Special Relativity → Quantum Mechanics

The SRQM Interpretation of Quantum Mechanics

A Tensor Study of Physical 4-Vectors

Using Special Relativity (SR) as a starting point, then noting a few empirical 4-Vector facts, one can derive the Principles that are normally considered to be Axioms of Quantum Mechanics (QM).

Since many of the QM Axioms are rather obscure, this seems a more logical and understandable paradigm than QM as a separate theory from SR, and sheds light on the origin and meaning of the QM Principles.

The SRQM or [SR→QM] Interpretation of Quantum Mechanics
A Tensor Study of Physical 4-Vectors

or: Why General Relativity (GR) is *NOT* wrong
or: Don't bet against Einstein ;)
or: QM, the easy way...

Recommended viewing:
via a .PDF Viewer/WebBrowser
with Fit-To-Page & Page-Up/Down
ex. Firefox Web Browser

*And yes,
I did the Math...*

Special Relativity → Quantum Mechanics
The SRQM Interpretation of Quantum Mechanics
A Tensor Study of Physical 4-Vectors

4-Vectors are a fantastic language/tool for describing the physics of both relativistic and quantum phenomena. They easily show many interesting properties and relations of our Universe, and do so in a simple and concise mathematical way. Due to their tensorial nature, these SR 4-Vectors are automatically coordinate-frame invariant, and can be used to generate *ALL* of the physical SR Lorentz Scalar tensors and higher-index-count SR tensors.
Let me repeat: You can mathematically build *ALL* the Lorentz Scalars and larger SR tensors from SR 4-Vectors.

4-Vectors are likewise easily shown to be related to the standard 3-vectors that are used in Newtonian classical mechanics, Maxwellian classical electromagnetism, and standard quantum theory.

Why 4-Vectors as opposed to some of the more abstract mathematical approaches to QM?
Because the components of 4-Vectors are things that can actually be empirically measured. Experiment is the ultimate arbiter of which theories actually correspond to reality. If your quantum logics and string theories give no testable/measurable predictions, then they are basically useless for real physics.

In this treatise, I will demonstrate how 4-Vectors are used in the context of Special Relativity, and then show that their use in Relativistic Quantum Mechanics is really not fundamentally different. Quantum Principles then emerge in a natural and elegant way.

I also introduce the SRQM Diagramming Method, an instructive graphical charting-method, which visually shows how the SRQM 4-Vectors, Lorentz 4-Scalars, and 4-Tensors are all related to each other. This symbolic representation clarifies a lot of physics and is a great tool for understanding and teaching.

SRQM
Some Physics Abbreviations & Notation

GR = General Relativity
SR = Special Relativity
CM = Classical Mechanics
EM = ElectroMagnetism
QM = Quantum Mechanics
RQM = Relativistic Quantum Mechanics
NRQM = Non-Relativistic Quantum Mechanics
QFT = Quantum Field Theory
QED = Quantum ElectroDynamics
RWE = Relativistic Wave Equation
KG = Klein-Gordon (Relativistic Quantum) Eqn
PDE = Partial Differential Equation
H = The Hamiltonian = $\gamma(\mathbf{P_T}\cdot\mathbf{U})$
L = The Lagrangian = $-(\mathbf{P_T}\cdot\mathbf{U})/\gamma$
∇ = 3-gradient = $(\partial_x,\partial_y,\partial_z) = (\partial/\partial x,\partial/\partial y,\partial/\partial z)$
∂ = 4-Gradient = $\partial^\mu = (\partial_t/c,-\nabla)$; $\partial_\mu = (\partial_t/c,\nabla)$
S = The Action (4-TotalMomentum $\mathbf{P_T} = -\partial[S]$)
Φ = The Phase (4-TotalWaveVector $\mathbf{K_T} = -\partial[\Phi]$)
τ = Proper Time (Invariant Rest Time) = t_o

$\boldsymbol{\beta}$ = Relativistic Beta = $\mathbf{v}/c = \{0..1\}\hat{\mathbf{n}}$; $\mathbf{v}$ = 3-velocity = $\{0..c\}\hat{\mathbf{n}}$
γ = Relativistic Gamma = $1/\sqrt{[1-\beta^2]} = 1/\sqrt{[1-\boldsymbol{\beta}\cdot\boldsymbol{\beta}]} = \{1..\infty\}$
D = Relativistic Doppler = $1/[\gamma(1-|\boldsymbol{\beta}|\cos[\theta])]$
$\delta^{ij} = \delta^i{}_j = \delta_{ij} = I_{(3)} = \{1$ if $i=j$, else $0\}$ 3D Kronecker delta
$\delta^{\mu\nu} = \delta^\mu{}_\nu = \delta_{\mu\nu} = I_{(4)} = \{1$ if $\mu=\nu$, else $0\}$ 4D Kronecker Delta
$\eta^{\mu\nu}\rightarrow\eta_{\mu\nu}\rightarrow\mathrm{Diag}[1,-I_{(3)}]_{rect}$ = Minkowski "Flat SpaceTime" Metric
$\eta^\mu{}_\nu = \delta^\mu{}_\nu = \mathrm{Diag}[1, I_{(3)}]$ = Interesting Index Raise:Lower Matchup
$\varepsilon^{ij}{}_k$ = 3D Levi-Civita anti-symmetric permutation symbol
$\varepsilon^{\mu\nu}{}_{\rho\sigma}$ = 4D Levi-Civita Anti-symmetric Permutation Symbol
{other upper:lower index combinations possible for Levi-Civita symbol}

Tensor-Index & 4-Vector Notation:

$A^i = \mathbf{a} = (a^1,a^2,a^3)$: 3-vector [Latin index {1..3}]
$A^\mu = \mathbf{A} = (a^0,a^1,a^2,a^3)$: 4-Vector [Greek index {0..3}]
$A^\mu B_\mu = A_\nu B^\nu = \mathbf{A}\cdot\mathbf{B}$: Einstein Sum : Dot Product : Inner Product
$A^\mu B^\nu = \mathbf{A}\otimes\mathbf{B}$: Tensor Product : Outer Product
$A^\mu B^\nu - A^\nu B^\mu = \mathbf{A}\wedge\mathbf{B}$: Wedge Product : Exterior Product
$A^\mu B^\nu - A^\mu B^\nu = 0^{\mu\nu}$: (2,0)-Zero Tensor
$A^\mu B^\nu - B^\nu A^\mu = [A^\mu,B^\nu] = [\mathbf{A},\mathbf{B}]$: Commutation
$A^\mu B^\nu - B^\mu A^\nu$ = ???

SRQM = The [SR→QM] Interpretation of Quantum Mechanics, by John B. Wilson

Special Relativity → Quantum Mechanics
The SRQM Interpretation: Links

SciRealm.org
John B. Wilson
SciRealm@aol.com
http://scirealm.org/SRQM.pdf

See also:
http://scirealm.org/SRQM.html (alt discussion)
http://scirealm.org/SRQM-RoadMap.html (main SRQM website)
http://scirealm.org/4Vectors.html (4-Vector study)
http://scirealm.org/SRQM-Tensors.html (Tensor & 4-Vector Calculator)
http://scirealm.org/SciCalculator.html (Complex-capable RPN Calculator)

or Google “SRQM”

http://scirealm.org/SRQM.pdf (this document)

Special Relativity → Quantum Mechanics
SRQM Diagramming Method

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson

The SRQM Diagramming Method shows the properties and relationships of various physical objects in a graphical way. This "flowchart" method aids understanding.

Representation: 4-Scalars by ellipses, 4-Vectors by rectangles, 4-Tensors by octagons. Physical/mathematical equations and descriptions inside each shape/object. Sometimes there will be additional clarifying descriptions around a shape/object.

Relationships: Lorentz Scalar Products or tensor compositions of different 4-Vectors are on simple lines between the related 4-Vectors. Lorentz Scalar Products of a single 4-Vector, or Invariants of Tensors, are next to that object and highlighted in a different color.

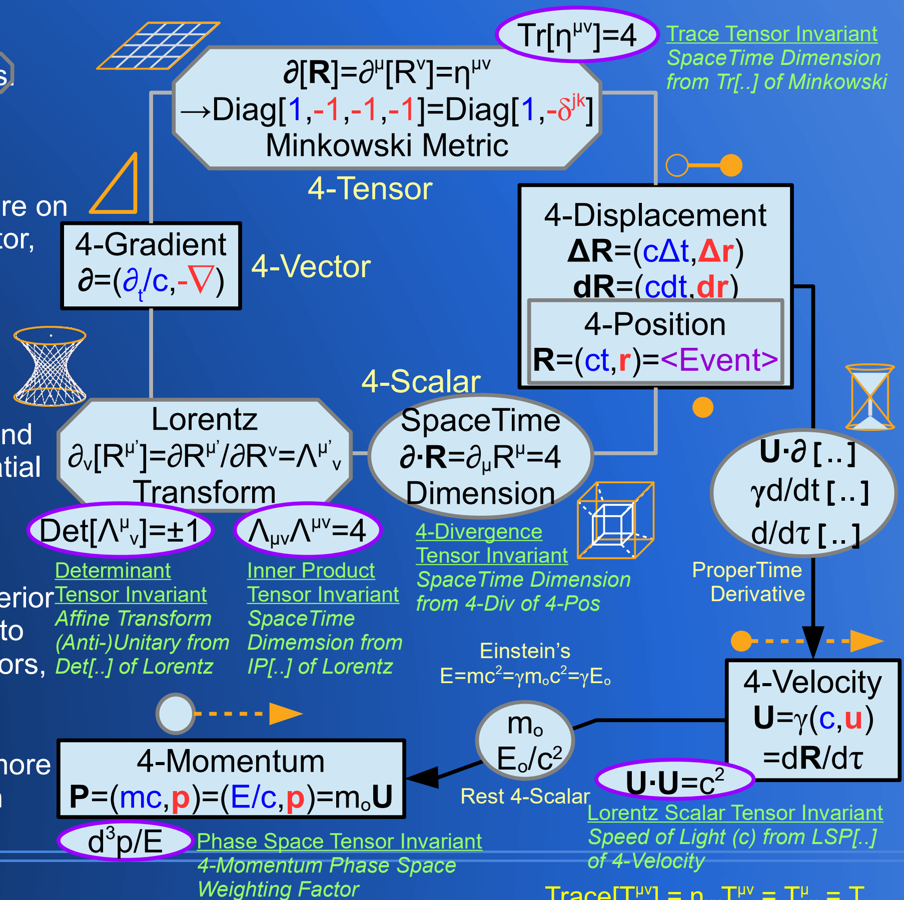
Flow: Objects that are some function of a Lorentz Scalar with another 4-Vector or 4-Tensor are on lines with arrows indicating the direction of flow. (ex. multiplication)

Properties: Some objects will also have a symbol representing its properties nearby, and sometimes there will be color highlighting within the object to emphasize temporal-spatial properties. I typically use blue=Temporal, red=Spatial, purple=mixed TimeSpace.

Alternate ways of writing 4-Vector expressions in physics:
($\mathbf{A}\cdot\mathbf{B}$) is a 4-Vector style, which uses vector-notation (ex. inner product "dot= · " or exterior product "wedge=^"), and is typically more compact, always using **bold** UPPERCASE to represent the 4-Vector, ex. ($\mathbf{A}\cdot\mathbf{B}$) = ($A^\mu \eta_{\mu\nu} B^\nu$), and **bold** lowercase to represent 3-vectors, ex. ($\mathbf{a}\cdot\mathbf{b}$) = ($a^j \delta_{jk} b^k$). Most 3-vector rules have analogues in 4-Vector mathematics.

($A^\mu \eta_{\mu\nu} B^\nu$) is a Ricci Calculus style, which uses tensor-index-notation and is useful for more complicated expressions, especially to clarify those expressions involving tensors with more than one index, such as the Faraday EM Tensor $F^{\mu\nu}$ = ($\partial^\mu A^\nu - \partial^\nu A^\mu$) = ($\partial$ ^ $\mathbf{A}$)

SRQM Diagramming Method



$Trace[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu{}_\mu = T$

$\mathbf{V}\cdot\mathbf{V} = V^\mu\eta_{\mu\nu}V^\nu = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$ = Lorentz Scalar

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^\mu{}_\nu$ or $T_\mu{}^\nu$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0,\mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0,-\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

SR → QM

4-Vector SRQM Interpretation of QM

SRQM Study: Physical/Mathematical Tensors

Tensor Types: 4-Scalar, 4-Vector, 4-Tensor

Component Types: Temporal, Spatial, Mixed

A Tensor Study of Physical 4-Vectors

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John B. Wilson

Matrix Format

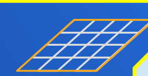
SRQM Diagram Format

Each 4D index = {0,1..3} = Tensor Rank 4

1 Temporal + 3 Spatial = 4 SpaceTime Dimensions

(m,n)-Tensor has:
(m) upper-indices
(n) lower-indices

SR:Minkowski Metric
$\partial[\mathbf{R}] = \partial^\mu R^\nu = \eta^{\mu\nu} = V^{\mu\nu} + H^{\mu\nu} \rightarrow$
$\mathrm{Diag}[1,-1,-1,-1] = \mathrm{Diag}[1,-I_{(3)}] = \mathrm{Diag}[1,-\delta^{jk}]$
{in Cartesian form} "Particle Physics" Convention
$\{\eta_{\mu\mu}\} = 1/\{\eta^{\mu\mu}\} : \eta_\mu{}^\nu = \delta_\mu{}^\nu$ $\mathrm{Tr}[\eta^{\mu\nu}]=4$



SR:Lorentz Transform
$\partial_\nu[R^{\mu'}] = \partial R^{\mu'}/\partial R^\nu = \Lambda^{\mu'}{}_\nu$
$\Lambda_\mu{}^\nu = (\Lambda^{-1})_\nu{}^\mu : \Lambda^\mu{}_\alpha \Lambda^\alpha{}_\nu = \eta^\mu{}_\nu = \delta^\mu{}_\nu$
$\eta_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta = \eta_{\alpha\beta}$
$\mathrm{Det}[\Lambda^\mu{}_\nu]=\pm 1$ $\Lambda_{\mu\nu}\Lambda^{\mu\nu}=4$

SR 4-Scalar S

S

SR 4-Scalar
(0,0)-Tensor
Lorentz Scalar S

Diagram Ellipse:
4-Scalars, 0 index
4*0 = 0 corners
4^0 = 1 component
(1)=1

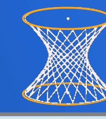
SR 4-Vector V^μ

V^0	V^1	V^2	V^3

SR 4-Vector
(1,0)-Tensor
$V^\mu = \mathbf{V} = (v^\mu) = (v^0,\mathbf{v}) = (v^0,v^i)$
$\rightarrow (v^t,v^x,v^y,v^z)$

Diagram Rectangle:
4-Vectors, 1 index
4*1 = 4 corners
4^1 = 4 components
(1+3)=4

SR 4-CoVector = "Dual" 4-Vector
(0,1)-Tensor aka. One-Form
$C_\mu = \eta_{\mu\sigma}C^\sigma = (c_\mu) = (c_0,c_i) \rightarrow (c_t,c_x,c_y,c_z)$
$= (c^0,-\mathbf{c}) = (c^0,-c^i) \rightarrow (c^t,-c^x,-c^y,-c^z)$



SpaceTime
$\partial\cdot\mathbf{R} = \partial_\mu R^\mu = 4$
Dimension

SR 4-Tensor $T^{\mu\nu}$

T^{00}	T^{01}	T^{02}	T^{03}
T^{10}	T^{11}	T^{12}	T^{13}
T^{20}	T^{21}	T^{22}	T^{23}
T^{30}	T^{31}	T^{32}	T^{33}

SR 4-Tensor
(2,0)-Tensor
$T^{\mu\nu} = \begin{bmatrix} T^{00}, & T^{0k} \\ T^{j0}, & T^{jk} \end{bmatrix}$
$\rightarrow$
$\begin{bmatrix} T^{tt}, & T^{tx}, & T^{ty}, & T^{tz} \\ T^{xt}, & T^{xx}, & T^{xy}, & T^{xz} \\ T^{yt}, & T^{yx}, & T^{yy}, & T^{yz} \\ T^{zt}, & T^{zx}, & T^{zy}, & T^{zz} \end{bmatrix}$

Diagram Octagon:
4-Tensors, 2 index
4*2 = 8 corners
4^2 = 16 components
(1+6+9)=16

6 AntiSymmetric
10 Symmetric
=============
16 General

SR
Mixed 4-Tensor
(1,1)-Tensor
$T_\mu{}^\nu = \eta_{\mu\rho}T^{\rho\nu}$
$= \begin{bmatrix} T_0{}^0, & T_0{}^k \\ T_j{}^0, & T_j{}^k \end{bmatrix}$
$= \begin{bmatrix} +T^{00}, & +T^{0k} \\ -T^{j0}, & -T^{jk} \end{bmatrix}$

SR
Mixed 4-Tensor
(1,1)-Tensor
$T^\mu{}_\nu = \eta_{\rho\nu}T^{\mu\rho}$
$= \begin{bmatrix} T^0{}_0, & T^0{}_k \\ T^j{}_0, & T^j{}_k \end{bmatrix}$
$= \begin{bmatrix} +T^{00}, & -T^{0k} \\ +T^{j0}, & -T^{jk} \end{bmatrix}$

SR
Lowered 4-Tensor
(0,2)-Tensor
$T_{\mu\nu} = \eta_{\mu\rho}\eta_{\nu\sigma}T^{\rho\sigma}$
$= \begin{bmatrix} T_{00}, & T_{0k} \\ T_{j0}, & T_{jk} \end{bmatrix}$
$= \begin{bmatrix} +T^{00}, & -T^{0k} \\ -T^{j0}, & +T^{jk} \end{bmatrix}$

Temporal region: blue
Spatial region: red
Mixed TimeSpace region: purple
The mnemonic being red and blue mixed make purple

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^\mu{}_\nu$ or $T_\mu{}^\nu$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0,\mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0,-\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Technically, all these objects are "SR 4-Tensors", but we usually reserve the name "4-Tensor" for objects with 2 or more indices, and use the (m,n)-Tensor notation to specify all the objects more precisely.

$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu{}_\mu = T$
$\mathbf{V}\cdot\mathbf{V} = V^\mu\eta_{\mu\nu}V^\nu = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$
= Lorentz Scalar

Special Relativity → Quantum Mechanics
SRQM Tensor Invariants

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson

One of the extremely important properties of Tensor Mathematics is the fact that there are numerous ways to generate Tensor Invariants. These Invariants lead to Physical Properties that are fundamental in our Universe. They are totally independent of the coordinate systems used to measure them. Thus, they represent symmetries that are inherent in the fabric of SpaceTime. See the Cayley-Hamilton Theorem.

Trace Tensor Invariant: $\mathrm{Tr}[T^{\mu\nu}] = T^{\mu}{}_{\mu}$

Inner Product Tensor Invariant: $\mathrm{IP}[T^{\mu\nu}] = T^{\mu\nu}T_{\mu\nu}$

Determinant Tensor Invariant: $\mathrm{Det}[T^{\mu\nu}] = \Pi[\text{ EigenValues }]$ for $T^{\mu}{}_{\nu}$

4-Divergence Tensor Invariant: $\text{4-Div}[T^{\mu}] = \partial_{\mu}T^{\mu} = \boldsymbol{\partial}\cdot\mathbf{T}$

Lorentz Scalar Product Tensor Invariant: $\mathrm{LSP}[T^{\mu},S^{\nu}] = T^{\mu}\eta_{\mu\nu}S^{\nu} = T_{\nu}S^{\nu} = \mathbf{T}\cdot\mathbf{S}$

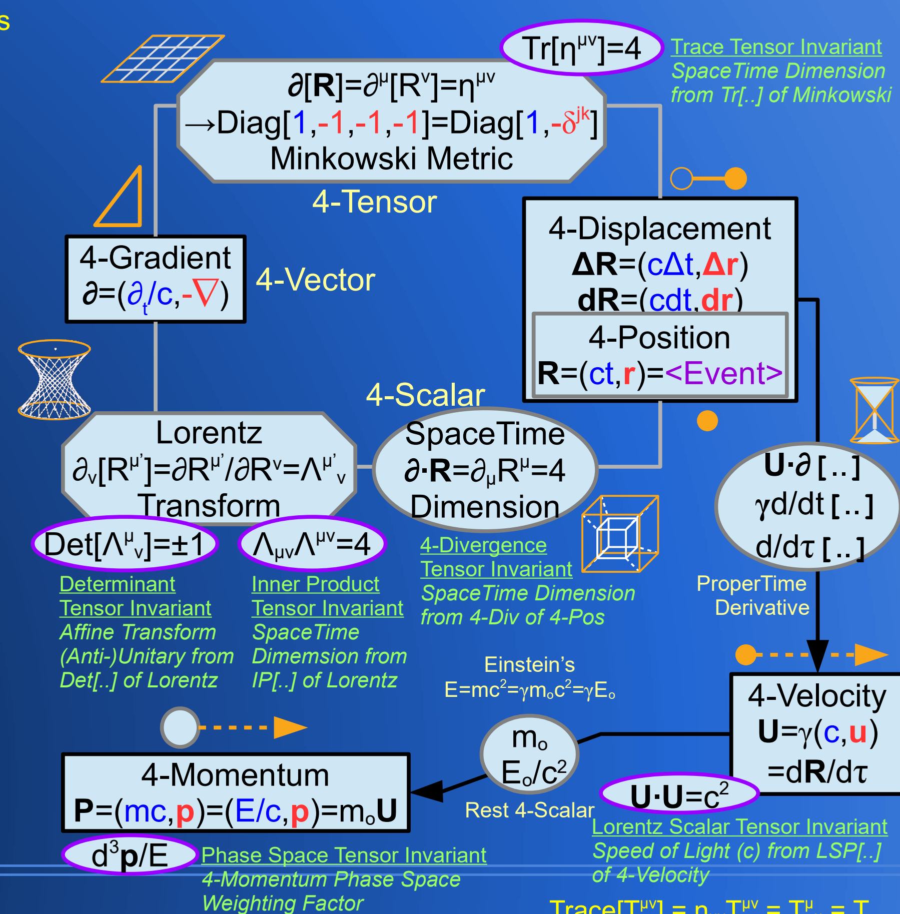
Phase Space Tensor Invariant: $\mathrm{PS}[T^{\mu}] = d^3\mathbf{t}/t^0$ if $\mathbf{T}\cdot\mathbf{T}$ = constant

The Ratio of 4-Vector Magnitudes (Ratio of Rest Value 4-Scalars): $\mathbf{T}\cdot\mathbf{T}/\mathbf{S}\cdot\mathbf{S} = (t^0_o/s^0_o)^2$

other Invariants are possible, including Tensor Eigenvalues, the various Asymmetric products, etc.

