G}$. Since $\mathbb{C}(H)$ is an Hilbert space, there exists a $z^* \in \mathbb{C}(H)$ such that $z_n \to z^*$ for $n \to \infty$. That is

$$F(z^*) = \lim_{n\to\infty} F(z_n) = \lim_{n\to\infty} G(z_n) = G(z^*)$$

i.e. $\mathrm{EQ}_{F,G}$ is an Hilbert space □

Proposition 29. *Fixed* $f \in H$, *the set*

$$\langle f\rangle := \{(x, \lambda f) \in \mathbb{C}(H) : x, \lambda \in \mathbb{R}\}$$

called ***fundamental subspace****, is a subspace of* $\mathbb{C}(H)$

Proof. The properties of the subspaces are obvious. Let's check the completness: let $\{(x_n, \lambda_n f)\}_{n\in\mathbb{N}} \subseteq \langle f\rangle$ be a Cauchy sequence. That is for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\forall n, m \in \mathbb{N}$

$$|(x_n, \lambda_n f) - (x_m, \lambda_m f)| < \varepsilon$$

that is $(x_n - x_m) < \varepsilon$ and $(\lambda_n - \lambda_m) < \varepsilon / \|f\|_H$. Since $\mathbb{R}$ is complete, there exist x^*, $\lambda^* \in \mathbb{R}$ such that $x_n \to x^*$ and $\lambda_n \to \lambda^*$ so $(x_n, \lambda_n f) \to (x^*, \lambda^* f) \in M$ □

Theorem 30. *Fixed $f \in H \setminus \{0\}$, the map*

$$T_f : \langle f \rangle \to \mathbb{C}(\mathbb{R})$$

$$(x, \lambda f) \mapsto (x, \lambda)$$

is a $\mathbb{C}-$isomorphism iff $\|f\|_H = 1$

Proof. If $\|f\|_H = 1$ then the product and the sum are conserved: if $(x_1, \lambda_1 f), (x_2, \lambda_2 f) \in \langle f \rangle$ then

$$T_f\big((x_1, \lambda_1 f)(x_2, \lambda_2 f)\big) = T_f\big(x_1 x_2 - \lambda_1\lambda_2, (x_1\lambda_2 + x_2\lambda_1)f\big) = (x_1x_2 - \lambda_1\lambda_2, x_1\lambda_2 + x_2\lambda_1)$$

$$T_f(x_1, \lambda_1 f)T_f(x_2, \lambda_2 f) = (x_1, \lambda_1)(x_2, \lambda_2) = (x_1x_2 - \lambda_1\lambda_2, x_1\lambda_2 + x_2\lambda_1)$$

as for the sum. The invertibility is obvious.
If T_f is a $\mathbb{C}-$isomorphism then it is an isometry too, that is