a) $T^{\alpha}{}_{\alpha}$ = Trace = Sum of EigenValues for (1,1)-Tensors
b) $T^{\alpha}{}_{[\alpha}T^{\beta}{}_{\beta]}$ = Asymm Bi-Product → Inner Product
c) $T^{\alpha}{}_{[\alpha}T^{\beta}{}_{\beta}T^{\gamma}{}_{\gamma]}$ = Asymm Tri-Product → ?Name?
d) $T^{\alpha}{}_{[\alpha}T^{\beta}{}_{\beta}T^{\gamma}{}_{\gamma}T^{\delta}{}_{\delta]}$ = Asymm Quad-Product → 4D Determinant = Product of EigenValues for (1,1)-Tensors

SRQM Diagramming Method



$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^{\mu}{}_{\mu} = T$
$\mathbf{V}\cdot\mathbf{V} = V^{\mu}\eta_{\mu\nu}V^{\nu} = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0_o)^2$
= Lorentz Scalar

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^{\mu}{}_{\nu}$ or $T_{\mu}{}^{\nu}$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^{\mu} = \mathbf{V} = (v^0,\mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_{\mu} = (v_0,-\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

SR → QM

4-Vector SRQM Interpretation of QM

SRQM Study: Physical/Mathematical Tensors
Tensor Types: 4-Scalar, 4-Vector, 4-Tensor
Examples – Venn Diagram

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson

Physical 4-Tensors: Objects which have Invariant 4D SpaceTime properties

0 index-count Tensors:

SR 4-Scalar (0,0)-Tensors Lorentz Scalar S

- EM Charge ($Q=\int \rho d^3\mathbf{x}$)
- Speed-of-Light ($c=\sqrt{[\mathbf{U}\cdot\mathbf{U}]}$)
- $d^3\mathbf{p}/E$
- ProperTime $\mathbf{U}\cdot\partial=d/d\tau=\gamma d/dt$ Derivative
- $V_o=\int\gamma d^3\mathbf{x}$
- SpaceTime $\partial\cdot\mathbf{R}=\partial_\mu R^\mu=4$ Dimension
- $\#_{dimensionless}$
- RestMass (m_o)
- Planck's Const (h)
- $\delta^4[\mathbf{X}-\mathbf{X}_o]$
- $d^4\mathbf{X}=cdt\cdot dx\cdot dy\cdot dz$
- $\Lambda_{\mu\nu}\Lambda^{\mu\nu}=4$
- $Tr[\eta^{\mu\nu}]=4$
- $Det[\Lambda^{\mu'}{}_\nu]=\pm1$

1 index-count Tensors:

SR 4-Vector (1,0)-Tensors

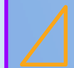
$V^\mu = \mathbf{V} = (v^\mu)$
$= (v^0,\mathbf{v}) = (v^0,v^i) \to (v^t,v^x,v^y,v^z)$

- 4-Position $\mathbf{R}=R^\mu=(ct,\mathbf{r})=$<Event> $\to(ct,x,y,z)$
- 4-Velocity $\mathbf{U}=U^\mu=(c,\mathbf{u})$ $=d\mathbf{R}/d\tau$
- 4-Momentum $\mathbf{P}=P^\mu=(mc,\mathbf{p})=m_o\mathbf{U}$ $=(E/c,\mathbf{p})=(E_o/c^2)\mathbf{U}$

SR 4-CoVector = "Dual" 4-Vector (0,1)-Tensors aka. One-Forms

$C_\mu = \eta_{\mu\sigma}C^\sigma = (c_\mu) = (c_0,c_i) \to (c_t,c_x,c_y,c_z)$
$= (c^0,-\mathbf{c}) = (c^0,-c^i) \to (c^t,-c^x,-c^y,-c^z)$

- Gradient One-Form $\partial_\mu=(\partial_t/c,\nabla)$ $\to(\partial_t/c,\partial_x,\partial_y,\partial_z)$ $=(\partial/c\partial t,\partial/\partial x,\partial/\partial y,\partial/\partial z)$

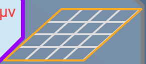


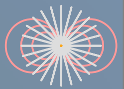
2 index-count Tensors:

SR 4-Tensor (2,0)-Tensors

$T^{\mu\nu} =$
$[\ T^{00},\ T^{0k}\ ]$
$[\ T^{j0}\ ,\ T^{jk}\ ]$

- Minkowski $\eta^{\mu\nu}=\partial^\mu[R^\nu]=\partial[\mathbf{R}]=V^{\mu\nu}+H^{\mu\nu}$ Metric
- Faraday EM 4-Tensor $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha = \partial \wedge \mathbf{A}$
- Perfect Fluid 4-Tensor $T^{\mu\nu} = (\rho_{eo})V^{\mu\nu} + (-p_o)H^{\mu\nu}$





SR Mixed 4-Tensor (1,1)-Tensors

$T^\mu{}_\nu = \eta_{\rho\nu}T^{\mu\rho}$
$=$
$[\ T^0{}_0, T^0{}_k\ ]$
$[\ T^j{}_0\ , T^j{}_k\ ]$

Projection Tensors $P^\mu{}_\nu$

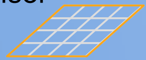
- Temporal Projection $P^\mu{}_\nu \to V^\mu{}_\nu$
- Spatial Projection $P^\mu{}_\nu \to H^\mu{}_\nu$

Lorentz $\partial_\nu[R^\mu] = \Lambda^{\mu'}{}_\nu$ Transforms

- Lorentz Boost $\Lambda^{\mu'}{}_\nu \to B^{\mu'}{}_\nu$
- Lorentz ParityInverse $\Lambda^{\mu'}{}_\nu \to P^{\mu'}{}_\nu$

SR Lowered 4-Tensor (0,2)-Tensors

$T_{\mu\nu} = \eta_{\mu\rho}\eta_{\nu\sigma}T^{\rho\sigma}$
$=$
$[\ T_{00}\ ,T_{0k}\ ]$
$[\ T_{j0}\ \ ,T_{jk}\ ]$



- Lowered Minkowski $\partial_\mu[R_\nu] = \eta_{\mu\nu} = (\ \cdot\ )$ Metric

Higher index-count Tensors:

- Riemann Curvature Tensor $R^\rho{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho{}_{\nu\sigma} - \partial_\nu\Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\sigma} \to 0^\rho{}_{\sigma\mu\nu}$ for SR "Flat" Minkowski SpaceTime
- Weyl (Conformal) Curvature Tensor $C^\rho{}_{\sigma\mu\nu}$ = Traceless part of Riemann $[R^\rho{}_{\sigma\mu\nu}]$

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^\mu{}_\nu$ or $T_\mu{}^\nu$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0,\mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0,-\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Ricci Decomposition of Riemann Tensor
$R^\rho{}_{\sigma\mu\nu} = S^\rho{}_{\sigma\mu\nu}$ (scalar part) $+ E^\rho{}_{\sigma\mu\nu}$ (semi-traceless part) $+ C^\rho{}_{\sigma\mu\nu}$ (traceless part)

$Trace[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu{}_\mu = T$
$\mathbf{V}\cdot\mathbf{V} = V^\mu\eta_{\mu\nu}V^\nu = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$
= Lorentz Scalar

SRQM 4-Vectors = (1,0)-Tensors
4-Tensors = (2+ index)-Tensors

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson

4-Vector = Type (1,0)-Tensor

4-Position $\mathbf{R} = R^\mu = (ct,\mathbf{r})$
4-Velocity $\mathbf{U} = U^\mu = \gamma(c,\mathbf{u})$
4-UnitTemporal $\mathbf{T} = T^\mu = \gamma(1,\boldsymbol{\beta})$
4-Momentum $\mathbf{P} = P^\mu = (E/c,\mathbf{p})$
4-TotalMomentum $\mathbf{P_T} = P_T^\mu = (E_T/c,\mathbf{p_T}) = (H/c,\mathbf{p_T})$
4-Force $\mathbf{F} = F^\mu = \gamma(\dot{E}/c,\mathbf{f})$
4-WaveVector $\mathbf{K} = K^\mu = (\omega/c,\mathbf{k})$
4-CurrentDensity $\mathbf{J} = J^\mu = (\rho c,\mathbf{j})$
4-VectorPotential $\mathbf{A} = A^\mu = (\varphi/c,\mathbf{a})$
4-Gradient $\partial_\mathbf{R} = \boldsymbol{\partial} = \partial^\mu = \partial/\partial R_\mu = (\partial_t/c,-\nabla)$
4-NumberFlux $\mathbf{N} = N^\mu = n(c,\mathbf{u}) = (nc,n\mathbf{u})$
4-Spin $\mathbf{S} = S^\mu = (s^0,\mathbf{s}) = (\mathbf{s}\cdot\boldsymbol{\beta},\mathbf{s})$

[Temporal : Spatial] components

[Time (t) : Space ($\mathbf{r}$)]
[Temporal "Velocity" Factor (γ) : Spatial Velocity ($\gamma\mathbf{u}$)]
[Temporal "Velocity" Factor (γ) : Spatial Velocity Factor ($\gamma\boldsymbol{\beta}$)]
[Energy (E) : 3-Momentum ($\mathbf{p}$)]
[TotalEnergy (E_T) = Hamiltonian (H) : 3-TotalMomentum ($\mathbf{p_T}$)]
[Power ($\gamma\dot{E}$) : Force ($\gamma\mathbf{f}$)]
[AngularFrequency (ω) : 3-AngularWaveNumber ($\mathbf{k}$)]
[ChargeDensity (ρ) : 3-CurrentDensity = 3-ChargeFlux ($\mathbf{j}$)]
[ScalarPotential (φ) : 3-VectorPotential ($\mathbf{a}$)], typically the EM versions (φ_{EM}) : ($\mathbf{a_{EM}}$)
[Time Differential (∂_t) : Spatial 3-Gradient(∇)]
[NumberDensity (n) : Spatial 3-NumberFlux ($\mathbf{n}=n\mathbf{u}$)]
[Temporal Spin (s^0) : Spatial 3-Spin ($\mathbf{s}$)]

4-Tensor = Type (2,0)-Tensor

[Temporal-Temporal : Temporal-Spatial : Spatial-Spatial] components

Faraday EM Tensor $F^{\mu\nu} = \begin{bmatrix} 0 & -e^j/c \\ +e^i/c & -\varepsilon^{ij}{}_k b^k \end{bmatrix}$ — [0 : 3-Electric-Field ($\mathbf{e} = e^i$) : 3-Magnetic-Field ($\mathbf{b} = b^k$)] — $F^{\mu\nu} = \boldsymbol{\partial}\wedge\mathbf{A} = \partial^\mu A^\nu - \partial^\nu A^\mu$

4-Angular Momentum Tensor $M^{\mu\nu} = \begin{bmatrix} 0 & -cn^j \\ +cn^i & -\varepsilon^{ij}{}_k l^k \end{bmatrix}$ — [0 : 3-Mass-Moment ($\mathbf{n} = n^i$) : 3-Angular-Momentum ($\mathbf{l} = l^k$)] — $M^{\mu\nu} = \mathbf{X}\wedge\mathbf{P} = X^\mu P^\nu - X^\nu P^\mu$

Minkowski Metric $\eta^{\mu\nu} = \mathrm{Diag}[1,-\mathbf{I}_{(3)}] = \mathrm{Diag}[1,-\delta^{jk}] = V^{\mu\nu}+H^{\mu\nu}$ — $[\,1 : 0 : -\mathbf{I}_{(3)}\,] = [\,1 : 0 : -\delta^{jk}\,]$ — $\eta^{\mu\nu} = \partial^\mu[R^\nu] = V^{\mu\nu} + H^{\mu\nu}$
Temporal Projection $V^{\mu\nu} = \mathrm{Diag}[1,\mathbf{0}]$ — $[\,1 : 0 : \mathbf{0}\,]$ — $V^{\mu\nu} = T^\mu T^\nu$
Spatial Projection $H^{\mu\nu} = \mathrm{Diag}[0,-\mathbf{I}_{(3)}]$ — $[\,0 : 0 : -\mathbf{I}_{(3)}\,]$ — $H^{\mu\nu} = \eta^{\mu\nu}-T^\mu T^\nu$

Perfect-Fluid Stress-Energy Tensor $T^{\mu\nu} \rightarrow \mathrm{Diag}[\rho_e,p,p,p]$ — $[\,\rho_e : 0 : p\mathbf{I}_{(3)}\,] = [\,\rho_e : 0 : p\delta^{jk}\,]$ — $T^{\mu\nu} = (\rho_{eo}+p_o)T^\mu T^\nu - (p_o)\partial^\mu[R^\nu]$
$T^{\mu\nu} = (\rho_{eo})V^{\mu\nu} + (-p_o)H^{\mu\nu}$

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^\mu{}_\nu$ or $T_\mu{}^\nu$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0,\mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0,-\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

4-Tensors can be constructed from the Tensor Products of 4-Vectors. Technically, 4-Tensors refer to all SR objects (4-Scalars, 4-Vectors, etc), but we typically reserve the name 4-Tensor for SR Tensors of 2 or more indices

SRQM 4-Scalars = (0,0)-Tensors = Lorentz Scalars → Physical Constants

4-Scalar = Type (0,0)-Tensor

RestTime:ProperTime (t_o) = (τ)	(τ) = [**R**·**U**]/[**U**·**U**] = [**R**·**R**]/[**R**·**U**] **Time as measured in the at-rest frame**
Speed of Light (c)	(c) = Sqrt[**U**·**U**] = [**T**·**U**] with 4-UnitTemporal **T** = $\gamma(1,\boldsymbol{\beta})$
RestMass (m_o)	(m_o) = [**P**·**U**]/[**U**·**U**] = [**P**·**R**]/[**U**·**R**] ($m_o \to m_e$) as Electron RestMass
RestEnergy (E_o)	(E_o) = [**P**·**U**]
RestAngFrequency (ω_o)	(ω_o) = [**K**·**U**]
RestChargeDensity (ρ_o)	(ρ_o) = [**J**·**U**]/[**U**·**U**] = (q)[**N**·**U**]/[**U**·**U**] = (q)(n_o)
RestScalarPotential (φ_o)	(φ_o) = [**A**·**U**], ($\varphi_o \to \varphi_{EM^o}$) as the EM version RestScalarPotential
ProperTimeDerivative (d/dτ)	(d/dτ) = [**U**·**∂**] = γ(d/dt) **Note that the 4-Gradient is to right of 4-Velocity**
RestNumberDensity (n_o)	(n_o) = [**N**·**U**]/[**U**·**U**]
SR Phase (Φ_{phase})	($\Phi_{phase,free}$) = -[**K**·**R**] = (**k**·**r** - ωt) : (Φ_{phase}) = -[$\mathbf{K_T}$·**R**] = ($\mathbf{k_T}$·**r** - ω_Tt)
SR Action (S_{action})	($S_{action,free}$) = -[**P**·**R**] = (**p**·**r** - Et) : (S_{action}) = -[$\mathbf{P_T}$·**R**] = ($\mathbf{p_T}$·**r** - E_Tt)
Planck Reduced Constant (ħ)	(ħ) = [**P**·**U**]/[**K**·**U**] = [**P**·**R**]/[**K**·**R**]
SpaceTime Dimension (4)	(4) = [**∂**·**R**] **4-Divergence of the 4-Position gives SR Dimension**
Electric : Magnetic Constants (ε_o : μ_o)	**∂**·$F^{\alpha\beta}$ = (μ_o)**J** = (1/$\varepsilon_o c^2$)**J** Maxwell EM Eqn. $\mu_o\varepsilon_o = 1/c^2$
EM Charge (q)	**U**·$F^{\alpha\beta}$ = (1/q)**F** Lorentz Force Eqn. (q→ -e) as Electron Charge
EM Charge (Q) *alt method*	(Q) = ∫ρd^3**x** = ∫$\rho_o\gamma d^3$**x** Integration over volume charge density
Particle # (N)	(N) = ∫$n d^3$**x** = ∫$n_o\gamma d^3$**x** Integration over volume number density
Rest Volume (V_o)	(V_o) = ∫γd^3**x** Integration over relativistic volume elements (Riemannian Volume Form)
RestEnergyDensity (ρ_{eo})	(ρ_{eo}) = $V_{\alpha\beta}T^{\alpha\beta}$ = Temporal "Vertical" Projection of Perfect Fluid Stress-Energy Tensor
RestPressure (p_o)	(p_o) = (-1/3)$H_{\alpha\beta}T^{\alpha\beta}$ = Spatial "Horizontal" Projection of Perfect Fluid Stress-Energy Tensor

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^{\mu}{}_{\nu}$ or $T_{\mu}{}^{\nu}$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor V^μ = **V** = (v^0,**v**)
SR 4-CoVector
(0,1)-Tensor V_μ = (v_0,-**v**)

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Lorentz Scalars can be constructed from the Lorentz Scalar Product of 4-Vectors

SR → QM

4-Vector SRQM Interpretation of QM

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4-Gradient
$\partial=\partial^\mu=(\partial_t/c,-\nabla)$
$\rightarrow(\partial_t/c,-\partial_x,-\partial_y,-\partial_z)$
$=(\partial/c\partial t,-\partial/\partial x,-\partial/\partial y,-\partial/\partial z)$

Gradient 4-Vector
$\partial^\mu=(\partial_t/c,-\nabla)$

$\partial_\mu=(\partial_t/c,\nabla)$
Gradient One-Form

4-Displacement
$\Delta\mathbf{R}=\Delta R^\mu=(c\Delta t,\Delta\mathbf{r})=\mathbf{R_2}-\mathbf{R_1}$ {finite}
$\mathbf{dR}=dR^\mu=(cdt,\mathbf{dr})$ {infintesimal}

4-Position
$\mathbf{R}=R^\mu=(ct,\mathbf{r})=$<Event>
$\rightarrow(ct,x,y,z)$
alt. notation $\mathbf{X}=X^\mu$

Lorentz Invariant, but not Poincaré Invariant

4-Velocity
$\mathbf{U}=U^\mu=\gamma(c,\mathbf{u})$
$=\mathbf{dR}/d\tau=c\mathbf{T}$

4-UnitTemporal
$\mathbf{T}=T^\mu=\gamma(1,\boldsymbol{\beta})$
$=\gamma(1,\mathbf{u}/c)=\mathbf{U}/c$

4-Acceleration
$\mathbf{A}=A^\mu=\gamma(c\gamma',\gamma'\mathbf{u}+\gamma\mathbf{a})$
$=\mathbf{dU}/d\tau=d^2\mathbf{R}/d\tau^2$

Minkowski
$\partial[\mathbf{R}]=\partial^\mu[R^\nu]=\eta^{\mu\nu}$
Metric

Lorentz
$\partial_\nu[R^{\mu'}]=\Lambda^{\mu'}{}_\nu$
Transform

SpaceTime
$\partial\cdot\mathbf{R}=\partial_\mu R^\mu=4$
Dimension

4-Momentum
$\mathbf{P}=P^\mu=(mc,\mathbf{p})=(mc,m\mathbf{u})=m_o\mathbf{U}$
$=(E/c,\mathbf{p})=(E_o/c^2)\mathbf{U}$

4-WaveVector
$\mathbf{K}=K^\mu=(\omega/c,\mathbf{k})=(\omega_o/c^2)\mathbf{U}$
$=(\omega/c,\omega\hat{\mathbf{n}}/v_{phase})=(1/c\text{Ŧ},\hat{\mathbf{n}}/\lambda)$

4-(EM)VectorPotential
$\mathbf{A}=A^\mu=(\varphi/c,\mathbf{a})=(\varphi_o/c^2)\mathbf{U}$
$\mathbf{A_{EM}}=A_{EM}{}^\mu=(\varphi_{EM}/c,\mathbf{a_{EM}})$

4-(Vector)PotentialMomentum
$\mathbf{Q}=Q^\mu=(q\varphi/c,q\mathbf{a})=(V/c,\mathbf{q})$
$=q\mathbf{A}=(q\varphi_o/c^2)\mathbf{U}=(V_o/c^2)\mathbf{U}$

4-ChargeFlux : 4-CurrentDensity
$\mathbf{J}=J^\mu=(\rho c,\mathbf{j})=\rho(c,\mathbf{u})=\rho_o\mathbf{U}$
$=qn_o\mathbf{U}=q\mathbf{N}$

4-(Dust)NumberFlux
$\mathbf{N}=N^\mu=(nc,\mathbf{n})=n(c,\mathbf{u})=n_o\mathbf{U}$

4-ThermalVector
4-InverseTemperatureMomentum
$\boldsymbol{\Theta}=\Theta^\mu=(\theta^0,\boldsymbol{\theta})=(c/k_BT,\mathbf{u}/k_BT)=(\theta_o/c)\mathbf{U}$
$=(1/k_BT)(c,\mathbf{u})=(1/k_B\gamma T)\mathbf{U}=(1/k_BT_o)\mathbf{U}$

4-Force
$\mathbf{F}=F^\mu=\gamma(\dot{E}/c,\mathbf{f})$
$=\mathbf{dP}/d\tau$

4-MassFlux
4-MomentumDensity
$\mathbf{G}=G^\mu=(\rho_mc,\mathbf{g})=\rho_m(c,\mathbf{u})$
$=m_o\mathbf{N}=n_om_o\mathbf{U}$

4-HeatEnergyFlux
$\mathbf{Q}=Q^\mu=(\rho_Ec,\mathbf{q})=\rho_E(c,\mathbf{u})$
$=E_o\mathbf{N}=n_oE_o\mathbf{U}=c^2\mathbf{G}$

4-PureEntropyFlux
$\mathbf{S_{ent\_pure}}=S_{ent\_pure}{}^\mu$
$=(S_{ent\_pure}{}^0,\mathbf{S_{ent\_pure}})$
$=S_{ent}\mathbf{N}=n_oS_{ent}\mathbf{U}$

4-HeatEntropyFlux
$\mathbf{S_{ent\_heat}}=(S_{ent\_heat}{}^0,\mathbf{S_{ent\_heat}})$
$=S_{ent}\mathbf{N}+\mathbf{Q}/T_o=S_{ent}\mathbf{N}+E_o\mathbf{N}/T_o$
$=n_o(S_{ent}+E_o/T_o)\mathbf{U}$

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^\mu{}_\nu$ or $T_\mu{}^\nu$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu=\mathbf{V}=(v^0,\mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu=(v_0,-\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

4-Vector $\mathbf{V}=V^\mu=(v^\mu)=(v^0,v^i)=(v^0,\mathbf{v})$
SR 4-Vector $\mathbf{V}=V^\mu=$ (scalar * $c^{\pm1}$, 3-vector)

$\text{Trace}[T^{\mu\nu}]=\eta_{\mu\nu}T^{\mu\nu}=T^\mu{}_\mu=T$
$\mathbf{V}\cdot\mathbf{V}=V^\mu\eta_{\mu\nu}V^\nu=[(v^0)^2-\mathbf{v}\cdot\mathbf{v}]=(v^0{}_o)^2$
= Lorentz Scalar

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←Discrete Continuous→
SR:Lorentz Transforms

Lorentz Identity Transform
$\Lambda^{\mu'}{}_{\nu} \to \eta^{\mu'}{}_{\nu} = \delta^{\mu'}{}_{\nu} = \mathrm{I}_{(4)}$

$$\begin{array}{c|cccc} & t & x & y & z \\ \hline t & 1 & 0 & 0 & 0 \\ x & 0 & 1 & 0 & 0 \\ y & 0 & 0 & 1 & 0 \\ z & 0 & 0 & 0 & 1 \end{array} = \begin{bmatrix} 1 & 0_j \\ 0^i & \delta^i{}_j \end{bmatrix}$$

Lorentz Time-Reverse Transform
$\Lambda^{\mu'}{}_{\nu} \to T^{\mu'}{}_{\nu} =$

$$\begin{array}{c|cccc} & t & x & y & z \\ \hline t & -1 & 0 & 0 & 0 \\ x & 0 & 1 & 0 & 0 \\ y & 0 & 0 & 1 & 0 \\ z & 0 & 0 & 0 & 1 \end{array} = \begin{bmatrix} -1 & 0_j \\ 0^i & \delta^i{}_j \end{bmatrix}$$

Lorentz Space-Parity Transform
$\Lambda^{\mu'}{}_{\nu} \to P^{\mu'}{}_{\nu} =$

$$\begin{array}{c|cccc} & t & x & y & z \\ \hline t & 1 & 0 & 0 & 0 \\ x & 0 & -1 & 0 & 0 \\ y & 0 & 0 & -1 & 0 \\ z & 0 & 0 & 0 & -1 \end{array} = \begin{bmatrix} 1 & 0_j \\ 0^i & -\delta^i{}_j \end{bmatrix}$$

Lorentz ComboPT Transform
$\Lambda^{\mu'}{}_{\nu} \to (PT)^{\mu'}{}_{\nu} = -\mathrm{I}_{(4)}$

$$\begin{array}{c|cccc} & t & x & y & z \\ \hline t & -1 & 0 & 0 & 0 \\ x & 0 & -1 & 0 & 0 \\ y & 0 & 0 & -1 & 0 \\ z & 0 & 0 & 0 & -1 \end{array} = \begin{bmatrix} -1 & 0_j \\ 0^i & -\delta^i{}_j \end{bmatrix}$$

Lorentz x-Boost Transform
$\Lambda^{\mu'}{}_{\nu} \to B^{\mu'}{}_{\nu} =$

$$\begin{array}{c|cccc} & t & x & y & z \\ \hline t & \gamma & -\beta\gamma & 0 & 0 \\ x & -\beta\gamma & \gamma & 0 & 0 \\ y & 0 & 0 & 1 & 0 \\ z & 0 & 0 & 0 & 1 \end{array} = \begin{array}{c|cccc} & t & x & y & z \\ \hline t & \cosh[\theta] & -\sinh[\theta] & 0 & 0 \\ x & -\sinh[\theta] & \cosh[\theta] & 0 & 0 \\ y & 0 & 0 & 1 & 0 \\ z & 0 & 0 & 0 & 1 \end{array}$$

General Time-Space Boost

$$\leftarrow \begin{bmatrix} \gamma & -\gamma\beta_j \\ -\gamma\beta^i & (\gamma-1)\beta^i\beta_j/(\boldsymbol{\beta}\cdot\boldsymbol{\beta})+\delta^i{}_j \end{bmatrix}$$

Symmetric Mixed 4-Tensor

SR:Minkowski Metric
$\partial[\mathbf{R}] = \partial^\mu R^\nu = \eta^{\mu\nu} = V^{\mu\nu} + H^{\mu\nu}$
$\to \mathrm{Diag}[1,-\mathrm{I}_{(3)}] = \mathrm{Diag}[1,-\delta^{jk}]$

$$\begin{array}{c|cccc} & t & x & y & z \\ \hline t & 1 & 0 & 0 & 0 \\ x & 0 & -1 & 0 & 0 \\ y & 0 & 0 & -1 & 0 \\ z & 0 & 0 & 0 & -1 \end{array} \quad \begin{bmatrix} 1 & 0^j \\ 0^i & -\delta^{ij} \end{bmatrix}$$

"Particle Physics" Convention
Symmetric

Lorentz z-Rotation Transform
$\Lambda^{\mu'}{}_{\nu} \to R^{\mu'}{}_{\nu} =$

$$\begin{array}{c|cccc} & t & x & y & z \\ \hline t & 1 & 0 & 0 & 0 \\ x & 0 & \cos[\theta] & -\sin[\theta] & 0 \\ y & 0 & \sin[\theta] & \cos[\theta] & 0 \\ z & 0 & 0 & 0 & 1 \end{array}$$

General Space-Space Rotation

$$\leftarrow \begin{bmatrix} 1 & 0_j \\ 0^i & (\delta^i{}_j - n^i n_j)\cos(\theta) - (\varepsilon^i{}_{jk} n^k)\sin(\theta) + n^i n_j \end{bmatrix}$$

Non-symmetric Mixed 4-Tensor

SpaceTime
$\partial \cdot \mathbf{R} = \partial_\mu R^\mu = 4$
Dimension

Lorentz Transform $\partial_\nu[R^{\mu'}] = \Lambda^{\mu'}{}_{\nu}$
$[\ \Lambda^{0'}{}_0, \Lambda^{0'}{}_j\ ]$ temporal-spatial-mixed
$[\ \Lambda^{i'}{}_0, \Lambda^{i'}{}_j\ ]$ components

Perfect Fluid
$T^{\mu\nu} = (\rho_{eo})V^{\mu\nu} + (-p_o)H^{\mu\nu}$

$$\begin{array}{c|cccc} & t & x & y & z \\ \hline t & \rho_e & 0 & 0 & 0 \\ x & 0 & p & 0 & 0 \\ y & 0 & 0 & p & 0 \\ z & 0 & 0 & 0 & p \end{array} \quad \begin{bmatrix} \rho_e & 0^j \\ 0^i & p\delta^{ij} \end{bmatrix}$$

4-Tensor
Symmetric

Faraday EM
$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha = \partial \wedge A$

$$\begin{array}{c|cccc} & t & x & y & z \\ \hline t & 0 & -e^x/c & -e^y/c & -e^z/c \\ x & +e^x/c & 0 & -b^z & +b^y \\ y & +e^y/c & +b^z & 0 & -b^x \\ z & +e^z/c & -b^y & +b^x & 0 \end{array} \quad \begin{bmatrix} 0 & -e^j/c \\ +e^i/c & -\varepsilon^{ij}{}_k b^k \end{bmatrix} \quad \begin{bmatrix} 0 & -\mathbf{e}/c \\ +\mathbf{e}^T/c & -\nabla\wedge\mathbf{a} \end{bmatrix}$$

4-Tensor
Anti-symmetric

4-AngularMomentum
$M^{\alpha\beta} = X^\alpha P^\beta - X^\beta P^\alpha = \mathbf{X} \wedge \mathbf{P}$

$$\begin{array}{c|cccc} & t & x & y & z \\ \hline t & 0 & -cn^x & -cn^y & -cn^z \\ x & +cn^x & 0 & +l^z & -l^y \\ y & +cn^y & -l^z & 0 & +l^x \\ z & +cn^z & +l^y & -l^x & 0 \end{array} \quad \begin{bmatrix} 0 & -cn^j \\ +cn^i & \varepsilon^{ij}{}_k l^k \end{bmatrix} \quad \begin{bmatrix} 0 & -c\mathbf{n} \\ +c\mathbf{n}^T & \mathbf{x}\wedge\mathbf{p} \end{bmatrix}$$

4-Tensor
Anti-symmetric

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^\mu{}_\nu$ or $T_\mu{}^\nu$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0,\mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0,-\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Note that all the Lorentz Transforms and the Minkowski Metric are dimensionless

$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu{}_\mu = T$
$\mathbf{V}\cdot\mathbf{V} = V^\mu\eta_{\mu\nu}V^\nu = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$
= Lorentz Scalar

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Temporal "(V)ertical" Projection (2,0)-Tensor

$\mathrm{Tr}[V^{\mu\nu}]=1$

$P^{\mu\nu} \rightarrow V^{\mu\nu} = T^{\mu}T^{\nu}$
$\rightarrow \mathrm{Diag}[1,0]$

$$\begin{array}{c|cccc} & t & x & y & z \\ \hline t & 1 & 0 & 0 & 0 \\ x & 0 & 0 & 0 & 0 \\ y & 0 & 0 & 0 & 0 \\ z & 0 & 0 & 0 & 0 \end{array} \qquad \begin{bmatrix} 1 & 0^j \\ 0^i & 0^{ij} \end{bmatrix}$$

4-Tensor
Symmetric

Spatial "(H)orizontal" Projection (2,0)-Tensor

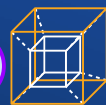
$\mathrm{Tr}[H^{\mu\nu}]=3$

$P^{\mu\nu} \rightarrow H^{\mu\nu} = \eta^{\mu\nu} - T^{\mu}T^{\nu}$
$\rightarrow \mathrm{Diag}[0,-\mathbf{I}_{(3)}]=\mathrm{Diag}[0,-\delta^{jk}]$

$$\begin{array}{c|cccc} & t & x & y & z \\ \hline t & 0 & 0 & 0 & 0 \\ x & 0 & -1 & 0 & 0 \\ y & 0 & 0 & -1 & 0 \\ z & 0 & 0 & 0 & -1 \end{array} \qquad \begin{bmatrix} 0 & 0^j \\ 0^i & -\delta^{ij} \end{bmatrix}$$

4-Tensor
Symmetric

SpaceTime
$\partial\cdot\mathbf{R}=\partial_{\mu}R^{\mu}=4$
Dimension



SR:Minkowski Metric

$\partial[\mathbf{R}] = \partial^{\mu}R^{\nu} = \eta^{\mu\nu} = V^{\mu\nu} + H^{\mu\nu}$
$\rightarrow \mathrm{Diag}[1,-\mathbf{I}_{(3)}]=\mathrm{Diag}[1,-\delta^{jk}]$

$$\begin{array}{c|cccc} & t & x & y & z \\ \hline t & 1 & 0 & 0 & 0 \\ x & 0 & -1 & 0 & 0 \\ y & 0 & 0 & -1 & 0 \\ z & 0 & 0 & 0 & -1 \end{array} \qquad \begin{bmatrix} 1 & 0^j \\ 0^i & -\delta^{ij} \end{bmatrix}$$

"Particle Physics" Convention
Symmetric

$\mathrm{Tr}[\eta^{\mu\nu}]=4$

Perfect Fluid

$\mathrm{Tr}[T^{\mu\nu}]=\rho_{eo}-3p_o$

$T^{\mu\nu} \rightarrow (\rho_{eo})V^{\mu\nu} + (-p_o)H^{\mu\nu}$

$$\begin{array}{c|cccc} & t & x & y & z \\ \hline t & \rho_e & 0 & 0 & 0 \\ x & 0 & p & 0 & 0 \\ y & 0 & 0 & p & 0 \\ z & 0 & 0 & 0 & p \end{array} \qquad \begin{bmatrix} \rho_e & 0^j \\ 0^i & p\delta^{ij} \end{bmatrix}$$

4-Tensor
Symmetric

$(p_o) = 0$ → (Cold) Matter-Dust

$(p_o) = -\rho_{eo}$ → Lambda Vacuum

$(p_o) = \rho_{eo}/3$ → Null-Dust=Photon Gas

$(\rho_{eo}) = (p_o) = 0$ → Zero:Nothing Vacuum

(Cold) Matter-Dust

$T^{\mu\nu} \rightarrow (\rho_{eo})V^{\mu\nu}$

(EoS) $w= 0$

$$\begin{array}{c|cccc} & t & x & y & z \\ \hline t & \rho_e & 0 & 0 & 0 \\ x & 0 & 0 & 0 & 0 \\ y & 0 & 0 & 0 & 0 \\ z & 0 & 0 & 0 & 0 \end{array} \qquad \begin{bmatrix} \rho_e & 0^j \\ 0^i & 0^{ij} \end{bmatrix}$$

4-Tensor
Symmetric

$\mathrm{Tr}[T^{\mu\nu}]=\rho_{eo}$

Null-Dust=Photon Gas

$T^{\mu\nu} \rightarrow (\rho_{eo})V^{\mu\nu} + (-\rho_{eo}/3)H^{\mu\nu}$

(EoS) $w=1/3$

$$\begin{array}{c|cccc} & t & x & y & z \\ \hline t & \rho_e & 0 & 0 & 0 \\ x & 0 & \rho_e/3 & 0 & 0 \\ y & 0 & 0 & \rho_e/3 & 0 \\ z & 0 & 0 & 0 & \rho_e/3 \end{array} \qquad \begin{bmatrix} \rho_e & 0^j \\ 0^i & (\rho_e/3)\delta^{ij} \end{bmatrix}$$

4-Tensor
Symmetric

$\mathrm{Tr}[T^{\mu\nu}]=0$

Lambda Vacuum

$T^{\mu\nu} \rightarrow (\rho_{eo})\eta^{\mu\nu} = (\Lambda)\eta^{\mu\nu}$

(EoS) $w= -1$

$$\begin{array}{c|cccc} & t & x & y & z \\ \hline t & \rho_e & 0 & 0 & 0 \\ x & 0 & -\rho_e & 0 & 0 \\ y & 0 & 0 & -\rho_e & 0 \\ z & 0 & 0 & 0 & -\rho_e \end{array} \qquad \begin{bmatrix} \rho_e & 0^j \\ 0^i & -\rho_e\delta^{ij} \end{bmatrix}$$

4-Tensor
Symmetric

$\mathrm{Tr}[T^{\mu\nu}]=4\rho_{eo}$

Faraday EM Tensor

$F^{\alpha\beta} = \partial^{\alpha}A^{\beta} - \partial^{\beta}A^{\alpha} = \partial \wedge A$

$$\begin{array}{c|cccc} & t & x & y & z \\ \hline t & 0 & -e^x/c & -e^y/c & -e^z/c \\ x & +e^x/c & 0 & -b^z & +b^y \\ y & +e^y/c & +b^z & 0 & -b^x \\ z & +e^z/c & -b^y & +b^x & 0 \end{array} \qquad \begin{bmatrix} 0 & -e^j/c \\ +e^i/c & -\varepsilon^{ij}{}_k b^k \end{bmatrix} \qquad \begin{bmatrix} 0 & -\mathbf{e}/c \\ +\mathbf{e}^T/c & -\nabla\wedge\mathbf{a} \end{bmatrix}$$

4-Tensor
Anti-symmetric

$\mathrm{Tr}[F^{\mu\nu}]=0$

EM (Maxwell) Stress-Energy Tensor

$T^{\mu\nu} \rightarrow (1/\mu_o)[F^{\mu\alpha}F^{\nu}{}_{\alpha}-(1/4)\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}]$

$$\begin{array}{c|cccc} & t & x & y & z \\ \hline t & \tfrac{1}{2}(\varepsilon_o e^2+b^2/\mu_o) & s^x/c & s^y/c & s^z/c \\ x & s^x/c & -\sigma^{xx} & -\sigma^{xy} & -\sigma^{xz} \\ y & s^y/c & -\sigma^{yx} & -\sigma^{yy} & -\sigma^{yz} \\ z & s^z/c & -\sigma^{zx} & -\sigma^{zy} & -\sigma^{zz} \end{array} \qquad \begin{bmatrix} \tfrac{1}{2}(\varepsilon_o e^2+b^2/\mu_o) & s^j/c \\ s^i/c & -\sigma^{ij} \end{bmatrix}$$

4-Tensor
Symmetric

$\mathrm{Tr}[T^{\mu\nu}]=0$

Zero:Nothing Vacuum

$T^{\mu\nu} \rightarrow 0^{\mu\nu}$

$$\begin{array}{c|cccc} & t & x & y & z \\ \hline t & 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 & 0 \\ y & 0 & 0 & 0 & 0 \\ z & 0 & 0 & 0 & 0 \end{array} \qquad \begin{bmatrix} 0 & 0^j \\ 0^i & 0^{ij} \end{bmatrix}$$

4-Tensor
Symmetric

$\mathrm{Tr}[T^{\mu\nu}]=0$

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^{\mu}{}_{\nu}$ or $T_{\mu}{}^{\nu}$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^{\mu} = \mathbf{V} = (v^0,\mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_{\mu} = (v_0,-\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Note that the Projection Tensors and the Minkowski Metric are dimensionless

Equation of State (EoS) $w = (p_o/\rho_{eo})$

$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^{\mu}{}_{\mu} = T$
$\mathbf{V}\cdot\mathbf{V} = V^{\mu}\eta_{\mu\nu}V^{\nu} = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$
$=$ Lorentz Scalar

SRQM Study: Physical 4-Tensors

Projection 4-Tensors

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson

$Tr[V^{\mu\nu}]=1$

Temporal "(V)ertical" Projection (2,0)-Tensor

$P^{\mu\nu} \rightarrow V^{\mu\nu} = T^\mu T^\nu$

$\rightarrow Diag[1,0]$

$$\begin{array}{c|cccc} & \underline{t} & \underline{x} & \underline{y} & \underline{z} \\ \hline \underline{t} & 1 & 0 & 0 & 0 \\ \underline{x} & 0 & 0 & 0 & 0 \\ \underline{y} & 0 & 0 & 0 & 0 \\ \underline{z} & 0 & 0 & 0 & 0 \end{array} \qquad \begin{bmatrix} 1 & 0^j \\ 0^i & 0^{ij} \end{bmatrix}$$

4-Tensor
Symmetric

$Tr[V^\mu{}_\nu]=1$

Temporal "(V)ertical" Projection (1,1)-Tensor

$P^\mu{}_\nu \rightarrow V^\mu{}_\nu = T^\mu T_\nu$

$\rightarrow Diag[1,0]$

$$\begin{array}{c|cccc} & \underline{t} & \underline{x} & \underline{y} & \underline{z} \\ \hline \underline{t} & 1 & 0 & 0 & 0 \\ \underline{x} & 0 & 0 & 0 & 0 \\ \underline{y} & 0 & 0 & 0 & 0 \\ \underline{z} & 0 & 0 & 0 & 0 \end{array} \qquad \begin{bmatrix} 1 & 0_j \\ 0^i & 0^i{}_j \end{bmatrix}$$

4-Tensor
Symmetric

$Tr[V_{\mu\nu}]=1$

Temporal "(V)ertical" Projection (0,2)-Tensor

$P_{\mu\nu} \rightarrow V_{\mu\nu} = T_\mu T_\nu$

$\rightarrow Diag[1,0]$

$$\begin{array}{c|cccc} & \underline{t} & \underline{x} & \underline{y} & \underline{z} \\ \hline \underline{t} & 1 & 0 & 0 & 0 \\ \underline{x} & 0 & 0 & 0 & 0 \\ \underline{y} & 0 & 0 & 0 & 0 \\ \underline{z} & 0 & 0 & 0 & 0 \end{array} \qquad \begin{bmatrix} 1 & 0_j \\ 0_i & 0_{ij} \end{bmatrix}$$

4-Tensor
Symmetric

$Tr[H^{\mu\nu}]=3$

Spatial "(H)orizontal" Projection (2,0)-Tensor

$P^{\mu\nu} \rightarrow H^{\mu\nu} = \eta^{\mu\nu} - T^\mu T^\nu$

$\rightarrow Diag[0,-\mathbf{I}_{(3)}]=Diag[0,-\delta^{jk}]$

$$\begin{array}{c|cccc} & \underline{t} & \underline{x} & \underline{y} & \underline{z} \\ \hline \underline{t} & 0 & 0 & 0 & 0 \\ \underline{x} & 0 & -1 & 0 & 0 \\ \underline{y} & 0 & 0 & -1 & 0 \\ \underline{z} & 0 & 0 & 0 & -1 \end{array} \qquad \begin{bmatrix} 0 & 0^j \\ 0^i & -\delta^{ij} \end{bmatrix}$$

4-Tensor
Symmetric

$Tr[H^\mu{}_\nu]=3$

Spatial "(H)orizontal" Projection (1,1)-Tensor

$P^\mu{}_\nu \rightarrow H^\mu{}_\nu = \eta^\mu{}_\nu - T^\mu T_\nu$

$\rightarrow Diag[0,\mathbf{I}_{(3)}]=Diag[0,\delta^j{}_k]$

$$\begin{array}{c|cccc} & \underline{t} & \underline{x} & \underline{y} & \underline{z} \\ \hline \underline{t} & 0 & 0 & 0 & 0 \\ \underline{x} & 0 & 1 & 0 & 0 \\ \underline{y} & 0 & 0 & 1 & 0 \\ \underline{z} & 0 & 0 & 0 & 1 \end{array} \qquad \begin{bmatrix} 0 & 0_j \\ 0^i & \delta^i{}_j \end{bmatrix}$$

4-Tensor
Symmetric

$Tr[H_{\mu\nu}]=3$

Spatial "(H)orizontal" Projection (0,2)-Tensor

$P_{\mu\nu} \rightarrow H_{\mu\nu} = \eta_{\mu\nu} - T_\mu T_\nu$

$\rightarrow Diag[0,-\mathbf{I}_{(3)}]=Diag[0,-\delta_{jk}]$

$$\begin{array}{c|cccc} & \underline{t} & \underline{x} & \underline{y} & \underline{z} \\ \hline \underline{t} & 0 & 0 & 0 & 0 \\ \underline{x} & 0 & -1 & 0 & 0 \\ \underline{y} & 0 & 0 & -1 & 0 \\ \underline{z} & 0 & 0 & 0 & -1 \end{array} \qquad \begin{bmatrix} 0 & 0_j \\ 0_i & -\delta_{ij} \end{bmatrix}$$

4-Tensor
Symmetric

Perfect Fluid

$T^{\mu\nu} \rightarrow (\rho_{eo})V^{\mu\nu} + (-p_o)H^{\mu\nu}$

$$\begin{array}{c|cccc} & \underline{t} & \underline{x} & \underline{y} & \underline{z} \\ \hline \underline{t} & \rho_e & 0 & 0 & 0 \\ \underline{x} & 0 & p & 0 & 0 \\ \underline{y} & 0 & 0 & p & 0 \\ \underline{z} & 0 & 0 & 0 & p \end{array} \qquad \begin{bmatrix} \rho_e & 0^j \\ 0^i & p\delta^{ij} \end{bmatrix}$$

4-Tensor
Symmetric

$Tr[T^{\mu\nu}]=\rho_{eo}-3p_o$

$P^\mu{}_\nu = P^{\mu\alpha}\eta_{\alpha\nu}$
$P_{\mu\nu} = P^{\alpha\beta}\eta_{\alpha\mu}\eta_{\beta\nu}$

The projection tensors can work on 4-Vectors to give a new 4-Vector, or on 4-Tensors to give either a 4-Scalar component or a new 4-Tensor.