$$|T_f(x, \lambda f)| = |(x, \lambda f)|$$

$$x^2 + \lambda^2 = x^2 + \lambda^2 \|f\|_H^2$$

$$\lambda(\|f\|_H - 1) = 0$$

and the result is proved □

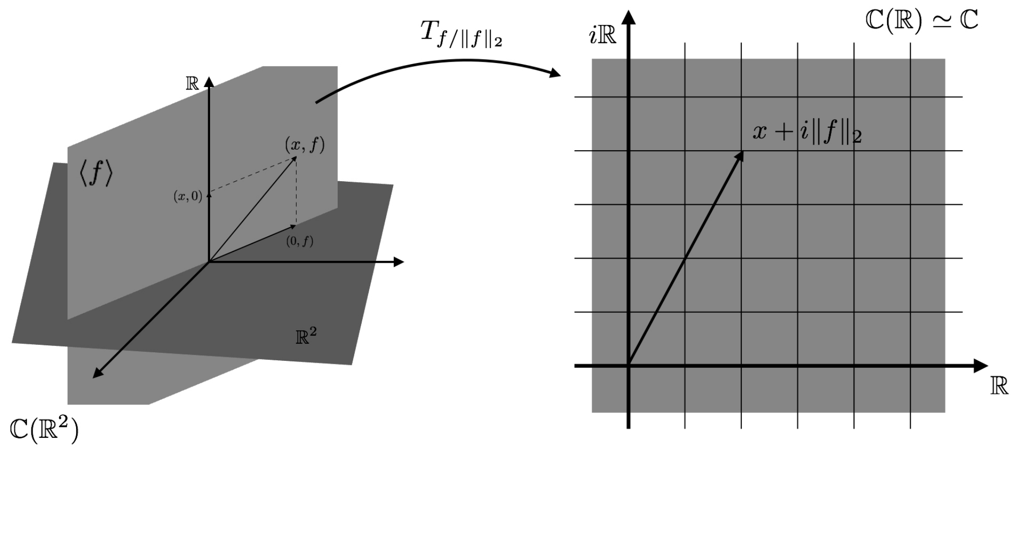

Figure 1: The action of $T_{f/\|f\|_2}$ on a foundamental subspace of $\mathbb{C}(\mathbb{R}^2)$

Proposition 31. *$\langle f \rangle \simeq \langle g \rangle$ for every $f, g \in H \setminus \{0\}$ and $\langle f \rangle = \langle g \rangle$ iff there exists $\mu \in \mathbb{R}$ such that $f = \mu g$*

Proof. The first item is obvious by the map $(x, \lambda f) \leftrightarrow (x, \lambda g)$; if $\langle f \rangle = \langle g \rangle$ than for all $\lambda \in \mathbb{R} \setminus \{0\}$ there exists $\lambda' \in \mathbb{R}$ such that

$$(x, \lambda f) = (x, \lambda' g) \quad \forall x \in \mathbb{R}$$

i.e. $\mu = \lambda'/\lambda$ □

Definition 5. Let's fix $f, g \in B_1(0) \subset \mathbb{C}(H)$, the **foundamental map** is the application

$$\Phi_{f,g} : \langle f \rangle \to \langle g \rangle$$
$$(x, \lambda f) \mapsto (x, \lambda g)$$

Observation 8. $\Phi_{f,g} = T_g^{-1} \circ T_f$

Corollary 32.

$$\mathbb{C}(H) = \bigcup_{f \in H \setminus \{0\}} \langle f \rangle$$

Proposition 33. *There exists a natural injection* $\mathbb{P}(H) \mapsto \mathbf{Gr}_2(\mathbb{C}(H))$[1]

Proof. Let's take the map

$$[f] \to \langle f \rangle \qquad \forall f \in H$$

Fixed $f, g \in H$ such that $\langle f \rangle = \langle g \rangle$, by Prop. 31 there exists $\lambda \in \mathbb{R}$ such that $f = \lambda g$, that is $[f] = [g]$ □

Theorem 34 (Fundamental of Pseudo-Complex Spaces)**.** *Let H be a non null Hilbert space. Then*

$$\forall z \in \mathbb{C}(H) \exists\, w \in \mathbb{C}(H) : z = w^2$$

Furthermore, $\langle f \rangle$ is autonomous if $f \in H$ such that $\|f\|_H = 1$

Proof. Let $z = (x, f)$ be in $\mathbb{C}(H)$. If $f = 0$ then we have two cases: if $x \geq 0$ then $w = (\sqrt{x}, 0)$; if $x < 0$ then there exists $g \in H$ such that $\|g\|_H = \sqrt{-x}$, that is $w = (0, g)$ is the square root of z. Otherwise, let's call $\tilde{f} = f / \|f\|_H$ it results that $\left\|\tilde{f}\right\|_H = 1$ and