4-UnitTemporal $T^\mu = \gamma(1,\boldsymbol{\beta})$
4-Generic $A^\nu = (a^0,\mathbf{a}) = (a^0,a^1,a^2,a^3)$

$$\begin{aligned} V^\mu{}_\nu A^\nu = (&1\cdot a^0,+0\cdot a^1+0\cdot a^2+0\cdot a^3, \\ &0\cdot a^0,+0\cdot a^1+0\cdot a^2+0\cdot a^3, \\ &0\cdot a^0,+0\cdot a^1+0\cdot a^2+0\cdot a^3, \\ &0\cdot a^0,+0\cdot a^1+0\cdot a^2+0\cdot a^3) = (a^0,0,0,0) = (a^0,\mathbf{0})\text{: Temporal Projection} \end{aligned}$$

$$\begin{aligned} H^\mu{}_\nu A^\nu = (&0\cdot a^0,+0\cdot a^1+0\cdot a^2+0\cdot a^3, \\ &0\cdot a^0,+1\cdot a^1+0\cdot a^2+0\cdot a^3, \\ &0\cdot a^0,+0\cdot a^1+1\cdot a^2+0\cdot a^3, \\ &0\cdot a^0,+0\cdot a^1+0\cdot a^2+1\cdot a^3) = (0,a^1,a^2,a^3) = (0,\mathbf{a})\text{: Spatial Projection} \end{aligned}$$

$V_{\mu\nu}T^{\mu\nu} = V_{\mu\nu}[(\rho_{eo})V^{\mu\nu} + (-p_o)H^{\mu\nu}] = (\rho_{eo})V_{\mu\nu}V^{\mu\nu} = (\rho_{eo})$
$H_{\mu\nu}T^{\mu\nu} = H_{\mu\nu}[(\rho_{eo})V^{\mu\nu} + (-p_o)H^{\mu\nu}] = (-p_o)H_{\mu\nu}H^{\mu\nu} = (-3p_o)$

$V^\mu{}_\alpha T^{\alpha\nu} = V^\mu{}_\alpha[(\rho_{eo})V^{\alpha\nu} + (-p_o)H^{\alpha\nu}] = (\rho_{eo})V^\mu{}_\alpha V^{\alpha\nu}+(0) = (\rho_{eo})V^{\mu\nu} = Diag[\rho_e,0,0,0]$
$H^\mu{}_\alpha T^{\alpha\nu} = H^\mu{}_\alpha[(\rho_{eo})V^{\alpha\nu} + (-p_o)H^{\alpha\nu}] = (0)+(-p_o)H^\mu{}_\alpha H^{\alpha\nu} = (-p_o)H^{\mu\nu} = Diag[0,p,p,p]$

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^\mu{}_\nu$ or $T_\mu{}^\nu$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0,\mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0,-\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Note that the Projection Tensors and the Minkowski Metric are dimensionless

$Trace[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu{}_\mu = T$
$\mathbf{V\cdot V} = V^\mu\eta_{\mu\nu}V^\nu = [(v^0)^2 - \mathbf{v\cdot v}] = (v^0{}_o)^2$
= Lorentz Scalar

4-Vector SRQM Interpretation of QM

SRQM Diagram:
Special Relativity → Quantum Mechanics RoadMap of SR→QM

A Tensor Study of Physical 4-Vectors

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SRQM:
SR→QM Interpretation Simplified

A Tensor Study of Physical 4-Vectors

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SRQM: The [SR→QM] Interpretation of Quantum Mechanics

SR Axioms: Invariant Interval + (c) as Physical Constant lead to SR, although technically SR is itself the low-curvature limiting-case of GR

$\{c,\tau,m_o,\hbar,i\}$: All Empirically Measured SR Lorentz Invariants and/or Mathematical Constants

Standard SR 4-Vectors: Related by these SR Lorentz Invariants

4-Position	$\mathbf{R} = (ct,\mathbf{r})$	= <Event>	$(\mathbf{R}\cdot\mathbf{R}) = (c\tau)^2$
4-Velocity	$\mathbf{U} = \gamma(c,\mathbf{u})$	$= (\mathbf{U}\cdot\partial)\mathbf{R}=({}^{d}/_{d\tau})\mathbf{R}=d\mathbf{R}/d\tau$	$(\mathbf{U}\cdot\mathbf{U}) = (c)^2$
4-Momentum	$\mathbf{P} = (E/c,\mathbf{p})$	$= m_o\mathbf{U}$	$(\mathbf{P}\cdot\mathbf{P}) = (m_oc)^2$
4-WaveVector	$\mathbf{K} = (\omega/c,\mathbf{k})$	$= \mathbf{P}/\hbar$	$(\mathbf{K}\cdot\mathbf{K}) = (m_oc/\hbar)^2$
4-Gradient	$\partial = (\partial_t/c,-\nabla)$	$= -i\mathbf{K}$	$(\partial\cdot\partial) = -(m_oc/\hbar)^2$ = KG Eqn→RQM $\xrightarrow{\lvert v\rvert<<c}$ QM

SR + Emipirically Measured Physical Constants lead to RQM via the Klein-Gordon Eqn, and thence to QM via the low-velocity limit { |**v**| << c }, giving the Schrödinger Eqn.
The relation also leads to the Dirac, Maxwell, Pauli, Proca, Weyl, & Scalar Wave QM Eqns.

SRQM 4-Vector Topics Covered

SR & QM via 4-Vector Diagrams

Mostly SR Stuff

4-Vector Basics
Paradigm Assumptions, Where is Quantum Gravity?
Minkowski SpaceTime, <Events>, WorldLines, Minkowski Metric
4-Scalars, 4-Vectors, 4-Tensors & Tensor Invariants
SR 4-Vector Connections
SR Lorentz Transforms, CPT Symmetry, Trace Identification, Antimatter
Fundamental Physical Constants = Lorentz Scalar Invariants
Projection Tensors: Temporal (V) & Spatial (H)
Stress-Energy Tensors, Perfect Fluids, Special Cases (Dust,Radiation,etc)
Invariant Intervals, Measurement, Causality, Relativity
SpaceTime Kinematics & Dynamics, ProperTime Derivative
Einstein's $E = mc^2 = \gamma m_o c^2 = \gamma E_o$, Rest Mass, Rest Energy, Invariants
SpaceTime Orthogonality: Time-like velocity, Space-like acceleration
Relativity of Simultaneity, Time Dilation, Length Contraction
SR Motion * Lorentz Scalar = Interesting Physical 4-Vector
SR Conservation Laws & Local Continuity Equations, Symmetries
Relativistic Doppler Effect, Relativistic Aberration Effect
SR Wave-Particle Relation, Invariant d'Alembertian Wave Eqn, SR Waves
SpaceTime is 4D: $\partial\cdot\mathbf{R}=\partial_\mu R^\mu=4$, $\Lambda_{\mu\nu}\Lambda^{\mu\nu}=4$, $Tr[\eta^{\mu\nu}]=4$, $\mathbf{A} = A^\mu = (a^0,a^1,a^2,a^3)$
Minimal Coupling = Interaction with a (Vector)Potential
Conservation of 4-TotalMomentum
SR Hamiltonian:Lagrangian Connection
Lagrangian, Lagrangian Density
Hamilton-Jacobi Equation (differential), Relativistic Action (integral)
Euler-Lagrange Equations
Relativistic Equations of Motion, Lorentz Force Equation
c^2 Invariant Relations, The Speed-of-Light (c)
Thermodynamic 4-Vectors, Unruh-Hawking Radiation, Particle Distributions

Mostly QM & SRQM Stuff

Relativistic Quantum Wave Equations
Klein-Gordon Equation/Relation
RoadMap from SR to QM: SR→QM, SRQM 4-Vector Connections
QM Schrödinger Relation
QM Axioms? - No, (QM Principles derived from SR) = SRQM
Relativistic Wave Equations: based on mass & spin & velocity
Klein-Gordon, Dirac, Proca, Maxwell, Weyl, Pauli, Schrödinger, etc.
Classical Limits $|\mathbf{v}|<<c$
Photon Polarization
Linear PDE's→
{Principle of Superposition, Hilbert Space, <Bra|,|Ket> Notation}
Canonical QM Commutation Relations – derived from SR
Heisenberg Uncertainty Principle (due to non-zero commutation)
Pauli Exclusion Principle (Fermion), Bose Aggregation Principle (Boson)
Complex 4-Vectors
CPT Theorem, Lorentz Invariance, Poincaré Invariance, Isometry
Hermetian Generators, Unitarity, Anti-Unitarity
QM → Classical Correspondence Principle, similar to SR → Classical
Quantum Probability
The Compton Effect = Photon:Electron Interaction (neglecting Spin Effects)
Photon Diffraction, Crystal-Electron Diffraction, The Kapitza-Dirac Effect
The ħ Relation, Einstein-de Broglie, Planck:Dirac
The Aharonov-Bohm Effect, The Josephson Junction Effect
Noether's Theorem, Continuous Symmetries, Conservation Laws
Dimensionless Quantities
Quantum Relativity: GR is ***NOT*** wrong, *Never bet against Einstein* :)
Quantum Mechanics is Derivable from Special Relativity, SR→QM, SRQM

SRQM = The [SR→QM] Interpretation of Quantum Mechanics
= Special Relativity → Quantum Mechanics

Special Relativity → Quantum Mechanics

Paradigm Background Assumptions (part 1)

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson

There are some paradigm assumptions that need to be cleared up:

Relativistic Physics **IS NOT** the generalization of Classical Physics.
Classical Physics **IS** the low-velocity limiting-case approximation of Relativistic Physics { $|\mathbf{v}| << c$ }.

This includes (Newtonian) Classical Mechanics and Classical QM, (meaning the non-relativistic Schrödinger QM Equation).
Classical EM is for the most part already compatible with Special Relativity.
However, Classical EM doesn't include intrinsic spin, even though spin is a result of SR Poincaré Invariance, not QM.

So far, in all of my research, if there was a way to get a result classically,
then there was usually a much simpler way to get the result using 4-Vectors and SRQM relativistic thinking.
Likewise, a lot of QM results make much more sense when approached from SRQM.

Einstein Energy:Mass Eqn: $\mathbf{P} = m_o\mathbf{U} \rightarrow \{ E = mc^2 = \gamma m_o c^2 = \gamma E_o : \mathbf{p} = m\mathbf{u} = \gamma m_o \mathbf{u} \}$
Hamiltonian: $H = \gamma(\mathbf{P_T}\cdot\mathbf{U})$ { Relativistic } $\rightarrow (T + V) = (E_{kinetic} + E_{potential})$ { Classical-limit only, $|\mathbf{u}| << c$ }
Lagrangian: $L = -(\mathbf{P_T}\cdot\mathbf{U})/\gamma$ { Relativistic } $\rightarrow (T - V) = (E_{kinetic} - E_{potential})$ { Classical-limit only, $|\mathbf{u}| << c$ }
SR Wave Eqn(differential format): $\mathbf{K_T} = -\partial[\Phi_{phase}] = \mathbf{P_T}/\hbar \rightarrow \{ \omega_T = -\partial_t[\Phi] : \mathbf{k_T} = \nabla[\Phi] \}$
Hamilton-Jacobi Eqn(differential format): $\mathbf{P_T} = -\partial[S_{action}] = \hbar\mathbf{K_T} \rightarrow \{ E_T = -\partial_t[S] : \mathbf{p_T} = \nabla[S] \}$
Action Equation(integral format): $\Delta S_{action} = -\int_{path}\mathbf{P_T}\cdot\mathbf{dX} = -\int_{path}(\mathbf{P_T}\cdot\mathbf{U})d\tau = \int_{path}L\, dt$
SR/QM Wave Equation(integral format): $\Delta\Phi_{phase} = -\int_{path}\mathbf{K_T}\cdot\mathbf{dX} = -\int_{path}(\mathbf{K_T}\cdot\mathbf{U})d\tau = \Delta S_{action}/\hbar$
Euler-Lagrange Equation: $(\mathbf{U} = (d/d\tau)\mathbf{R}) \rightarrow (\partial_\mathbf{R} = (d/d\tau)\partial_\mathbf{U})$
Hamilton's Equations: $(d/d\tau)[\mathbf{X}] = (\partial/\partial\mathbf{P_T})[H_o]$ & $(d/d\tau)[\mathbf{P_T}] = (\partial/\partial\mathbf{X})[H_o]$
d'Alembertian Wave Equation: $\partial\cdot\partial = (\partial_t/c)^2 - \nabla\cdot\nabla$, with solutions $\sim \Sigma_n e^{\pm(\mathbf{K_n}\cdot\mathbf{X})}$

Einstein-de Broglie Relation: $\mathbf{P} = \hbar\mathbf{K} \rightarrow \{ E = \hbar\omega : \mathbf{p} = \hbar\mathbf{k} \}$
Complex Plane-wave Relation: $\mathbf{K} = i\partial \rightarrow \{ \omega = i\partial_t : \mathbf{k} = -i\nabla \}$
Schrödinger Relations: $\mathbf{P} = i\hbar\partial \rightarrow \{ E = i\hbar\partial_t : \mathbf{p} = -i\hbar\nabla \}$
Canonical QM Commutation Relations inc. QM Time-Energy:
$[P^\mu,X^\nu] = i\hbar\eta^{\mu\nu} \rightarrow \{ [x^0,p^0] = [t,E] = -i\hbar : [x^j,p^k] = i\hbar\delta^{jk} \}$
Minimal Coupling: $\mathbf{P} = \mathbf{P_T} - q\mathbf{A} \rightarrow \{ E = E_T - q\varphi : \mathbf{p} = \mathbf{p_T} - q\mathbf{a} \}$
Josephson Junction Relation(differential format): $\mathbf{A} = -(\hbar/q)\partial[\Delta\Phi_{pot}]$
Aharonov-Bohm Relation(integral format): $\Delta\Phi_{pot} = -(q/\hbar)\int_{path}\mathbf{A}\cdot\mathbf{dX}$
Compton Scattering: $\Delta\lambda = (\lambda' - \lambda) = (\hbar/m_o c)(1 - \cos[\theta])$
Klein-Gordon Relativistic Quantum Wave Eqn: $\partial\cdot\partial = -(m_o c/\hbar)^2$

4-Vector formulations are all extremely easy to derive in SRQM and are all relativistically covariant.

Special Relativity → Quantum Mechanics
Paradigm Background Assumptions (part 2)

There are some paradigm assumptions that need to be cleared up:

SR 4D Physical 4-Vectors *ARE NOT* generalizations of Classical/Quantum 3D Physical 3-vectors.
While a "mathematical" Euclidean (n+1)D-vector is the generalization of a Euclidean (n)D-vector,
the "physical" analogy ends there.

Minkowskian SR 4-Vectors *ARE* the primitive elements of 4D Minkowski SR SpaceTime.
Classical/Quantum Physical 3-vectors are just the spatial components of SR Physical 4-Vectors.
There is also a fundamentally-related Classical/Quantum Physical scalar related to each 3-vector,
which is just the temporal component scalar of a given SR Physical 4-Vector.

ex. 4-Position $\mathbf{R} = (r^\mu) = (r^0,\mathbf{r}) = (ct,\mathbf{r}) \rightarrow (ct,x,y,z)$: 4-Momentum $\mathbf{P} = (p^\mu) = (p^0,\mathbf{p}) = (E/c,\mathbf{p}) \rightarrow (E/c,p^x,p^y,p^z)$

These Classical/Quantum {scalar}+{**3-vector**} are the dual {temporal}+{**spatial**} components
of a single SR 4-Vector = (temporal scalar * $c^{\pm 1}$, **spatial 3-vector**)
with SR lightspeed factor ($c^{\pm 1}$) to give correct overall dimensional units.

While different observers may see different "values" of the
Classical/Quantum components (v^0,v^1,v^2,v^3) from their point-of-view in SpaceTime,
each will see the same actual SR 4-Vector **V** and its magnitude |**V**| at a given <Event> in SpaceTime.

Special Relativity → Quantum Mechanics
Paradigm Background Assumptions (part 3)

There are some paradigm assumptions that need to be cleared up:

We will **NOT** be employing the commonly-(mis)used Newtonian classical limits $\{c \to \infty\}$ and $\{\hbar \to 0\}$.
Neither of these is a valid physical assumption, for the following reasons:

[1]

Both (c) and (ħ) are unchanging Physical Constants and Lorentz Invariants.
Taking a limit where these change is non-physical. They are CONSTANT.
Many, many experiments verify that these constants have not changed over the lifetime of the universe.
This is one reason for the 2019 Redefinition of SI Base Units on Fundamental Constants ($c, \hbar, e, k_B, N_A, K_{CD}, \Delta\nu_{Cs}$).

[2]

Let $E = pc$. If $c \to \infty$, then $E \to \infty$. Then Classical EM light rays/waves have infinite energy.
Let $E = \hbar\omega$. If $\hbar \to 0$, then $E \to 0$. Then Classical EM light rays/waves have zero energy.

Obviously neither of these is true in the Newtonian limit.
In Classical EM and Classical Mechanics, (c) remains a large but finite constant.
Likewise, (ħ) remains very small but never becomes zero.

The correct way to take the limits is via:
The low-velocity non-relativistic limit $\{ |\mathbf{v}| \ll c \}$, which is a physically-occurring situation.
The Hamilton-Jacobi non-quantum limit $\{ \hbar|\nabla\cdot\mathbf{p}| \ll (\mathbf{p}\cdot\mathbf{p}) \}$, which is a physically-occurring situation.

Special Relativity → Quantum Mechanics
Paradigm Background Assumptions (part 4)

There are some paradigm assumptions that need to be cleared up:

We will *NOT* be implementing the common {→lazy and extremely misguided} convention of setting physical constants to the value of (dimensionless) unity, often called "Natural Units", to hide them from equations; nor using mass (m) instead of (m_o) as the RestMass. Likewise for other components vs Lorentz Scalars with naughts, like energy (E) vs (E_o) as the RestEnergy.

One sees this very often in the literature. The usual excuse cited is "For the sake of brevity".
Well, the "sake of brevity" forsakes "clarity"

The *ONLY* situation in which setting constants to unity is practical or advisable is in numerical simulation. When teaching physics, or trying to understand physics, it helps when equations are dimensionally correct. In other words, the technique of dimensional analysis is a powerful tool that should not be disdained.
i.e. Brevity aids speed of computation, Clarity aids understanding.

The situation of using "naught = $_o$" for rest-values, such as (m_o) for RestMass and (E_o) for RestEnergy:
Is intrinsic to SR, is a very good idea, absolutely adds clarity, identifies Lorentz Scalar Invariants, and will be explained in more detail later.
Essentially, the relativistic gamma (γ) pairs with a (Lorentz scalar:rest value $_o$) to make a relativistic component: $m = \gamma m_o$, $E = \gamma E_o$
Note the multiple equivalent ways that one can write 4-Vectors using these rules:

4-Momentum $\mathbf{P} = P^\mu = (p^\mu) = (mc,p^i) = (mc,\mathbf{p}) = m_o\mathbf{U} = m_o\gamma(c,\mathbf{u}) = \gamma m_o(c,\mathbf{u}) = m(c,\mathbf{u}) = (mc,m\mathbf{u}) = (mc,\mathbf{p})$
$= (E/c,\mathbf{p}) = (E_o/c^2)\mathbf{U} = (E_o/c^2)\gamma(c,\mathbf{u}) = \gamma(E_o/c^2)(c,\mathbf{u}) = (E/c^2)(c,\mathbf{u}) = (E/c,E\mathbf{u}/c^2) = (E/c,\mathbf{p})$

It is damn hard enough just to get the minus-signs right in GR/SR, as there are different metric-conventions available.
BTW, I prefer the "Particle Physics" Metric-Convention (+,-,-,-). {Makes rest values positive, fewer minus signs to deal with}

Show the physical constants and naughts in the work. They deserve the respect and you will benefit.
You can always set constants to unity later, when you are doing your numerical simulations.

Special Relativity → Quantum Mechanics

Paradigm Background Assumptions (part 5)

There are some paradigm assumptions that need to be cleared up:

Many physics books say that the Electric field **E** and the Magnetic field **B** are the “real” physical objects, and that the EM scalar-potential φ and the EM 3-vector-potential **A** are just “calculational/mathematical” artifacts.

Neither of these statements is relativistically correct.

All of these physical EM properties: {**E**,**B**,φ,**A**} are actually just the components of SR tensors, and as such, their magnitudes will vary in different observers’ reference-frames.
The truly SR invariant physical objects are:
The 4-Gradient ∂, the 4-VectorPotential **A**, and their combination via exterior (wedge=^) product into the Faraday EM Tensor $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha = \partial \wedge \mathbf{A}$

Given this SR knowledge, we demote the physical property symbols, (the tensor components) to their lower-case equivalents {**e**,**b**,φ,**a**}.

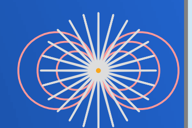
Temporal-spatial components of 4-Tensor $F^{\alpha\beta}$: electric 3-vector field **e**.
Spatial-spatial components of 4-Tensor $F^{\alpha\beta}$: magnetic 3-vector field **b**.
Temporal component of 4-Vector **A**: EM scalar-potential φ.
Spatial components of 4-Vector **A**: EM 3-vector-potential **a**.

Note that the speed-of-light (c) plays a prominent role in the component definitions.
Also, QM requires the 4-VectorPotential **A** as explanation of the Aharonov-Bohm Effect.
Again, all the higher-index-count SR tensors can be built from fundamental 4-Vectors.

4-Gradient
$\partial = \partial^\mu = (\partial_t/c, -\nabla)$
$\rightarrow (\partial_t/c, -\partial_x, -\partial_y, -\partial_z)$

Faraday EM Tensor
$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$
$= \partial \wedge \mathbf{A}$
$\rightarrow$

$$\begin{bmatrix} F^{tt} & F^{tx} & F^{ty} & F^{tz} \\ F^{xt} & F^{xx} & F^{xy} & F^{xz} \\ F^{yt} & F^{yx} & F^{yy} & F^{yz} \\ F^{zt} & F^{zx} & F^{zy} & F^{zz} \end{bmatrix} = \begin{bmatrix} 0 & -e^x/c & -e^y/c & -e^z/c \\ +e^x/c & 0 & -b^z & +b^y \\ +e^y/c & +b^z & 0 & -b^x \\ +e^z/c & -b^y & +b^x & 0 \end{bmatrix} = \begin{bmatrix} 0 & , -e^j/c \\ +e^i/c, & -\varepsilon^{ij}{}_k b^k \end{bmatrix} = \begin{bmatrix} 0 & , -\mathbf{e}/c \\ +\mathbf{e}^T/c, & -\nabla \wedge \mathbf{a} \end{bmatrix}$$



4-(EM)VectorPotential
$\mathbf{A} = A^\mu = (\varphi/c, \mathbf{a})$
$\mathbf{A}_{EM} = A_{EM}{}^\mu = (\varphi_{EM}/c, \mathbf{a}_{EM})$

Special Relativity → Quantum Mechanics
Paradigm Background Assumptions (part 6)

There are some paradigm assumptions that need to be cleared up:

A number of QM philosophies make the assertion that particle "properties" do not "exist" until measured. The assertion is based on the Heisenberg Uncertainty Principle, and more specifically on quantum non-zero commutation, in which a measurement on one property of a particle alters a non-commuting property of the same particle.

That is an incorrect analysis. Properties define particles: what they do, how they interact with other particles. Particles and their properties "exist" independently of human intervention or observation. The correct way to analyze this is to understand what a measurement is: the arrangement of some number of fundamental particles in a particular manner as to allow an observer to get information about one or more of the subject particle's properties. Typically this involves "counting" spacetime events and using SR invariant intervals as a basis of measurement.

Some properties are indeed non-commuting. This simply means that it is not possible to arrange a set of particles in such a way as to measure (ie. obtain "complete" information about) both of the subject particle's non-commuting properties at the same spacetime event. The measurement arrangement events can be done at best sequentially, and the temporal order of these events makes a difference in observed results. EPR-Bell, however, allows one to "infer" properties on a subject particle by making a measurement on a different {space-like separated but entangled} particle.

So, a better way to think about it is this: The "measurement" of a property does not "exist" until a physical setup event is arranged. Non-commuting properties require different physical arrangements in order for all to be measured, and the temporally-first measurement alters the particle's properties in a minimum sort of way, which affects the latter measurement. All observers agree on the order of temporally-separated spacetime events.However, individual observers may have different sets of partial information about the same particle(s).

This makes way more sense than the subjective belief that a particle's property doesn't exist until it is observed, which is about as unscientific and laughable a statement as I can imagine.

Relativity is the system of measurement that QM has been looking for

Special Relativity → Quantum Mechanics
Paradigm Background Assumptions (part 7)

There are some paradigm assumptions that need to be cleared up:

Correct Notation is critical for understanding physics

Unfortunately, there are a number of "sloppy" notations in relativistic and quantum physics.

Incorrect: Using T^{ii} as a Trace of tensor T^{ij}, or $T^{\mu\mu}$ as a Trace of tensor $T^{\mu\nu}$
T^{ii} is just the diagonal part of 3-tensor T^{ij}, the components: $T^{ii} = \mathrm{Diag}[T^{11},T^{22},T^{33}]$
$T_i^{\ i}$ is the Trace of 3-tensor T^{ij}: $T_i^{\ i} = T_1^{\ 1}+T_2^{\ 2}+T_3^{\ 3} = \text{3-trace}[T^{ij}] = T^{11}+T^{22}+T^{33}$ in the Euclidean Metric $E^{ij} = \delta^{ij}$

$T^{\mu\mu}$ is just the diagonal part of 4-Tensor $T^{\mu\nu}$, the components: $T^{\mu\mu} = \mathrm{Diag}[T^{00},T^{11},T^{22},T^{33}]$
$T_\mu^{\ \mu}$ is the Trace of 4-Tensor $T^{\mu\nu}$: $T_\mu^{\ \mu} = T_0^{\ 0}+T_1^{\ 1}+T_2^{\ 2}+T_3^{\ 3} = \text{4-Trace}[T^{\mu\nu}] = T^{00}-T^{11}-T^{22}-T^{33}$ in the Minkowskian Metric $\eta^{\mu\nu}$

Incorrect: Hiding factors of (c) in relativistic equations, ex. $E = m$
The use of "natural units" leads to a lot of ambiguity, and one loses the ability to do dimensional analysis.
Wrong: $E=m$: Energy is *not* identical to mass.
Correct: $E=mc^2$: Energy is related to mass via the speed-of-light, ie. mass is a type of concentrated energy.

Incorrect: Using m instead of m_o for rest mass, Using E instead of E_o for rest energy
Correct: $E = mc^2 = \gamma m_o c^2 = \gamma E_o$
E & m are relativistic internal components of 4-Momentum $\mathbf{P}=(mc,\mathbf{p})=(E/c,\mathbf{p})$ which vary in different reference-frames.
E_o & m_o are Lorentz Scalar Invariants, the rest values, which are the same, even in different reference-frames: $\mathbf{P}=m_o\mathbf{U}=(E_o/c^2)\mathbf{U}$

Special Relativity → Quantum Mechanics

Paradigm Background Assumptions (part 8)

There are some paradigm assumptions that need to be cleared up:

Wrong: $[x^i,p^j] = i\hbar\delta^{ij}$
Right: $[x^j,p^k] = i\hbar\delta^{jk}$
Better: $[P^\mu,X^\nu] = i\hbar\eta^{\mu\nu}$

Incorrect: Using the same symbol for a tensor-index and a component
The biggest offender in many books for this one is quantum commutation.
Unclear because (i) means two different things in one equation.
Better: (i = √[-1]) is the imaginary unit ; { j,k } are tensor-indicies
In general, any equation which uses complex-number math should reserve (i) for the imaginary, not as a tensor-index.

Incorrect: Using the 4-Gradient notation incorrectly
The 4-Gradient is a 4-Vector, a (1,0)-Tensor, which uses an upper index, and has a negative spatial component in SR.
The Gradient One-Form, its natural tensor form, a (0,1)-Tensor, uses a lower index in SR.
4-Gradient: $\partial=\partial^\mu=(\partial_t/c,-\nabla)$ Gradient One-Form: $\partial_\mu=(\partial_t/c,\nabla)$

Incorrect: Mixing styles in 4-Vector naming conventions
There is pretty much universal agreement on the 4-Momentum $\mathbf{P}=P^\mu=(E/c,\mathbf{p})=(mc,\mathbf{p})$
Do not in the same document use 4-Potential $\mathbf{A}=(\varphi,\mathbf{A})$: This is wrong on many levels.
The correct form is 4-VectorPotential $\mathbf{A}=A^\mu=(\varphi/c,\mathbf{a})$, with (φ) as the scalar-potential & (**a**) as the 3-vector-potential
<u>For all 4-Vectors, one should use a consistent notation:</u>
The Upper-Case SpaceTime 4-Vector Names match the lower-case spatial 3-vector names
There is a (c) factor in the temporal component to give overall matching dimensional units for the entire 4-Vector
4-Vector components are typically lower-case with a few historical exceptions, mainly energy E, energy-density e or ρ

Old Paradigm: QM (as I was taught)
SR and QM as separate theories

A Tensor Study of Physical 4-Vectors

SciRealm.org
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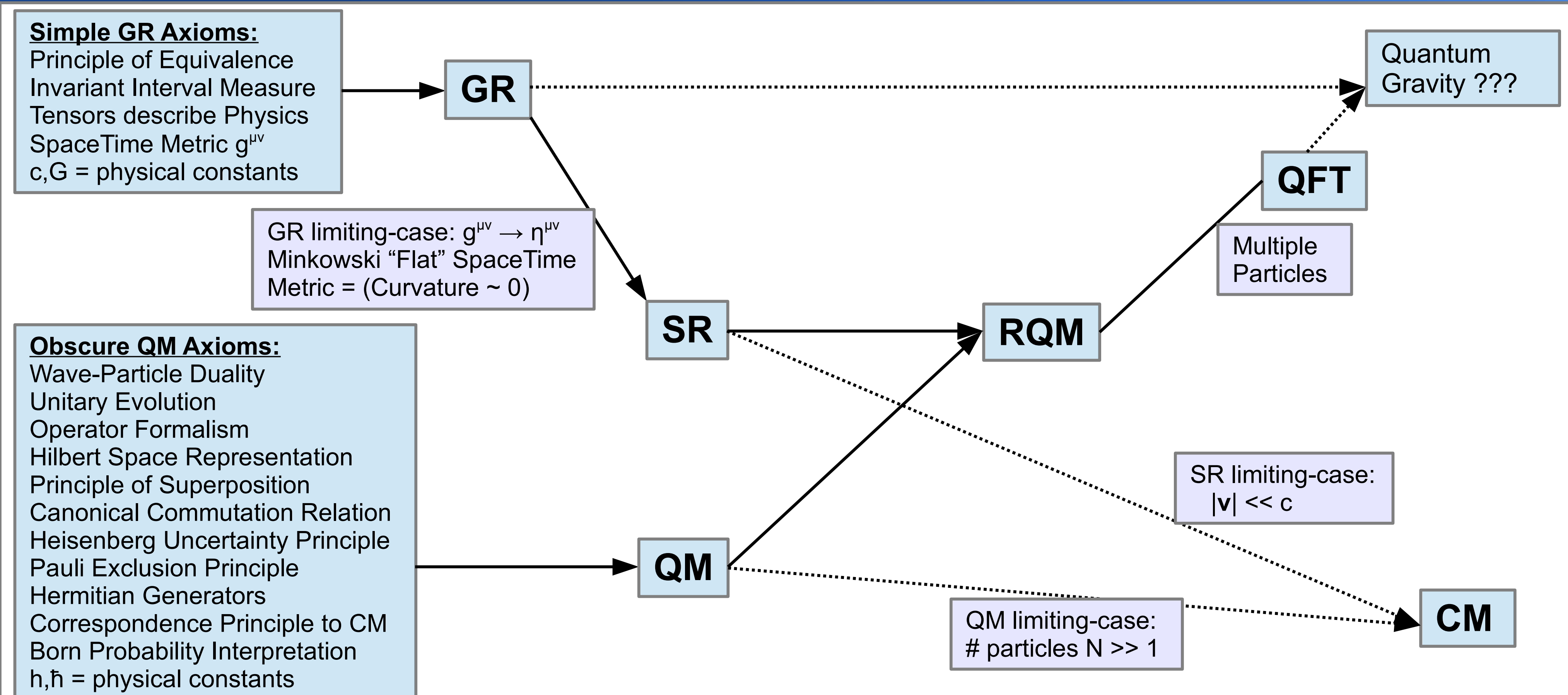

This was the QM paradigm that I was taught while in Grad School; everyone trying for Quantum Gravity

Old Paradigm: QM (years later)

SR and QM still as separate theories

QM limiting-case better defined, still no QG

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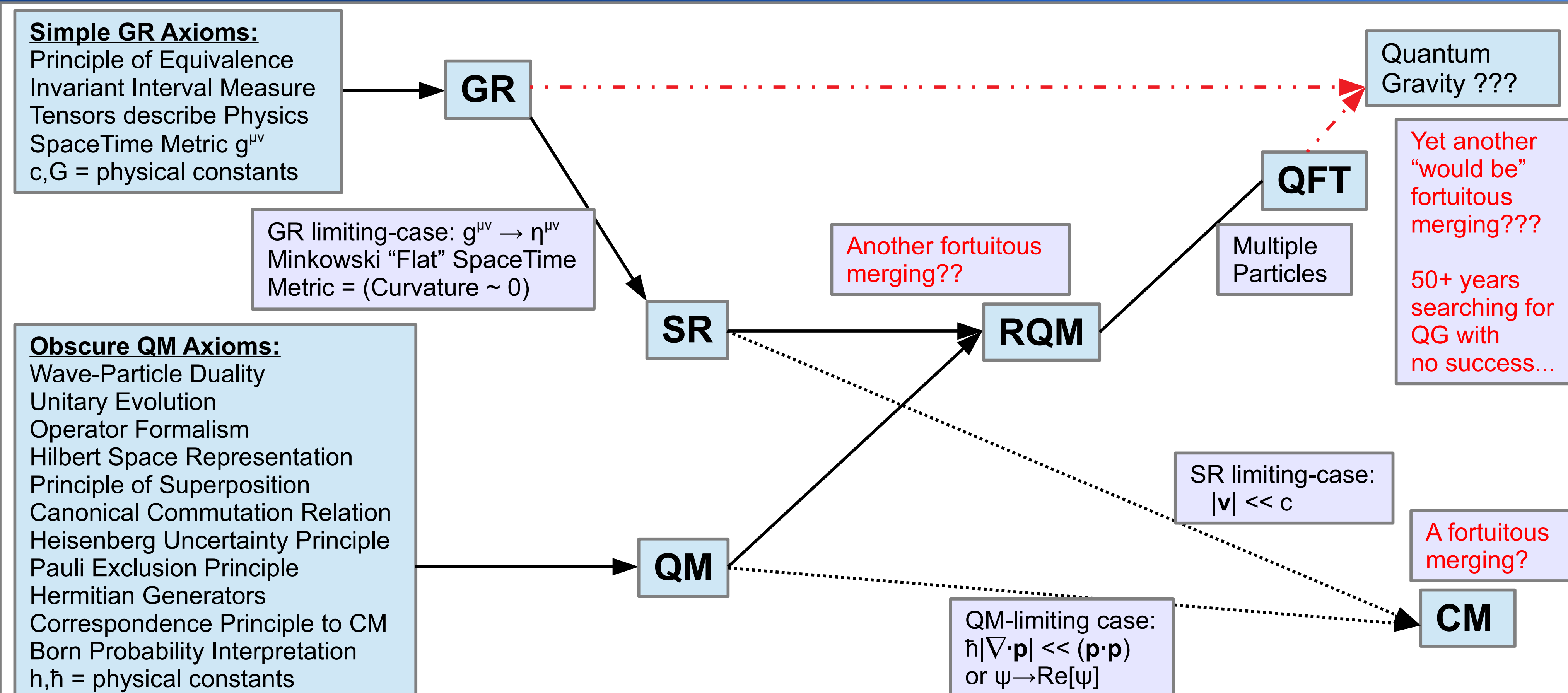
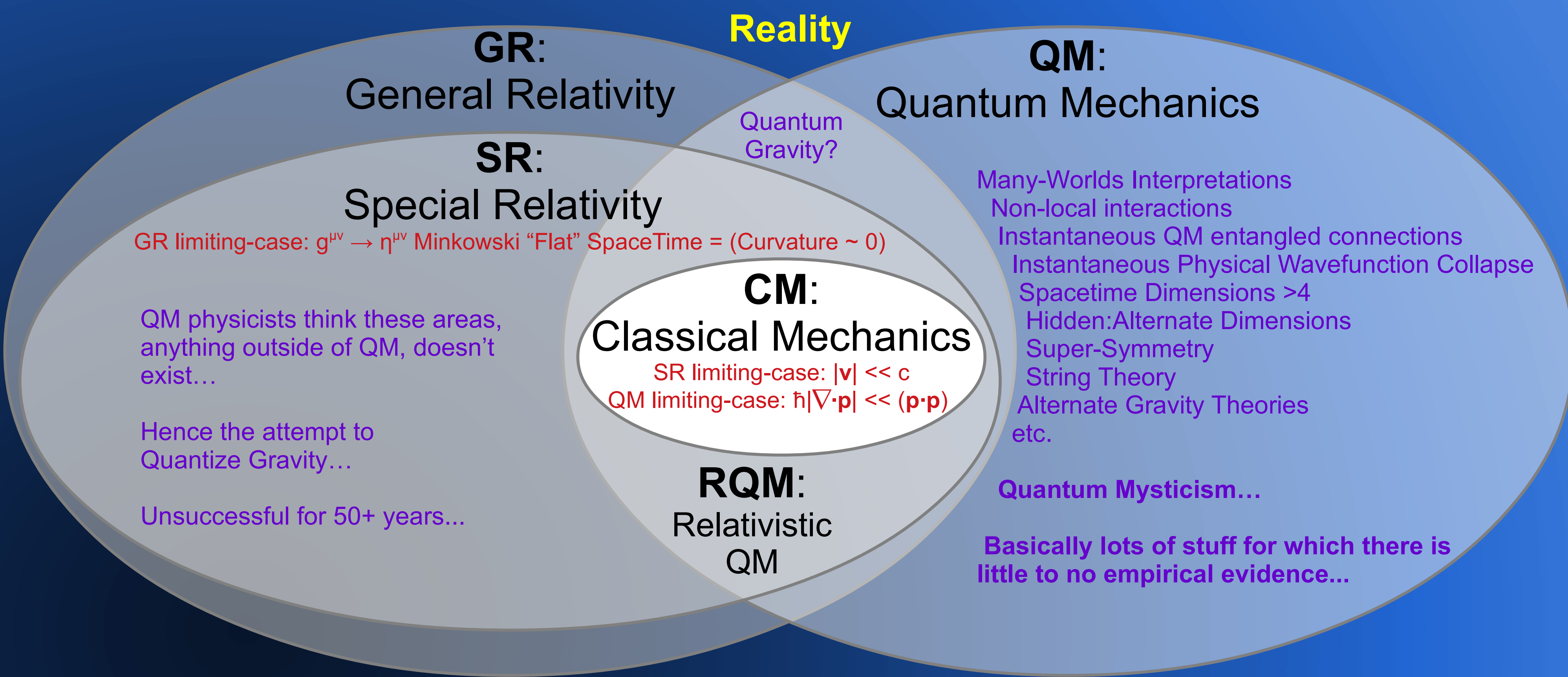


It is known that QM + SR "join nicely" together to form RQM, but problems with RQM + GR...

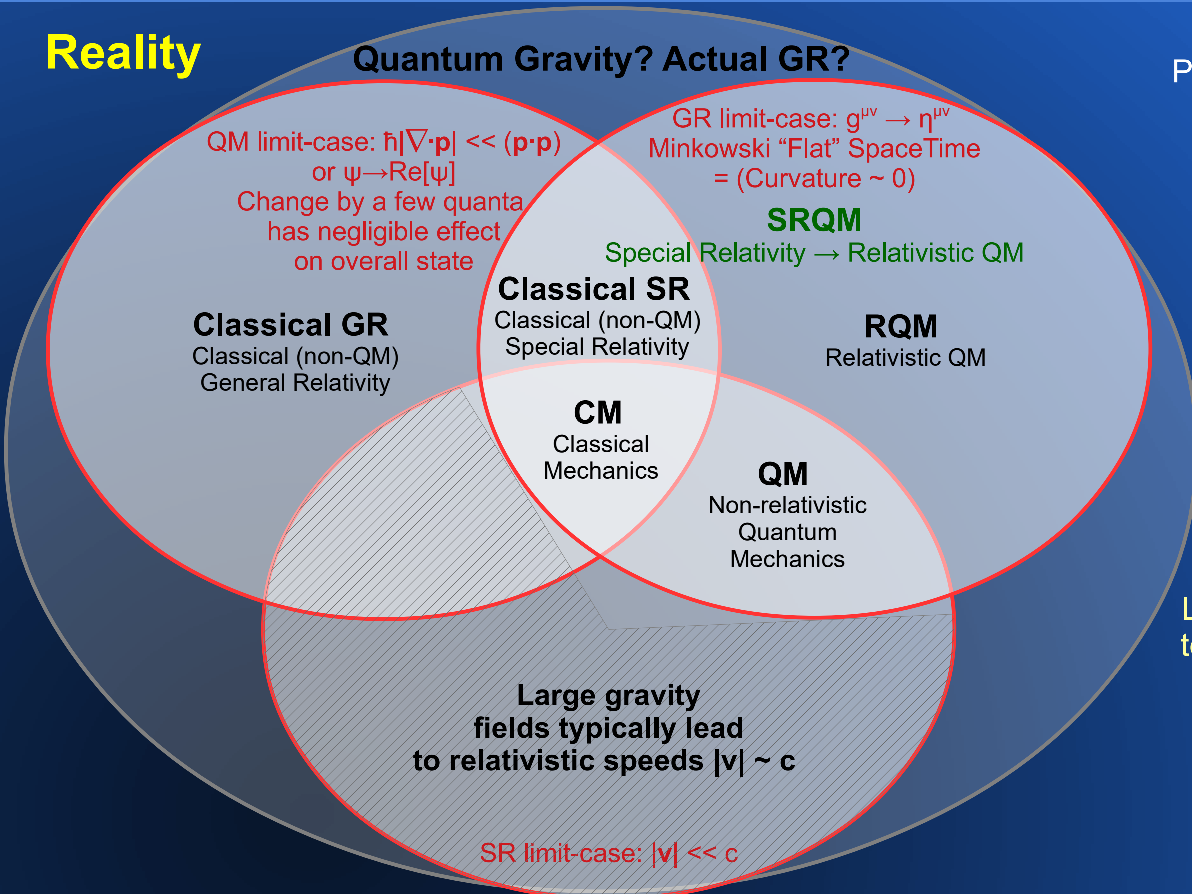
Physical Theories as Venn Diagram
Which regions are real?



Many QM physicists believe that the regions outside of QM don't exist…
SRQM Interpretation would say that the regions outside of GR probably don't exist...

Physical Limit-Cases as Venn Diagram

Which limit-regions use which physics?



Instead of taking the Physical Theories as set, examine Physical Reality and then apply various limiting-conditions.

What do we then call the various regions?

As we move inwards from any region on the diagram, we are adding more stringent conditions which give physical limiting-cases of "larger" theories.

If one is in Classical GR, one can get Classical SR by moving toward the Minkowski SpaceTime limit.

If one is in RQM, one can get Classical SR by moving toward the Hamilton-Jacobi non-QM limit, or to standard QM by moving toward the SR low-velocity limit.

Looking at it this way, I can define SRQM to be equivalent to Minkowski SpaceTime, which contains RQM, and leads to Classical SR, or QM, or CM by taking additional limits.

<u>My assertion:</u>
There is no "Quantized Gravity"
Actual GR contains SRQM and Classical GR.
Perhaps "Gravitizing QM"...

Special Relativity → Quantum Mechanics
Background: Proven Physics

Both General Relativity (GR) and Special Relativity (SR) have passed very stringent tests of multiple varieties.
Likewise, Relativistic Quantum Mechanics (RQM) and Quantum Mechanics (QM) have passed all tests within their realms of validity: {generally micro-scale systems, but a few special macro-scale systems ex. Bose-Einstein condensates, superfluids, etc.}.