$$z = \left(x, \|f\|_H \tilde{f}\right)$$

so we can write

$$z \in \left\langle \tilde{f} \right\rangle$$

Then

$$T_{\tilde{f}}(z) = (x, \|f\|_H)$$

Let's take $\psi : \mathbb{C}(\mathbb{R}) \to \mathbb{C}$ such that $\psi(x, y) = x + iy$. It is clear that ψ is an isomorphism and preserves products. So there exists $w \in \mathbb{C}$ such that $w^2 = x + i\|f\|_H$. Since $T_{\tilde{f}}$ is a $\mathbb{C}-$isomorphism, it implies that

$$z = T_{\tilde{f}}^{-1}(\psi^{-1}(w^2)) = T_{\tilde{f}}^{-1}(\psi^{-1}(w))^2$$

i.e. $T_{\tilde{f}}^{-1}(\psi^{-1}(w))$ is a square root of z □

Corollary 35. *Let H be a non null Hilbert space. Then*

$$\forall n \geq 1 \forall z \in \mathbb{C}(H) \exists\, w \in \mathbb{C}(H) : z = w^n$$

Proof. The proof is similar to the previous one in which we use that $T(z^n) = T(z)^n$ where T is a $\mathbb{C}-$morphism □

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[1] Grassmannian space

1.5 Commutative Lifting

Definition 6. A map $F : \mathbb{C}(H) \to \mathbb{C}(H')$ is **commutative** if there exist an application $\Lambda : H \to H'$ called **shift** and a complex map $\tilde{F} : \mathbb{C}(\mathbb{R}) \to \mathbb{C}(\mathbb{R})$ such that the next diagrams are commutative for all $f \in H$

$$\begin{array}{ccc} \langle f \rangle \subseteq \mathbb{C}(H) & \xrightarrow{F} & \langle \Lambda(f) \rangle \subseteq \mathbb{C}(H') \\ \Big\downarrow T_{f/\|f\|_H} & & \Big\downarrow T_{\Lambda(f)/\|\Lambda(f)\|_{H'}} \\ \mathbb{C}(\mathbb{R}) & \xrightarrow{\tilde{F}} & \mathbb{C}(\mathbb{R}) \end{array}$$

If $H = H'$, F is called **commutative lifting** of $\tilde{F}$ into $\mathbb{C}(H)$, indicated by

$$F := \mathrm{Lift}_{\Lambda,H}\left(\tilde{F}\right)$$

where $\Lambda : H \to H$

Theorem 36. $\mathrm{Lift}_{\Lambda,H} : \mathbb{C}(\mathbb{R})^{\mathbb{C}(\mathbb{R})} \to \mathbb{C}(H)^{\mathbb{C}(H)}$ *is a group homomorphism and preserves the product*

Proof. For all $F, G : \mathbb{C}(\mathbb{R}) \to \mathbb{C}(\mathbb{R})$ and $\forall (x, f) \in \mathbb{C}(H)$, where $\hat{f} := f/\|f\|_H$

$$\mathrm{Lift}_{\Lambda,H}(F + G)(x, f) = T^{-1}_{\hat{\Lambda f}} \circ (F+G) \circ T_{\hat{f}} = T^{-1}_{\hat{\Lambda f}}(F(T_{\hat{f}}) + G(T_{\hat{f}})) = T^{-1}_{\hat{\Lambda f}}(F(T_{\hat{f}})) + T^{-1}_{\hat{\Lambda f}}(G(T_{\hat{f}})) =$$

$$= \left(T^{-1}_{\hat{\Lambda f}} \circ F \circ T_{\hat{f}}\right) + \left(T^{-1}_{\hat{\Lambda f}} \circ G \circ T_{\hat{f}}\right) = \mathrm{Lift}_{\Lambda,H}(F) + \mathrm{Lift}_{\Lambda,H}(G)$$

$0 : \mathbb{C}(\mathbb{R}) \to \mathbb{C}(\mathbb{R})$ is the identity element, so

$$\mathrm{Lift}_{\Lambda,H}(0) = T^{-1}_{\hat{\Lambda f}} \circ 0 \circ T_{\hat{f}} = 0$$