To date, however, there is no experimental indication that quantum effects "alter" the fundamentals of either SR or GR. Likewise, there are no known violations, QM or otherwise, of Local Lorentz Invariance (LLI) nor of Local Position/Poincaré Invariance (LPI).
In fact, in all known experiments where both SR/GR and QM are present, QM respects the principles of SR/GR, whereas SR/GR modify the results of QM. All tested quantum-level particles, atoms, isotopes, super-positions, spin-states, etc. obey GR's Universality of FreeFall & Equivalence Principle and SR's { $E = mc^2$ } and speed-of-light (c) communication limit. Quantum-level atomic clocks are used to measure gravitational red:blue-shift effects. i.e. GR gravitational frequency-shift (time-dilation) alters atomic=quantum-level timing.

Some might argue that QM modifies the results of SR, such as via non-commuting measurements. However, that is an alteration of CM expectations, not SR expectations. In fact, there is a basic non-zero commutation relation fully within SR: $[\partial^\mu, X^\nu] = \eta^{\mu\nu}$ which will be derived from purely SR Principles in this treatise.

On the other hand, GR Gravity *does* induce changes in quantum interference patterns and hence modifies QM:
See the COW gravity-induced neutron QM interference experiments and the LIGO gravitational-wave detections via QM interferometry.
Likewise, SR induces fine-structure splitting of spectral lines of atoms, “quantum” spin, spin magnetic moments, spin-statistics (fermions & bosons), antimatter, QED, Lamb shift, etc. - essentially requiring QM to be RQM to be valid.

Some QM scientists say that quantum entanglement is "non-local", but you still can't send any real messages/signals/information/particles faster than SR's speed-of-light (c). The only “non-local” aspect is the alteration of probabilities based on knowledge gained via measurement. A local measurement can alter the “partial information” known about a distant (entangled) system. There is no FTL communication to or alteration of the distant particle. Getting a Stern-Gerlach “up” here doesn’t cause the distant entangled particle to suddenly start moving “down”. One only knows “now” that it “will” go down if the distant experimenter actually performs the measurement.

QM respects the principles of SR/GR, whereas SR/GR modify the results of QM

Special Relativity → Quantum Mechanics
Background: GR Principles

Principles/Axioms and Mathematical Consequences of GR:

Equivalence Principle: Inertial Motion = Geodesic Motion, Universality/Equivalency of Free-Fall, $\text{Mass}_{\text{inertial}} = \text{Mass}_{\text{gravitational}}$

Relativity Principle: SpaceTime M has a Lorentzian Metric $g^{\mu\nu}$, SR:Minkowski Space rules apply locally

General Covariance Principle: Tensors describe Physics, Laws of Physics are independent of Coordinate System

Invariance Principle: Invariant Interval Measure comes from Tensor Invariance Properties

Causality Principle: Minkowski Diagram/Light-Cone give {Time-Like, Light-Light(null), Space-Like} Measures and Causality Conditions

Einstein:Riemann's Ideas about Matter & Curvature:
Riemann(g) has 20 independent components = too many
Ricci(g) has 10 independent components = enough to describe gravitational field

{c,G} are Fundamental Physical Constants

To date, there are no violations of any of these GR Principles.

GR limiting-case: $g^{\mu\nu} \to \eta^{\mu\nu}$
Minkowski "Flat" SpaceTime
Metric = (Curvature ~ 0)

All known experiments to date comply with all of these Principles, including QM

Old Paradigm: QM (for comparison)

SR and QM still as separate theories
QM limiting-case better defined, still no QG

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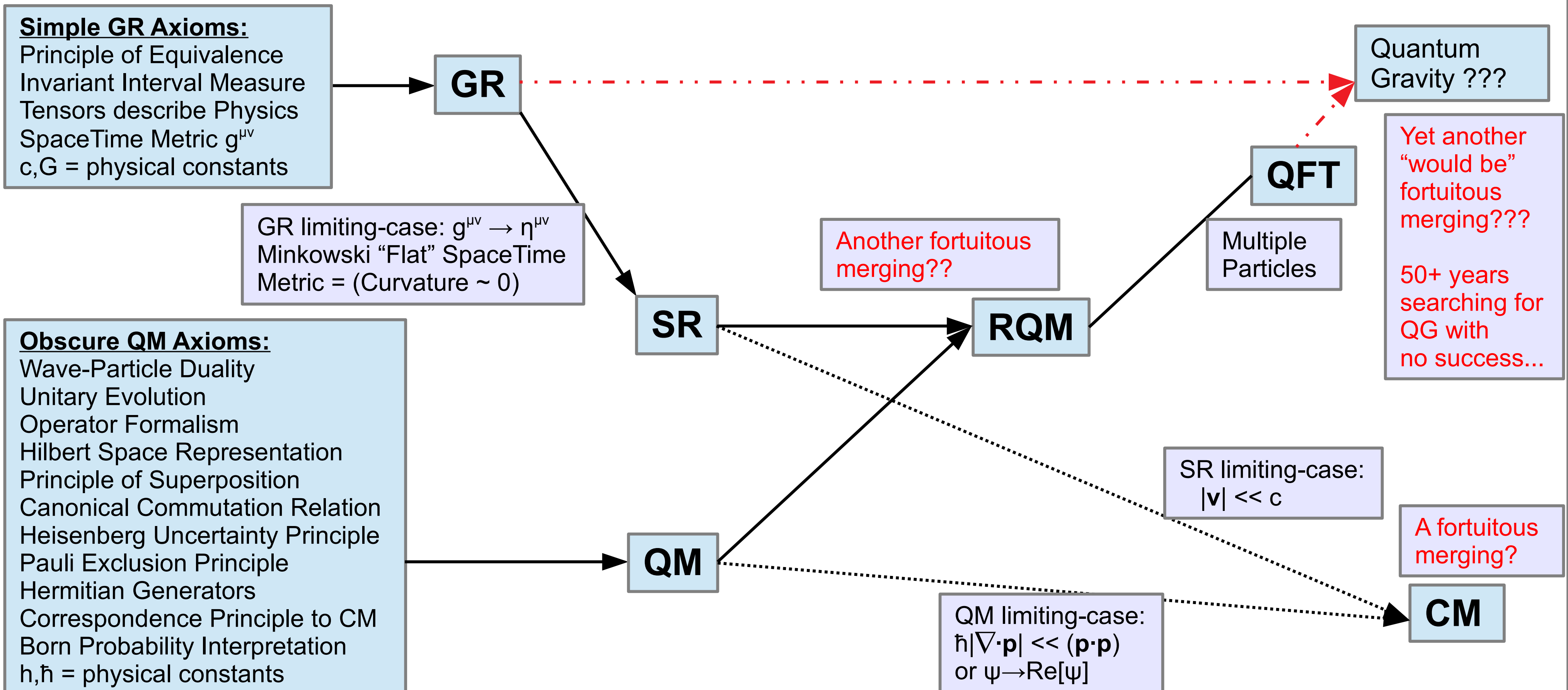

It is known that QM + SR "join nicely" together to form RQM, but problems with RQM + GR...

New Paradigm: SRQM or [SR→QM]

QM derived from SR + a few empirical facts

Simple GR Axioms:
Principle of Equivalence
Invariant Interval Measure
Tensors describe Physics
SpaceTime Metric $g^{\mu\nu}$
c,G = physical constants

GR

GR limiting-case: $g^{\mu\nu} \rightarrow \eta^{\mu\nu}$
Minkowski "Flat" SpaceTime
Metric = (Curvature ~ 0)

(properties)
SR 4-vector:
$\mathbf{R}=(ct,\mathbf{r})$
$\mathbf{U}=\gamma(c,\mathbf{u})$
$\mathbf{P}=(E/c,\mathbf{p})$
$\mathbf{K}=(\omega/c,\mathbf{k})$
$\partial=(\partial_t/c,-\nabla)$

(relations)
SR 4-vector:
R=<Event>
$\mathbf{U}=d\mathbf{R}/d\tau$
$\mathbf{P}=(m_o)\mathbf{U}$
$\mathbf{K}=(1/\hbar)\mathbf{P}$
$\partial=(-i)\mathbf{K}$

~~Quantum Gravity ???~~

QFT

Multiple Particles

SR

SRQM

RQM

SR limiting-case:
$|\mathbf{v}| << c$

QM

Derived RQM **Principles:**
Wave-Particle Duality
Unitary Evolution
Operator Formalism
Hilbert Space Representation
Principle of Superposition
Canonical Commutation Relation
Heisenberg Uncertainty Principle
Pauli Exclusion Principle
Hermitian Generators
h,ħ = physical constants

Derived QM **Principles:**
Correspondence Principle to CM
Born Probability Interpretation

QM limiting-case:
$\{\hbar|\nabla\cdot\mathbf{p}| << (\mathbf{p}\cdot\mathbf{p})\}$ or $\{\psi\rightarrow Re[\psi]\}$
Change by a few quanta has negligible effect on overall state

CM

This new paradigm explains why RQM "miraculously fits" SR, but not necessarily GR

New Paradigm: SRQM w/ EM
QM, EM, CM derived from SR + a few empirical facts

Simple GR Axioms:
Principle of Equivalence:
Invariant Interval Measure
Tensors describe Physics
SpaceTime Metric $g^{\mu\nu}$
c,G = physical constants

GR

GR limiting-case: $g^{\mu\nu} \to \eta^{\mu\nu}$
Minkowski “Flat” SpaceTime
Metric = (Curvature ~ 0)

(properties)
SR 4-vector & EM tensor:
$\mathbf{R}=(ct,\mathbf{r})$ $\mathbf{A}=(\varphi/c,\mathbf{a})$
$\mathbf{U}=\gamma(c,\mathbf{u})$ $\mathbf{J}=(c\rho,\mathbf{j})$
$\mathbf{P}=(E/c,\mathbf{p})$
$\mathbf{K}=(\omega/c,\mathbf{k})$ $F^{\alpha\beta}=[\ 0\ ,\ -e^j/c\ ]$
$\boldsymbol{\partial}=(\partial_t/c,-\boldsymbol{\nabla})$ $[+e^i/c,-\varepsilon^{ij}{}_k b^k]$
$\mathbf{F}=\gamma(E'/c,\mathbf{f})$
$\mathbf{N}=n(c,\mathbf{u})$

(relations)
SR 4-vector & EM tensor:
$\mathbf{R}=$<Event> $\mathbf{A}=(\varphi_o/c^2)\mathbf{U}$
$\mathbf{U}=d\mathbf{R}/d\tau$ $\mathbf{J}=(\rho_o)\mathbf{U}=(q)\mathbf{N}$
$\mathbf{P}=(m_o)\mathbf{U}$ $\mathbf{K}=-\boldsymbol{\partial}[\Phi_{phase}]$
$\mathbf{K}=(1/\hbar)\mathbf{P}$ $F^{\alpha\beta}=\partial^\alpha A^\beta-\partial^\beta A^\alpha$
$\boldsymbol{\partial}=(-i)\mathbf{K}$ $\mathbf{U}\cdot F^{\alpha\beta}=(1/q)\mathbf{F}$
$\mathbf{F}=d\mathbf{P}/d\tau$ $\boldsymbol{\partial}\cdot F^{\alpha\beta}=(\mu_o)\mathbf{J}$
$\mathbf{N}=(n_o)\mathbf{U}$ $\boldsymbol{\partial}\cdot\mathbf{J}=0$

~~Quantum Gravity ???~~

QFT

Multiple Particles

SR

SRQM

RQM
QED

SR limiting-case:
$|\mathbf{v}| \ll c$

Derived RQM **Principles:**
Wave-Particle Duality
Unitary Evolution
Operator Formalism
Hilbert Space Representation
Principle of Superposition
Canonical Commutation Relation
Heisenberg Uncertainty Principle
Pauli Exclusion Principle
Hermitian Generators
h,ħ = physical constants

QM
EM w/ spin

Derived QM **Principles:**
Correspondence Principle to CM
Born Probability Interpretation

QM limiting-case:
$\{\hbar|\boldsymbol{\nabla}\cdot\mathbf{p}| \ll (\mathbf{p}\cdot\mathbf{p})\}$ or $\{\psi \to \mathrm{Re}[\psi]\}$
Change by a few quanta has negligible effect on overall state

CM $q=0$
EM $q\neq0$

This new paradigm explains why RQM “miraculously fits” SR, but not necessarily GR

Classical SR w/ EM Paradigm (for comparison)

CM & EM derived from SR + a few empirical facts

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Simple GR Axioms:
Principle of Equivalence
Invariant Interval Measure
Tensors describe Physics
SpaceTime Metric $g^{\mu\nu}$
c,G = physical constants

GR

GR limiting-case: $g^{\mu\nu} \rightarrow \eta^{\mu\nu}$
Minkowski "Flat" SpaceTime
Metric = (Curvature ~ 0)

SR

(properties)
SR 4-vector & EM tensor:
$\mathbf{R}=(ct,\mathbf{r})$ $\quad \mathbf{A}=(\varphi/c,\mathbf{a})$
$\mathbf{U}=\gamma(c,\mathbf{u})$ $\quad \mathbf{J}=(c\rho,\mathbf{j})$
$\mathbf{P}=(E/c,\mathbf{p})$
$\mathbf{K}=(\omega/c,\mathbf{k})$ $\quad F^{\alpha\beta}=[\ 0\ ,\ -e^j/c\ ]$
$\boldsymbol{\partial}=(\partial_t/c,-\nabla)$ $\quad [+e^i/c,-\varepsilon^{ij}{}_k b^k]$
$\mathbf{F}=\gamma(E'/c,\mathbf{f})$
$\mathbf{N}=n(c,\mathbf{u})$

(relations)
SR 4-vector & EM tensor:
$\mathbf{R}=$<Event> $\quad \mathbf{A}=(\varphi_o/c^2)\mathbf{U}$
$\mathbf{U}=d\mathbf{R}/d\tau$ $\quad \mathbf{J}=(\rho_o)\mathbf{U}=(q)\mathbf{N}$
$\mathbf{P}=(m_o)\mathbf{U}$ $\quad \mathbf{K}=-\boldsymbol{\partial}[\Phi_{phase}]$
[redacted] $\quad F^{\alpha\beta}=\partial^\alpha A^\beta-\partial^\beta A^\alpha$
$\boldsymbol{\partial}=(-i)\mathbf{K}$ $\quad \mathbf{U}\cdot F^{\alpha\beta}=(1/q)\mathbf{F}$
$\mathbf{F}=d\mathbf{P}/d\tau$ $\quad \boldsymbol{\partial}\cdot F^{\alpha\beta}=(\mu_o)\mathbf{J}$
$\mathbf{N}=(n_o)\mathbf{U}$ $\quad \boldsymbol{\partial}\cdot\mathbf{J}=0$

SR EM

SR limiting-case:
$|\mathbf{v}| << c$

CM q=0
EM q≠0

The entire classical SR→EM,CM structure is based on the limiting-case of quantum effects being negligible.

Notice that only the SR 4-Vector relation: $\mathbf{K}=(1/\hbar)\mathbf{P}$
is missing from the Classical Interpretation…

All of the SR 4-Vectors, including ($\mathbf{K}$ & $\boldsymbol{\partial}$),
are still present in the Classical setting.

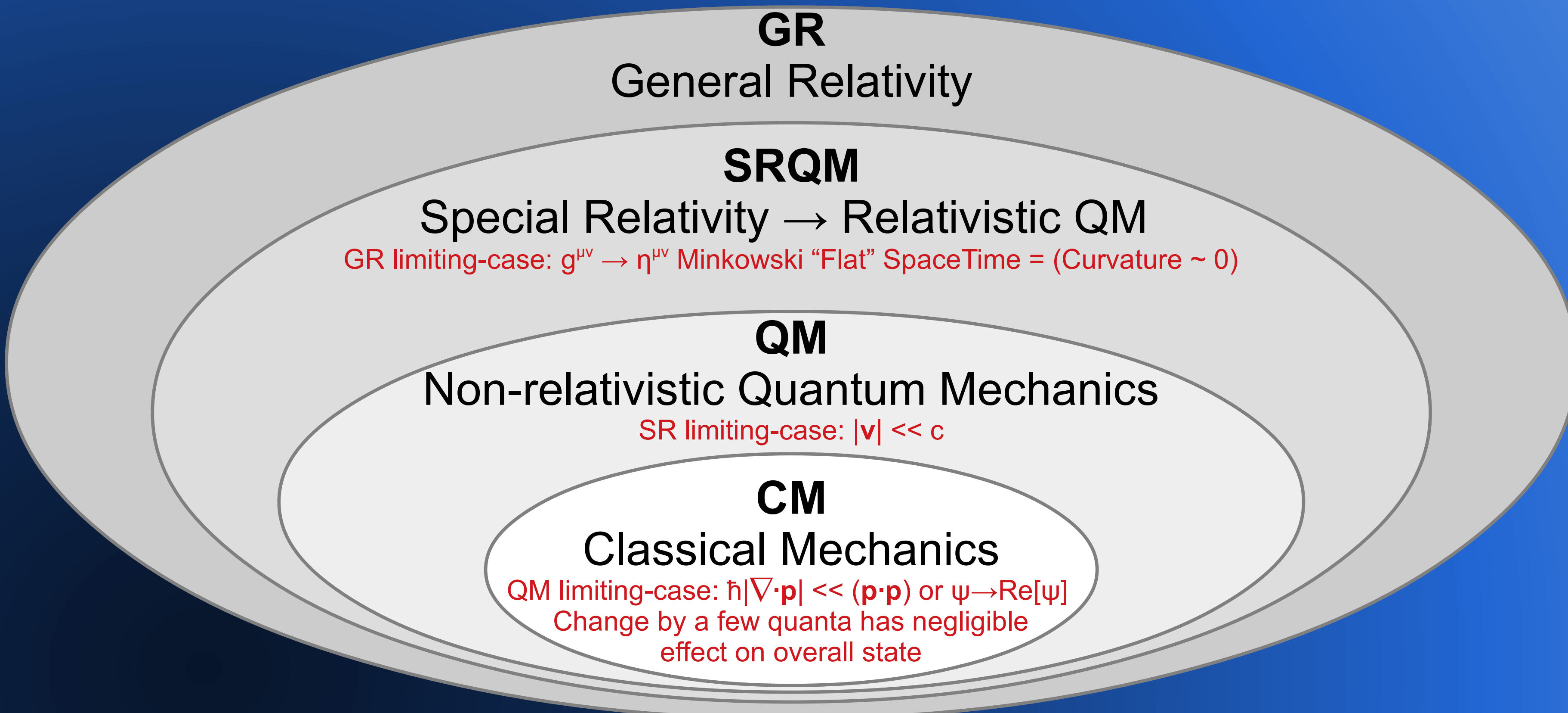
$\mathbf{K}$ is used in the Relativistic Doppler Effect and EM waves.
$\boldsymbol{\partial}$ is used in the SR Conservation/Continuity Equations,
Maxwell Equations, Hamilton-Jacobi, Lorenz Gauge, etc.
$\boldsymbol{\partial}=(-i)\mathbf{K}$ may be somewhat controversial, but it is the equation for complex plane-waves, which are in classical EM (in real form).

Background Inherent Assumption

QM limiting-case:
$\{\hbar|\nabla\cdot\mathbf{p}| << (\mathbf{p}\cdot\mathbf{p})\}$ or $\{\psi\rightarrow\mathrm{Re}[\psi]\}$
Hamilton-Jacobi non-quantum
Change by a few quanta has negligible effect on overall state

This {Classical=non-QM} SR→EM,CM paradigm has been working successfully for decades

New Paradigm:
SRQM View as Venn Diagram



The SRQM view: Each level (range of validity) is a subset of the larger level.

New Paradigm:
SRQM View w/ EM as Venn Diagram

A Tensor Study of Physical 4-Vectors
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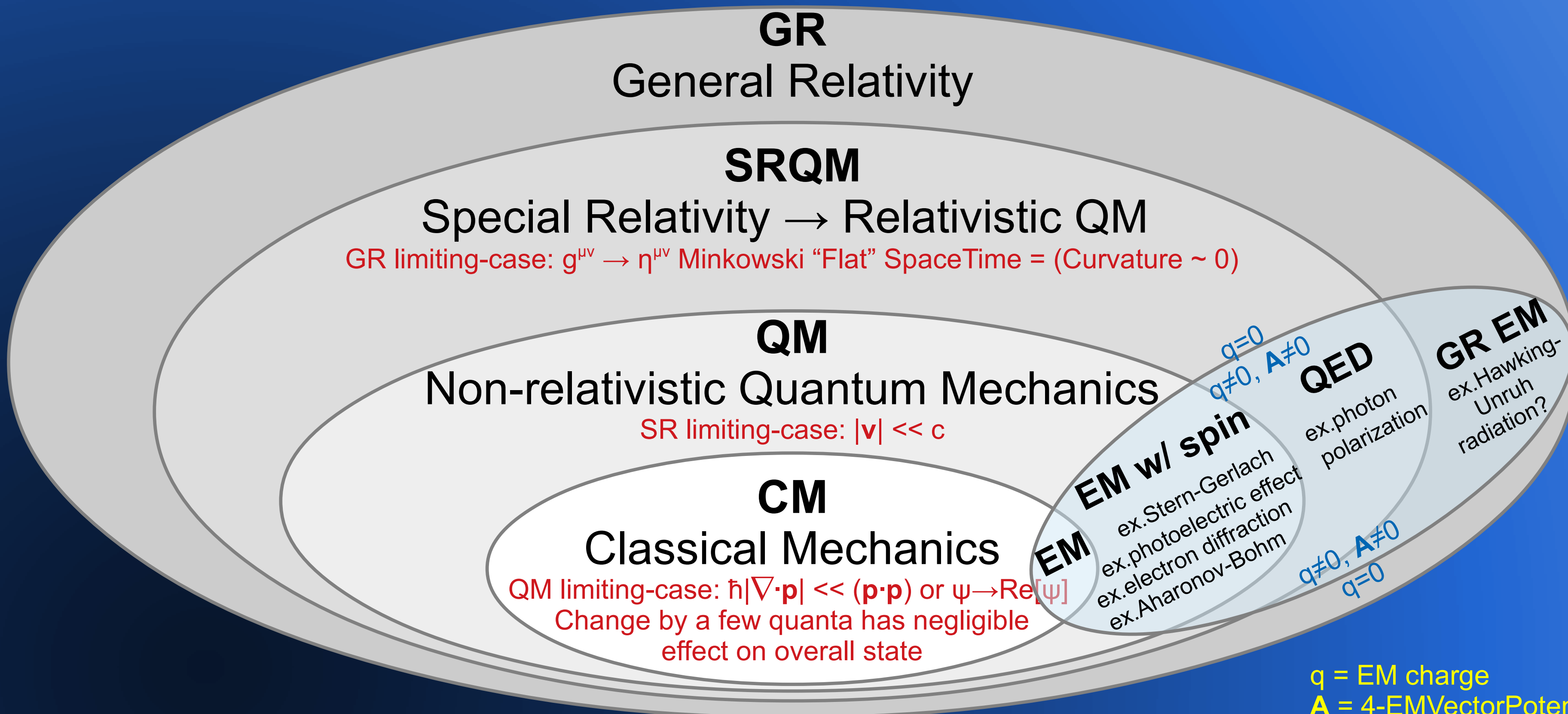
John B. Wilson



q = EM charge
A = 4-EMVectorPotential

The SRQM view: Each level (range of validity) is a subset of the larger level

SR language beautifully expressed with Physical 4-Vectors

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Newton's laws of classical physics are greatly simplified by the use of physical 3-vector notation, which converts 3 separate space components, which may be different in various coordinate systems, into a single invariant object, a vector:
The basis values of these components can differ, yet still refer to the same overall 3-vector object.