Moreover

$$\mathrm{Lift}_{\Lambda,H}(F \cdot G)(x, f) = T^{-1}_{\hat{\Lambda f}} \circ (F \cdot G) \circ T_{\hat{f}} = T^{-1}_{\hat{\Lambda f}}(F(T_{\hat{f}}) \cdot G(T_{\hat{f}})) = T^{-1}_{\hat{\Lambda f}}(F(T_{\hat{f}})) \cdot T^{-1}_{\hat{\Lambda f}}(G(T_{\hat{f}})) =$$

$$= \left(T^{-1}_{\hat{\Lambda f}} \circ F \circ T_{\hat{f}}\right) \cdot \left(T^{-1}_{\hat{\Lambda f}} \circ G \circ T_{\hat{f}}\right) = \mathrm{Lift}_{\Lambda,H}(F) \cdot \mathrm{Lift}_{\Lambda,H}(G)$$

□

EXAMPLE 8. z^2 is a commutative lifting because if $\tilde{F}(w) = w^2$ and $\Lambda = \mathrm{id}_H$ then

$$T^{-1}_{f/\|f\|_H} \tilde{F}\left(x, \|f\|_H\right) = T^{-1}_{f/\|f\|_H}\left(x, \|f\|_H\right)^2 = T^{-1}_{f/\|f\|_H}\left(x^2 - \|f\|^2_H, 2x\|f\|_H\right) = \left(x^2 - \|f\|^2_H, 2xf\right) = (x, f)^2$$

So z^2 is the commutative lifting of the complex square power

EXAMPLE 9. The map $z \mapsto z^n$ is a commutative lifting because if $\tilde{F}(w) = w^n$ and $\Lambda = \mathrm{id}_H$ then, by strong induction

$$(x, f)^{n+1} = (x, f)(x, f)^n = T^{-1}_{f/\|f\|_H}\left(x, \|f\|_H\right) T^{-1}_{f/\|f\|_H}\left(x, \|f\|_H\right)^n =$$

$$= T^{-1}_{f/\|f\|_H}\left(\left(x, \|f\|_H\right)\left(x, \|f\|_H\right)^n\right) = T^{-1}_{f/\|f\|_H}\left(x, \|f\|_H\right)^{n+1}$$

So z^n is the commutative lifting of the complex $n-$power

EXAMPLE 10. Every constant function $F : \mathbb{C}(H) \to \mathbb{C}(H)$ is a commutative lifting: indeed if $F(x, f) = (y, g)$ we can take $\Lambda : H \to H$ such that $\Lambda(f) = g$ and $\tilde{F}(x, \|f\|_H) = (y, \|g\|_H)$

Theorem 37. *Let $F : \mathbb{C}(H) \to \mathbb{C}(H)$ be a pseudo-complex map, then F is a commutative lifting iff*

$$F\,|_{\langle g\rangle} = \Phi_{f,g} \circ F\,|_{\langle f\rangle} \qquad \forall f, g \in B_1(0)$$

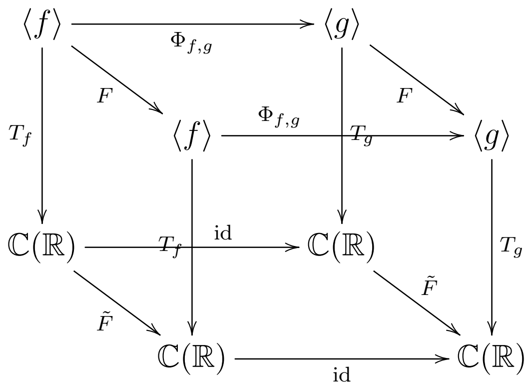
Proof. We can rewrite the last line into the next mode

$$F \circ \Phi_{f,g}\,|_{\langle f\rangle} = \Phi_{f,g} \circ F\,|_{\langle f\rangle}$$