3-vector = 3D (1,0)-tensor
$\mathbf{a} = a^i = (a^1,a^2,a^3)$

→ (a^x,a^y,a^z) Cartesian/Rectangular 3D basis
→ (a^r,a^θ,a^z) Polar/Cylindrical 3D basis
→ (a^r,a^θ,a^Φ) Spherical 3D basis

$\mathbf{a}\cdot\mathbf{a} = a^j\delta_{jk}a^k = |\mathbf{a}|^2$

$\mathbf{A}\cdot\mathbf{A} = A^\mu\eta_{\mu\nu}A^\nu = (a^0)^2 - \mathbf{a}\cdot\mathbf{a} = (a^0_o)^2$

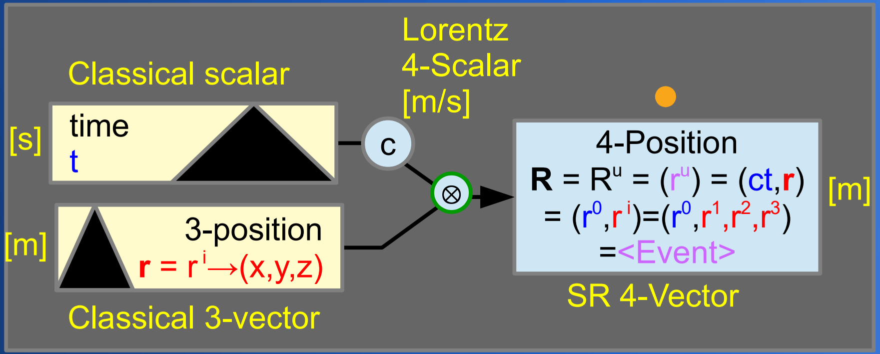
The scalar products of either type are basis-independent

4-Vector = 4D (1,0)-Tensor
$\mathbf{A} = A^\mu = (a^\mu) = (a^0,a^i) = (a^0,\mathbf{a}) = (a^0,a^1,a^2,a^3)$

→ (a^t,a^x,a^y,a^z) Cartesian/Rectangular 4D basis
→ (a^t,a^r,a^θ,a^z) Polar/Cylindrical 4D basis
→ $(a^t,a^r,a^\theta,a^\varphi)$ Spherical 4D basis

Classical 3D objects styled this way to emphasize that they are actually just the separated components of SR 4-Vectors. The triangle/wedge (3 sides) represents splitting the components into a scalar and 3-vector.

SR is able to expand the concept of mathematical vectors into the Physical 4-Vector, which combines both (time) and (space) components into a single TimeSpace object:
These 4-Vectors are elements of Minkowski 4D SR SpaceTime.
Typically there is a speed-of-light factor (c)
in the temporal component to make the dimensional units match.
eg. $\mathbf{R} = (ct,\mathbf{r})$: overall dimensional units of [length] = SI Unit [m]
This also allows the 4-Vector name to match up with the 3-vector name.



<u>In this presentation:</u>
I use the (+,-,-,-) metric signature, giving $\mathbf{A}\cdot\mathbf{A} = A^\mu\eta_{\mu\nu}A^\nu = [(a^0)^2 - \mathbf{a}\cdot\mathbf{a}] = (a^0_o)^2$
4-Vectors will use Upper-Case Letters, ex. **A**; 3-vectors will use lower-case letters, ex. **a**
Vectors of both types will be in **bold** font; components and scalars in normal font and usually lower-case. 4-Vector name will match 3-vector name.
Tensor form will usually be normal font with a tensor index, ex. A^μ or a^i, with Greek TimeSpace index (0,1..3); Latin SpaceOnly index (1..3)

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^\mu{}_\nu$ or $T_\mu{}^\nu$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0,\mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0,-\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Classical (scalar 3-vector)
Galilean Invariant
Not Lorentz Invariant

$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu{}_\mu = T$
$\mathbf{V}\cdot\mathbf{V} = V^\mu\eta_{\mu\nu}V^\nu = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0_o)^2$
= Lorentz Scalar

SR 4-Vectors & Lorentz Scalars
Frame-Invariant Equations
SRQM Diagramming Method

SciRealm.org
John B. Wilson

4-Vectors are type (1,0)-Tensors, Lorentz {4-}Scalars are type (0,0)-Tensors, 4-CoVectors are type (0,1)-Tensors, (m,n)-Tensors have (m) upper-indices and (n) lower-indices. V^μ, S, C_μ, $T^{\alpha\beta\gamma..\{m\ indicies\}}{}_{\mu\nu..\{n\ indicies\}}$

Any equation which employs only Tensors, such as those with only 4-Vectors and Lorentz 4-Scalars, (ex. $\mathbf{P} = m_o\mathbf{U}$) is automatically Frame-Invariant, or coordinate-frame-independent. One's frame-of-reference plays no role in the form of the overall equations. This is also known as being "Manifestly-Invariant". This is exactly what Einstein meant by his postulate:
"The laws of physics should have the same form for all inertial observers". Use of the RestFrame-naught ($_o$) helps show this.

4-Vector = 4D (1,0)-Tensor
$\mathbf{A} = A^\mu = (a^\mu) = (a^0,a^i) = (a^0,\mathbf{a}) = (a^0,a^1,a^2,a^3) \rightarrow (a^t,a^x,a^y,a^z)$

$\mathbf{A}\cdot\mathbf{A} = A^\mu\eta_{\mu\nu}A^\nu = (a^0)^2-\mathbf{a}\cdot\mathbf{a} = (a^0{}_o)^2$

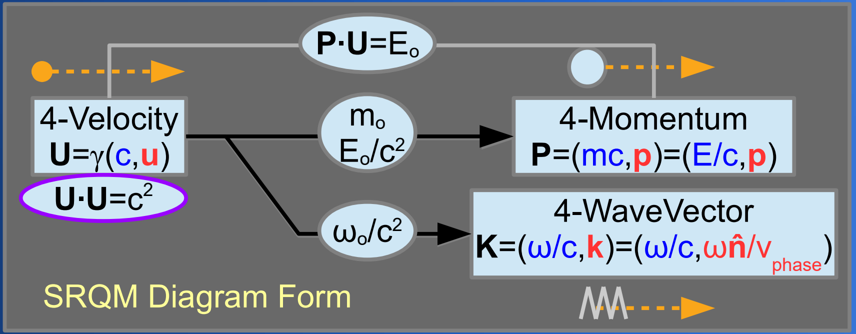
The components (a^0,a^1,a^2,a^3) of the 4-Vector **A** can vary depending on the observer and their choice of coordinate system, but the 4-Vector **A** itself is invariant. Equations using only 4-Tensors, 4-Vectors, and Lorentz 4-Scalars are true for all inertial observers. The SRQM Diagramming Method makes this easy to see in a visual format, and will be used throughout this treatise.
The following examples are frame-invariant equations:

$\mathbf{U}\cdot\mathbf{U} = (c)^2$
$\mathbf{U} = \gamma(c,\mathbf{u})$
$\mathbf{P} = (mc,\mathbf{p}) = (E/c,\mathbf{p}) = m_o\mathbf{U} = (E_o/c^2)\mathbf{U}$
$\mathbf{K} = (\omega/c,\mathbf{k}) = (\omega_o/c^2)\mathbf{U}$
$\mathbf{P}\cdot\mathbf{U} = E_o$

Equation Form

The SRQM Diagram Form has all of the info of the Equation Form, but shows overall relationships and symmetries among the 4-Vectors much more clearly.

Blue: Temporal components
Red: Spatial components
Purple: Mixed TimeSpace components



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SR 4-Vectors are primitive elements of Minkowski SpaceTime (4D)

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson

We want to be clear, however, that SR 4-Vectors are **NOT** generalizations of Classical or Quantum 3-vectors.

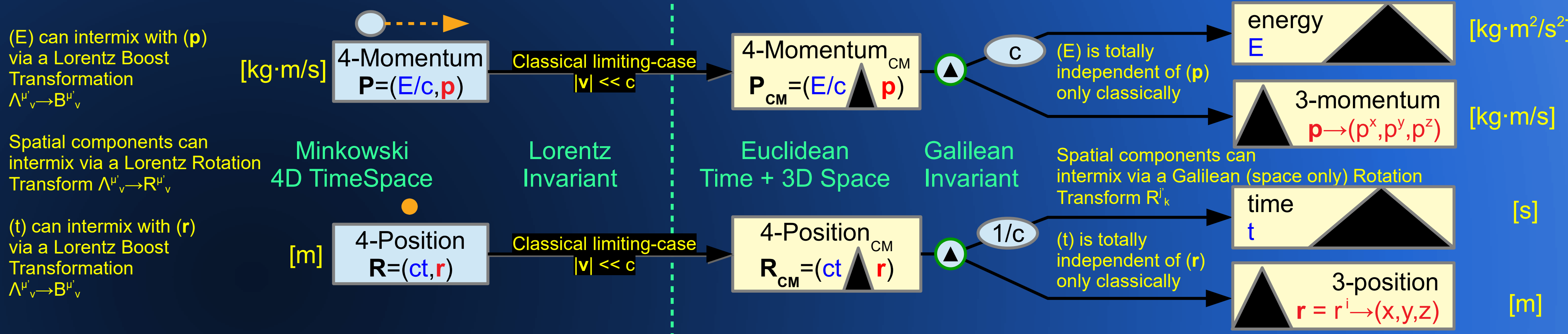
SR 4-Vectors are the primitive elements of Minkowski SpaceTime (4D) which incorporate both:
a {temporal scalar element} and a {**spatial 3-vector element**} as components.
4-Vector **A** = $(a^0,a^1,a^2,a^3) = (a^0,\mathbf{a}) \rightarrow (a^t,a^x,a^y,a^z)$ with scalar (a^t) & 3-vector $\mathbf{a} \rightarrow (a^x,a^y,a^z)$

It is the Classical or Quantum 3-vector (**a**) which is a limiting-case approximation of the spatial part of SR 4-Vector (**A**) for { |**v**| << c }.

i.e. The Energy (E) and 3-momentum (**p**) as "separate" entities occurs only in the low-velocity limit { |**v**| << c } of the Lorentz Boost Transform.
They are actually part of a single 4D entity: the 4-Momentum **P** = (E/c,**p**); with the components: temporal (E), spatial (**p**), dependent on a frame-of-reference, while the overall 4-Vector **P** is invariant. Likewise with (t) and (**r**) in the 4-Position **R**.

SR is Minkowskian; obeys Lorentz Invariance.

CM is Euclidean; obeys Galilean Invariance.



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Relations among SR 4-Vectors are Manifestly Invariant

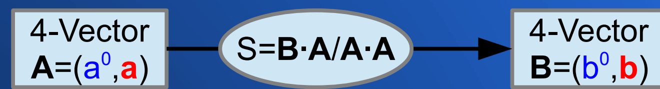
A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson

Relations among 4-Vectors and Lorentz 4-Scalars are Manifestly Invariant, meaning that they are true in all inertial reference frames.

Consider a particle at a SpaceTime <Event> that has properties described by 4-Vectors **A** and **B**:

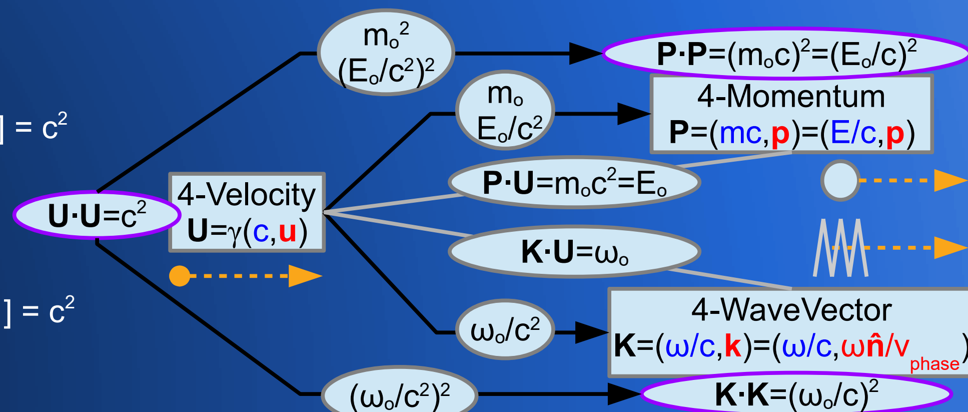
One possible relationship is that the two 4-Vectors are related by a Lorentz 4-Scalar (S): ex. **B** = (S) **A**.
How can one determine this? Answer: Make an experiment that empirically measures the tensor invariant [**B·A** / **A·A**].
If **B** = (S) **A**
B·A = (S) **A·A**
(S) = [**B·A** / **A·A**]   Note that this basically a vector projection.



Run the experiment many times. If you always get the same result for (S), then it is likely that the relationship is true, and invariant.

Example: Measure (S_P) = [**P·U** / **U·U**] for a given particle type.
Repeated measurement always give (S_P) = m_o
This makes sense because we know [**P·U**] = γ(E - **p·u**) = E_o and [**U·U**] = c^2
Thus, 4-Momentum **P** = (E_o/c^2)**U** = (m_o)**U** = (m_o)*4-Velocity **U**

Example: Measure (S_K) = [**K·U** / **U·U**] for a given particle type.
Repeated measurement always give (S_K) = (ω_o/c^2)
This makes sense because we know [**K·U**] = γ(ω - **k·u**) = ω_o and [**U·U**] = c^2
Thus, 4-WaveVector **K** = (ω_o/c^2)**U** = (ω_o/c^2)*4-Velocity **U**



Since **P** and **K** are both related to **U**, this would also mean that the 4-Momentum **P** is related to the 4-WaveVector **K** in a particular manner for each given particle type...

Some SR Mathematical Tools
Definitions and Approximations

A Tensor Study of Physical 4-Vectors

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John B. Wilson

$\boldsymbol{\beta} = \mathbf{v}/c$: dimensionless Velocity Beta Factor { $\beta=(0..1)$; rest at ($\beta=0$); speed-of-light (c) at ($\beta=1$) }
$\gamma = 1/\sqrt{[1-\beta^2]} = 1/\sqrt{[1-\boldsymbol{\beta}\cdot\boldsymbol{\beta}]}$: dimensionless Lorentz Relativistic Gamma Factor { $\gamma=(1..\infty)$; rest at ($\gamma=1$); speed-of-light (c) at ($\gamma=\infty$) }

$(1+x)^n \sim (1 + nx + O[x^2])$ for { $|x| << 1$ } Approximation used for SR→Classical limiting-cases

Lorentz Transformation $\Lambda^{\mu'}{}_{\nu} = \partial X^{\mu'}/\partial X^{\nu} = \partial_{\nu}[X^{\mu'}]$: a relativistic frame-shift, such as a rotation or velocity boost
It transforms a 4-Vector in the following way: $X^{\mu'} = \Lambda^{\mu'}{}_{\nu} X^{\nu}$: with Einstein summation over the paired indicies
A typical Lorentz Boost Transformation $\Lambda^{\mu'}{}_{\nu} \rightarrow B^{\mu'}{}_{\nu}$ for a linear-velocity frame-shift (x,t)-Boost in the $\hat{x}$-direction:

Lorentz x-Boost Transform $\Lambda^{\mu'}{}_{\nu} \rightarrow B^{\mu'}{}_{\nu} =$

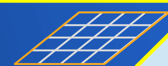
$$\begin{array}{c|cccc} & t & x & y & z \\ \hline t & \gamma & -\beta\gamma & 0 & 0 \\ x & -\beta\gamma & \gamma & 0 & 0 \\ y & 0 & 0 & 1 & 0 \\ z & 0 & 0 & 0 & 1 \end{array}$$

SR:Minkowski Metric
$\partial[\mathbf{R}] = \partial^{\mu}R^{\nu} = \eta^{\mu\nu} = V^{\mu\nu} + H^{\mu\nu}$
$\rightarrow \mathrm{Diag}[1,-I_{(3)}] = \mathrm{Diag}[1,-\delta^{jk}]$

$$\begin{array}{c|cccc} & t & x & y & z \\ \hline t & 1 & 0 & 0 & 0 \\ x & 0 & -1 & 0 & 0 \\ y & 0 & 0 & -1 & 0 \\ z & 0 & 0 & 0 & -1 \end{array} \qquad \begin{bmatrix} 1 & 0^{j} \\ 0^{i} & -\delta^{ij} \end{bmatrix}$$

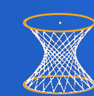
"Particle Physics" Convention
Symmetric

SR:Minkowski Metric
$\partial[\mathbf{R}] = \partial^{\mu}R^{\nu} = \eta^{\mu\nu} = V^{\mu\nu} + H^{\mu\nu} \rightarrow$
$\mathrm{Diag}[1,-1,-1,-1] = \mathrm{Diag}[1,-I_{(3)}] = \mathrm{Diag}[1,-\delta^{jk}]$
{in Cartesian form} "Particle Physics" Convention
$\{\eta_{\mu\mu}\} = 1/\{\eta^{\mu\mu}\} : \eta_{\mu}{}^{\nu} = \delta_{\mu}{}^{\nu}$ $\mathrm{Tr}[\eta^{\mu\nu}]=4$



SR:Lorentz Transform
$\partial_{\nu}[R^{\mu'}] = \partial R^{\mu'}/\partial R^{\nu} = \Lambda^{\mu'}{}_{\nu}$
$\Lambda^{\mu}{}_{\nu} = (\Lambda^{-1})_{\nu}{}^{\mu} : \Lambda^{\mu}{}_{\alpha}\Lambda^{\alpha}{}_{\nu} = \eta^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu}$
$\eta_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta} = \eta_{\alpha\beta}$
$\mathrm{Det}[\Lambda^{\mu}{}_{\nu}]=\pm1$ $\Lambda_{\mu\nu}\Lambda^{\mu\nu}=4$

Original $A^{\nu} = (a^t, a^x, a^y, a^z)$
Boosted $A^{\mu'} = (a^t, a^x, a^y, a^z)' = \Lambda^{\mu'}{}_{\nu}A^{\nu} \rightarrow B^{\mu'}{}_{\nu}A^{\nu} = (\gamma a^t - \gamma\beta a^x, -\gamma\beta a^t + \gamma a^x, a^y, a^z)$ {for $\hat{x}$-boost Lorentz Transform}



SpaceTime
$\partial\cdot\mathbf{R} = \partial_{\mu}R^{\mu} = 4$
Dimension

$\mathbf{A'}\cdot\mathbf{B'} = (\Lambda^{\mu'}{}_{\nu}A^{\nu})\cdot(\Lambda^{\rho'}{}_{\sigma}B^{\sigma}) = \mathbf{A}\cdot\mathbf{B} = A^{\mu}\eta_{\mu\nu}B^{\nu} = A^{\mu}B_{\mu} = A_{\nu}B^{\nu} = \Sigma_{\nu=0..3}[a_{\nu}b^{\nu}] = \Sigma_{u=0..3}[a^{u}b_{u}] = (a^0b_0 + a^1b_1 + a^2b_2 + a^3b_3)$
$= (a^0b^0 - \mathbf{a}\cdot\mathbf{b}) = (a^0b^0 - a^1b^1 - a^2b^2 - a^3b^3)$
using the Einstein summation convention where upper:lower paired-indices are summed over

$\partial[\mathbf{X}] = \partial^{\mu}[X^{\nu}] = (\partial_t/c,-\nabla)(ct,\mathbf{x}) = \mathrm{Diag}[\partial_t/c[ct],-\nabla[\mathbf{x}]] = \mathrm{Diag}[1,-\mathbf{I}_{(3)}] = \mathrm{Diag}[1,-1,-1,-1] = \eta^{\mu\nu}$ Minkowski "Flat" SpaceTime Metric

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A Tensor Study of Physical 4-Vectors

SRQM Diagram: The Basis of Classical SR Physics Special Relativity via 4-Vectors

SciRealm.org
John B. Wilson

Focus on three of the main SR Physical 4-Vectors.

4-Position $\mathbf{R}=R^{\mu}=(ct,\mathbf{r})=$<Event> — <Event> Location

4-Velocity $\mathbf{U}=U^{\mu}=\gamma(c,\mathbf{u})$ — <Event> Motion

4-Gradient $\partial=\partial^{\mu}=(\partial_t/c,-\nabla)$ — <Event> Alteration

4-Displacement
$\Delta\mathbf{R}=(c\Delta t,\Delta\mathbf{r})$
$d\mathbf{R}=(cdt,d\mathbf{r})$
4-Position
$\mathbf{R}=(ct,\mathbf{r})$

4-Gradient
$\partial=(\partial_t/c,-\nabla)$
$=(\partial_t/c,-\partial_x,-\partial_y,-\partial_z)$
$=(\partial/c\partial t,-\partial/\partial x,-\partial/\partial y,-\partial/\partial z)$

4-Velocity
$\mathbf{U}=\gamma(c,\mathbf{u})$
$=d\mathbf{R}/d\tau$

SRQM Diagram

These three give some of the main classical results of Special Relativity, including SR concepts like:

The Minkowski Metric, SpaceTime Dimension = 4, Lorentz Transformations
<Events>, Invariant Interval Measure, Causality (Temporal Ordering)
The Invariant Speed-of-Light (c)
Invariant ProperTime (clock at rest), Invariant ProperLength (ruler at rest)
Time Dilation (clock moving), Length Contraction (ruler moving)
Relativity of Simultaneity, Minkowski Diagrams, Light Cone
Use of the Lorentz Scalar Product to make Lorentz Invariants
Invariant SR Wave Equations, via the d'Alembertian
Continuity Equations
etc.

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SRQM Diagram:

The Basis of Classical SR Physics Special Relativity via 4-Vectors

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson

The Basis of most all Classical SR Physics is in the SR Minkowski Metric of "Flat" SpaceTime, which can be generated from the 4-Position and 4-Gradient, and determines the measurement between <Event>'s.