$\Rightarrow$) By hypothesis F is a commutative lifting, i.e. there exist $\tilde{F} : \mathbb{C}(\mathbb{R}) \to \mathbb{C}(\mathbb{R})$ such that

$$F = \mathrm{Lift}_{\mathrm{id}_H, H}\left(\tilde{F}\right)$$

Looking at the next diagrams

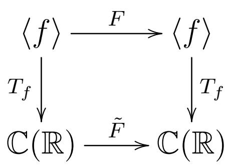


we can observe that $\Phi_{f,g} = T_g^{-1} \circ \mathrm{id} \circ T_f$, so

$$\Phi_{f,g} \circ F\,|_{\langle f\rangle} = \left(T_g^{-1} \circ \mathrm{id} \circ T_f\right) \circ \left(T_f^{-1} \circ \tilde{F} \circ T_f\right) = T_g^{-1} \circ \tilde{F} \circ T_f =$$

$$= T_g^{-1} \circ \tilde{F} \circ \mathrm{id} \circ \mathrm{id} \circ T_f = \left(T_g^{-1} \circ \tilde{F} \circ T_g\right) \circ \left(T_g^{-1} \circ \mathrm{id} \circ T_f\right) = F \circ \Phi_{f,g}\,|_{\langle f\rangle}$$

and so the first proof.
$\Leftarrow$) We can observe, by hypothesis, that $F(\langle f\rangle) \subseteq \langle f\rangle$ because Φ has got $\langle f\rangle$ as a domain. We have to find a $\tilde{F} : \mathbb{C}(\mathbb{R}) \to \mathbb{C}(\mathbb{R})$ such that the diagrams



are commutative. Let's pick $\tilde{F} := T_f \circ \Phi_{f,g}^{-1} \circ F \circ \Phi_{f,g} \circ T_f^{-1}$, so

$$\tilde{F} \circ T_f = \left(T_f \circ \Phi_{f,g}^{-1} \circ F \circ \Phi_{f,g} \circ T_f^{-1}\right) \circ T_f = T_f \circ \Phi_{f,g}^{-1} \circ \left(F \circ \Phi_{f,g}\right) =$$

$$= T_f \circ \Phi_{f,g}^{-1} \circ \left(\Phi_{f,g} \circ F\right) = T_f \circ \left(\Phi_{f,g}^{-1} \circ \Phi_{f,g}\right) \circ F = T_f \circ F$$

which proves the result $\square$

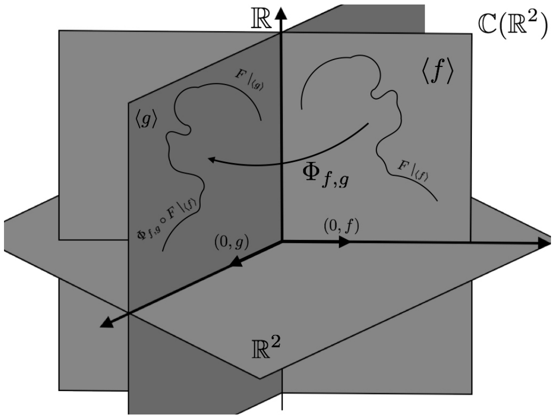


Figure 2: An example of the action of $\Phi_{f,g}$ on a commutative function F in $\mathbb{C}(\mathbb{R}^2)$

Proposition 38. *If* $\Lambda_1, \Lambda_2 : H \to H$ *then forall* $(x, f) \in \mathbb{C}(H)$

$$T_{\widehat{\Lambda_2\Lambda_1 f}} \circ \mathrm{Lift}_{\Lambda_2\Lambda_1,H} = T_{\widehat{\Lambda_1 f}} \circ \mathrm{Lift}_{\Lambda_1,H}$$

1.6 Product Spaces

Definition 7. Let $k \in \mathbb{N}$ be a integer with $k \geq 1$. We define

$$\mathbb{C}^1(H) := \mathbb{C}(H) \quad ; \quad \mathbb{C}^{k+1}(H) := \mathbb{C}\Big(\mathbb{C}^k(H)\Big)$$

Proposition 39. *The transformation* $T : \mathbb{C}(H) \to \mathbb{C}(\mathbb{C}(H))$ *such that*

$$(x, f) \mapsto (x, (0, f))$$

is in $\mathscr{C}(H, \mathbb{C}(H))$

Proof. For all $(x, f), (y, g) \in \mathbb{C}(H)$

$$T(x, f) + T(y, f) = (x, (0, f)) + (y, (0, g)) =$$

$$= (x + y, (0, f + g)) = T(x + y, f + g) = T((x, f) + (y, g))$$

Let's verify C2

$$T(x, f) \cdot T(y, g) = (x, (0, f)) \cdot (y, (0, g)) = \big(xy - \langle f \mid g\rangle, (0, xg) + (0, yf)\big) =$$

$$= \big(xy - \langle f \mid g\rangle, (0, xg + yf)\big) = T(xy - \langle f \mid g\rangle, xg + yf) = T\big((x, f) \cdot (y, g)\big)$$

□

Observation 9. If $\{(H_n, \langle\cdot \mid \cdot\rangle_{H_n})\}_{n=1,\dots,N}$ is a finite family of Hilbert spaces on $\mathbb{R}$ then $H := H_1 \times \cdots \times H_n$ is an Hilbert space with dot product $\langle\cdot \mid \cdot\rangle_H : H^2 \to \mathbb{R}$ such that for all $(f_1, \dots, f_N), (g_1, \dots, g_N) \in H$

$$\langle(f_1, \dots, f_N) \mid (g_1, \dots, g_N)\rangle_H := \sum_{k=1}^{N} \langle f_k \mid g_k\rangle_{H_k}$$

Theorem 40. *For all* $k \in \mathbb{N} \setminus \{0\}$

$$\mathbb{C}(H)^k \simeq \mathbb{C}^k(H^k)$$

Proof. Observe that $\mathbb{C}^k(H^k) \simeq \mathbb{R}^k \times H^k$ and $\mathbb{C}(H)^k \simeq (\mathbb{R} \times H)^k$ $\square$

Proposition 41. *For all integer* $k > 1$

$$\mathbb{C}^k(H) \simeq \mathbb{C}(\mathbb{R}^{k-1} \times H)$$

Definition 8. The **product space** of $\mathbb{C}(H)$ with $\mathbb{C}(H')$ is

$$\mathbb{C}(H) \times \mathbb{C}(H') := \mathbb{C}(\mathbb{C}(H \times H'))$$

with the projections $\pi_1 : \mathbb{C}(\mathbb{C}(H \times H')) \to \mathbb{C}(H)$ e $\pi_2 : \mathbb{C}(\mathbb{C}(H \times H')) \to \mathbb{C}(H')$ such that

$$\pi_1\big(x, (y, (f, g))\big) = (x, f) \quad ; \quad \pi_2\big(x, (y, (f, g))\big) = (y, g)$$

for all $(x, f) \in \mathbb{C}(H)$ and forall $(y, g) \in \mathbb{C}(H')$

Conclusions

The benefit of this approach consists on having universal properties that do not depend directly on the choice of H; the handicap is the *weak* associative property as shown in Proposition 7, which creates zero divisors (th. 12). Another interesting item is the characterization of $\mathbb{C}-$morphisms, which look like ring homomorphisms and have only one direction in the sense of injectivity. Our goals are now the generalization of the exponential function and building a pseudo-complex derivation, with which verify if it is possible to extend Cauchy-Riemann equations and so the concept of holomorphy

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