This Metric provides the relations between the main 4-Vectors of SR: 4-Position **R**, 4-Gradient ∂, 4-Velocity **U**.

The Tensor Invariants of these 4-Vectors give the:
Invariant Interval Measures & Causality, from **R·R**
Invariant d'Alembertian Wave Equation, from ∂·∂
Invariant Magnitude LightSpeed (c), from **U·U**

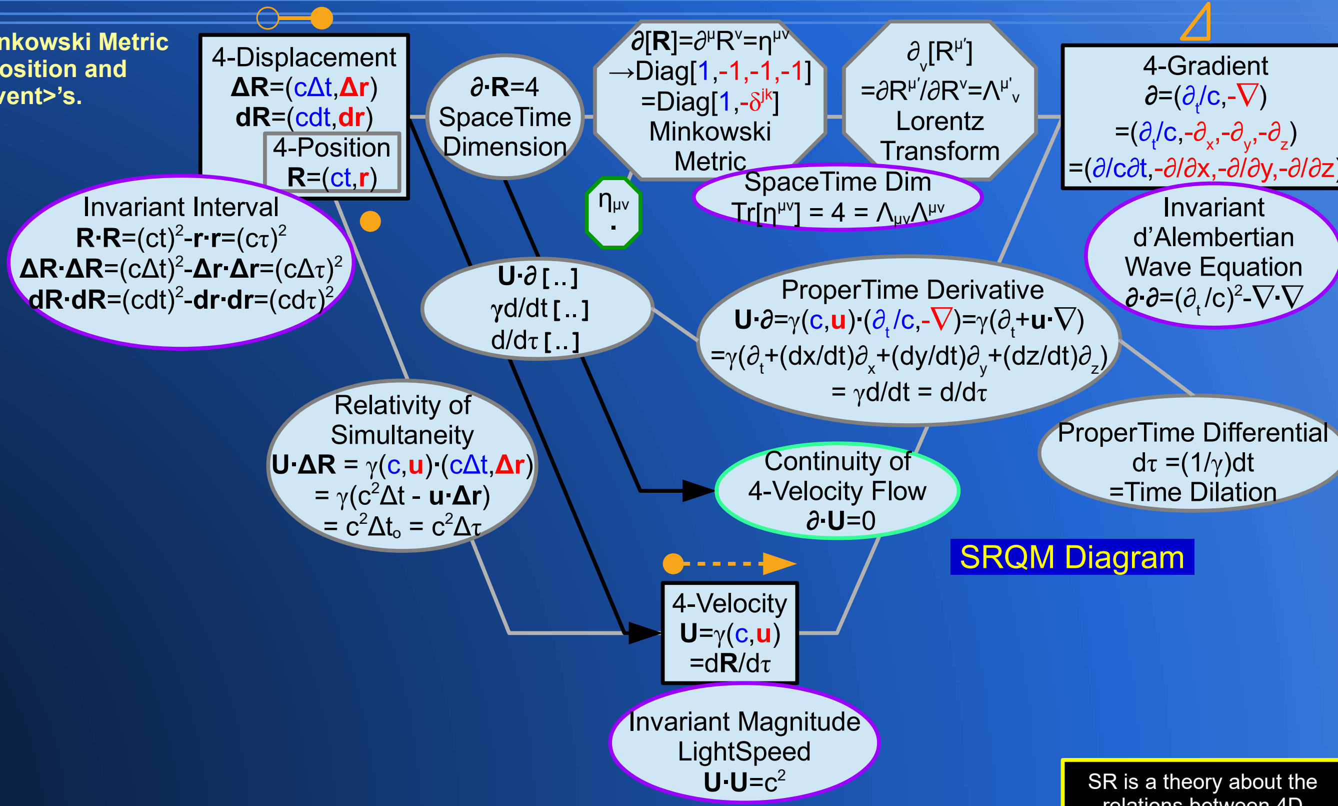
The relation between 4-Gradient ∂ and 4-Position **R** gives the Dimension of SpaceTime (4), the Minkowski Metric $\eta^{\mu\nu}$, and the Lorentz Transformations $\Lambda^{\mu'}{}_{\nu}$.

The relation between 4-Gradient ∂ and 4-Velocity **U** gives the ProperTime Derivative d/dτ. Rearranging gives the ProperTime Differential dτ, which leads to Time Dilation & Length Contraction.

The ProperTime Derivative d/dτ:
acting on 4-Position **R** gives 4-Velocity **U**
acting on the SpaceTime Dimension Lorentz Scalar gives the Continuity of 4-Velocity Flow.

The relation between 4-Displacement **ΔR** and 4-Velocity **U** gives Relativity of Simultaneity.

One of the most important properties is the Tensor Invariant Lorentz Scalar Product (dot = ·), provided by the lowered- index form of the Minkowski Metric $\eta_{\mu\nu}$.



From here, each object will be examined in turn...

SR is a theory about the relations between 4D SpaceTime <Event>'s, ie. how they are measured

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SR → QM

4-Vector SRQM Interpretation of QM

SRQM Diagram:
The Basis of Classical SR Physics
4-Position, 4-Displacement, 4-Differential

A Tensor Study of Physical 4-Vectors

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4-Displacement $\mathbf{\Delta R}=(c\Delta t,\mathbf{\Delta r})=\mathbf{U}\Delta\tau=\mathbf{R_2}-\mathbf{R_1}=(ct_2-ct_1,\mathbf{r_2}-\mathbf{r_1})$: {finite}
4-Differential $\mathbf{dR}=(cdt,\mathbf{dr})=\mathbf{U}d\tau$: {infintesimal}
4-Position $\mathbf{R}=(ct,\mathbf{r})$=<Event>

The 4-Position is essentially one of the fundamental 4-Vectors of SR. It is the SpaceTime location of an <Event>, the basic element of Minkowski SpaceTime: a time (t) & a place (**r**) → (when,where) = (ct,**r**).

The 4-Position relates time to space via the fundamental physical constant (c): the speed-of-light = "(c)elerity ; (c)eleritas", which is used to give consistent dimensional units across all SR 4-Vectors.

The 4-Position is a specific type of 4-Displacement, for which one of the endpoints is the origin, or 4-Zero.

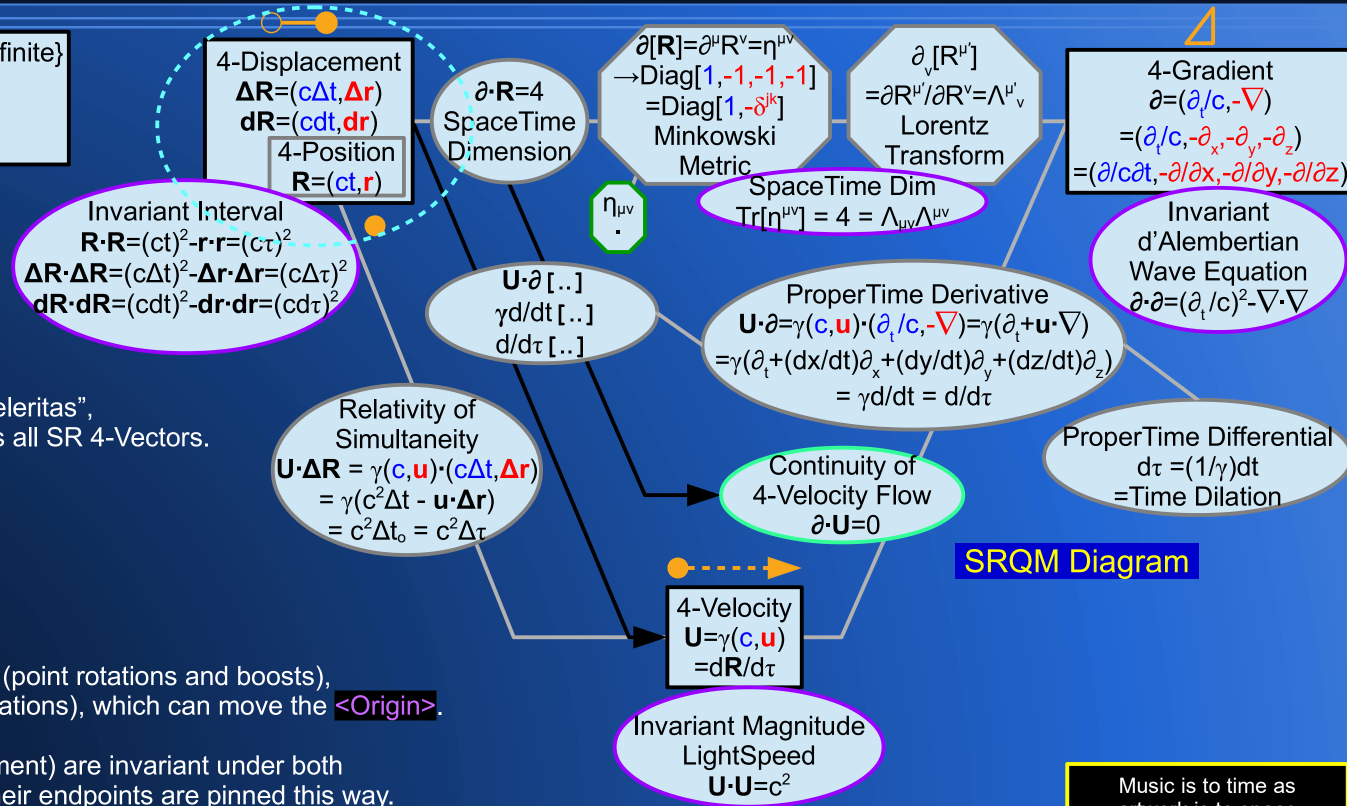
$\mathbf{R_2}\to\mathbf{R}$, $\mathbf{R_1}\to\mathbf{Z}$
$\mathbf{\Delta R} = \mathbf{R_2} - \mathbf{R_1} \to \mathbf{R} - \mathbf{Z} = \mathbf{R}$

4-Zero
$\mathbf{Z}=(0,\mathbf{0})=(0,0,0,0)$=<Origin>

As such, the 4-Position and 4-Zero are Lorentz Invariant (point rotations and boosts), but not Poincaré Invariant (Lorentz + time & space translations), which can move the <Origin>.

The general 4-Displacement and 4-Differential(Displacement) are invariant under both Lorentz and Poincaré transformations, since neither of their endpoints are pinned this way.

The 4-Differential(Displacement) is just the infinitesimal version of the finite 4-Displacement, and is used in the calculus of SR. $\mathbf{U}=d\mathbf{R}/d\tau$: $d\mathbf{R}=\mathbf{U}d\tau$



Music is to time as artwork is to space

4-Creativity
☼ = (Music , Artwork)

4-Position $\mathbf{R}=(ct,\mathbf{r})$=<Event>
$\mathbf{R} = \int d\mathbf{R} = \int \mathbf{U}d\tau = \int\gamma(c,\mathbf{u})d\tau = \int(c,\mathbf{u})\gamma d\tau = \int(c,\mathbf{u})dt = (ct,\mathbf{r})$
$\mathbf{R} = \Sigma\mathbf{\Delta R} = \Sigma\mathbf{U}\Delta\tau = \Sigma\gamma(c,\mathbf{u})\Delta\tau = \Sigma(c,\mathbf{u})\gamma\Delta\tau = \Sigma(c,\mathbf{u})\Delta t = (ct,\mathbf{r})$

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SR → QM

4-Vector SRQM Interpretation of QM

SRQM Diagram:

The Basis of Classical SR Physics

Invariant Intervals, Causality, TimeSpace

A Tensor Study of Physical 4-Vectors

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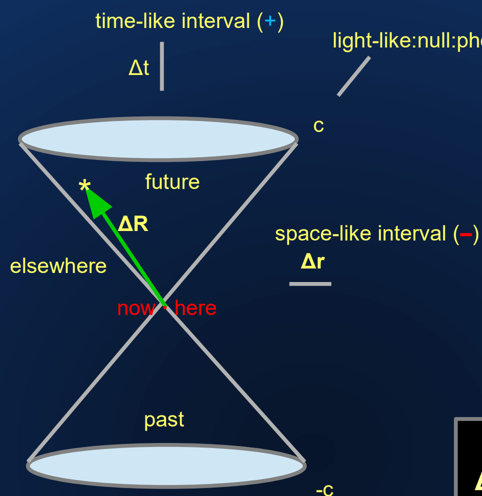
4-Displacement $\mathbf{\Delta R}=(c\Delta t,\mathbf{\Delta r})=\mathbf{U}\Delta\tau=\mathbf{R_2}-\mathbf{R_1}=(ct_2-ct_1,\mathbf{r_2}-\mathbf{r_1})$: {finite}
4-Differential $\mathbf{dR}=(cdt,\mathbf{dr})=\mathbf{U}d\tau$: {infintesimal}
4-Position $\mathbf{R}=(ct,\mathbf{r})$=<Event>

The Invariant Interval is the Lorentz Scalar Product of the {4-Position, 4-Displacement, 4-Differential} with itself, giving a magnitude-squared.

$\mathbf{R\cdot R}=(ct)^2-\mathbf{r\cdot r}=(ct_o)^2=(c\tau)^2$
$\mathbf{\Delta R\cdot\Delta R}=(c\Delta t)^2-\mathbf{\Delta r\cdot\Delta r}=(c\Delta t_o)^2=(c\Delta\tau)^2$
$\mathbf{dR\cdot dR}=(cdt)^2-\mathbf{dr\cdot dr}=(cdt_o)^2=(cd\tau)^2$



LightCone

4-Displacement $\mathbf{\Delta R}=(c\Delta t,\mathbf{\Delta r})$ $\mathbf{dR}=(cdt,\mathbf{dr})$
4-Position $\mathbf{R}=(ct,\mathbf{r})$

Invariant Interval
$\mathbf{R\cdot R}=(ct)^2-\mathbf{r\cdot r}=(c\tau)^2$
$\mathbf{\Delta R\cdot\Delta R}=(c\Delta t)^2-\mathbf{\Delta r\cdot\Delta r}=(c\Delta\tau)^2$
$\mathbf{dR\cdot dR}=(cdt)^2-\mathbf{dr\cdot dr}=(cd\tau)^2$

$\partial\cdot\mathbf{R}=4$
SpaceTime Dimension

$\partial[\mathbf{R}]=\partial^\mu R^\nu=\eta^{\mu\nu}$
$\rightarrow\mathrm{Diag}[1,-1,-1,-1]$
$=\mathrm{Diag}[1,-\delta^{jk}]$
Minkowski Metric

$\eta_{\mu\nu}$

SpaceTime Dim
$\mathrm{Tr}[\eta^{\mu\nu}]=4=\Lambda_{\mu\nu}\Lambda^{\mu\nu}$

$\partial_\nu[R^{\mu'}]$
$=\partial R^{\mu'}/\partial R^\nu=\Lambda^{\mu'}{}_\nu$
Lorentz Transform

4-Gradient
$\partial=(\partial_t/c,-\nabla)$
$=(\partial_t/c,-\partial_x,-\partial_y,-\partial_z)$
$=(\partial/c\partial t,-\partial/\partial x,-\partial/\partial y,-\partial/\partial z)$

Invariant d'Alembertian Wave Equation
$\partial\cdot\partial=(\partial_t/c)^2-\nabla\cdot\nabla$

$\mathbf{U}\cdot\partial$ [..]
$\gamma d/dt$ [..]
$d/d\tau$ [..]

ProperTime Derivative
$\mathbf{U}\cdot\partial=\gamma(c,\mathbf{u})\cdot(\partial_t/c,-\nabla)=\gamma(\partial_t+\mathbf{u}\cdot\nabla)$
$=\gamma(\partial_t+(dx/dt)\partial_x+(dy/dt)\partial_y+(dz/dt)\partial_z)$
$=\gamma d/dt=d/d\tau$

ProperTime Differential
$d\tau=(1/\gamma)dt$
=Time Dilation

Relativity of Simultaneity
$\mathbf{U\cdot\Delta R}=\gamma(c,\mathbf{u})\cdot(c\Delta t,\mathbf{\Delta r})$
$=\gamma(c^2\Delta t-\mathbf{u\cdot\Delta r})$
$=c^2\Delta t_o=c^2\Delta\tau$

Continuity of 4-Velocity Flow
$\partial\cdot\mathbf{U}=0$

SRQM Diagram

4-Velocity
$\mathbf{U}=\gamma(c,\mathbf{u})$
$=d\mathbf{R}/d\tau$

Invariant Magnitude LightSpeed
$\mathbf{U\cdot U}=c^2$

The 4D intervals are invariant, meaning that all observers must agree on their magnitudes, regardless of differing reference frames. This leads to the idea of ProperTime (Δτ), which is the time-displacement measured by a clock at-rest. This also leads to the various Causality Conditions of SR, and the concept of the (Minkowski Diagram) Light Cone.
The differential form **dR·dR** is apparently also still true in GR.

$\mathbf{\Delta R\cdot\Delta R}=[(c\Delta t)^2-\mathbf{\Delta r\cdot\Delta r}]=$	$(c\Delta\tau)^2$	Time-like:Temporal	(+) {causal = temporally-ordered}
	(0)	Light-like:Null:Photonic	(0) {causal, maximum signal speed ($\lvert\mathbf{\Delta r}/\Delta t\rvert=c$)}
	$-(\Delta r_o)^2$	Space-like:Spatial	(–) {non-causal, spatially-extended}

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^\mu{}_\nu$ or $T_\mu{}^\nu$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu=\mathbf{V}=(v^0,\mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu=(v_0,-\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

$\mathrm{Trace}[T^{\mu\nu}]=\eta_{\mu\nu}T^{\mu\nu}=T^\mu{}_\mu=T$
$\mathbf{V\cdot V}=V^\mu\eta_{\mu\nu}V^\nu=[(v^0)^2-\mathbf{v\cdot v}]=(v^0{}_o)^2$
= Lorentz Scalar

SR → QM

A Tensor Study of Physical 4-Vectors

SRQM Diagram:
The Basis of Classical SR Physics
SpaceTime Dimension = 4D

4-Vector SRQM Interpretation of QM

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4-Gradient $\partial=(\partial_t/c,-\nabla)$

4-Position $\mathbf{R}=(ct,\mathbf{r})$=<Event>

$\partial\cdot\mathbf{R}$ = 4 : The 4-Divergence SpaceTime Dimension Relation

$\partial\cdot\mathbf{R}$
$= (\partial_t/c,-\nabla)\cdot(ct,\mathbf{r})$
$= [(\partial_t/c)*(ct) - (-\nabla)\cdot(\mathbf{r})]$
$= (\partial_t[t] + \nabla\cdot\mathbf{r})$
$= (\partial_t[t] + \partial_x[x] + \partial_y[y] + \partial_z[z])$
$= (\partial[t]/\partial t + \partial[x]/\partial x + \partial[y]/\partial y + \partial[z]/\partial z)$
$= (1+1+1+1)$
$= 4$

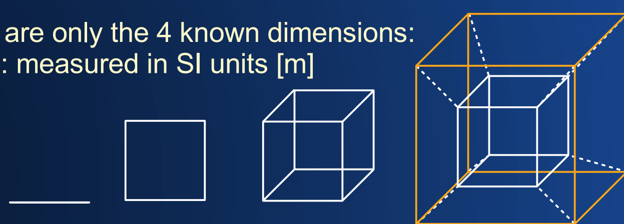
$(\partial\cdot\mathbf{R}) = (\partial^\alpha\cdot R^\beta) = (\partial^\alpha\eta_{\alpha\beta}R^\beta) = \eta_{\alpha\beta}(\partial^\alpha R^\beta) = \eta_{\alpha\beta}(\eta^{\alpha\beta}) = \eta_\beta{}^\beta = \eta_\alpha{}^\alpha = \delta_\alpha{}^\alpha = 4$

This Tensor Invariant Lorentz Scalar relation gives the dimension of SpaceTime. The only way there can more dimensions is if there is another SpaceTime direction available. The 4-Divergence is also used in SR Conservation Laws, ex. $(\partial\cdot\mathbf{J}) = 0$

All empirical evidence to-date indicates that there are only the 4 known dimensions:
1 temporal (t): measured in SI units = [s], with (ct): measured in SI units [m]
3 spatial (x, y, z) : measured in SI units = [m]

These are of course the ones that appear in the

4-Position $\mathbf{R}=(ct,\mathbf{r})\to(ct,x,y,z)$: measured in SI units [m]



1D (x) 2D (x,y) 3D (x,y,z) 4D (ct,x,y,z)

The Tesseract, a 4D cube, symbolizes 4D SpaceTime

SpaceTime $\partial\cdot\mathbf{R}=\partial^\mu\eta_{\mu\nu}R^\nu=\partial_\nu R^\nu=4$ Dimension

Invariant Interval
$\mathbf{R}\cdot\mathbf{R}=(ct)^2-\mathbf{r}\cdot\mathbf{r}=(c\tau)^2$
$\Delta\mathbf{R}\cdot\Delta\mathbf{R}=(c\Delta t)^2-\Delta\mathbf{r}\cdot\Delta\mathbf{r}=(c\Delta\tau)^2$
$d\mathbf{R}\cdot d\mathbf{R}=(cdt)^2-d\mathbf{r}\cdot d\mathbf{r}=(cd\tau)^2$

4-Displacement
$\Delta\mathbf{R}=(c\Delta t,\Delta\mathbf{r})$
$d\mathbf{R}=(cdt,d\mathbf{r})$

4-Position $\mathbf{R}=(ct,\mathbf{r})$

$\partial\cdot\mathbf{R}=4$ SpaceTime Dimension

$\partial[\mathbf{R}]=\partial^\mu R^\nu=\eta^{\mu\nu}$
$\to$Diag[1,-1,-1,-1]
=Diag[1,$-\delta^{jk}$]
Minkowski Metric

$\partial_\nu[R^{\mu'}]$
$=\partial R^{\mu'}/\partial R^\nu=\Lambda^{\mu'}{}_\nu$
Lorentz Transform

$\eta_{\mu\nu}\cdot$

SpaceTime Dim
$Tr[\eta^{\mu\nu}] = 4 = \Lambda_{\mu\nu}\Lambda^{\mu\nu}$

4-Gradient
$\partial=(\partial_t/c,-\nabla)$
$=(\partial_t/c,-\partial_x,-\partial_y,-\partial_z)$
$=(\partial/c\partial t,-\partial/\partial x,-\partial/\partial y,-\partial/\partial z)$

Invariant d'Alembertian Wave Equation
$\partial\cdot\partial=(\partial_t/c)^2-\nabla\cdot\nabla$

$\mathbf{U}\cdot\partial$ [..]
$\gamma d/dt$ [..]
$d/d\tau$ [..]

ProperTime Derivative
$\mathbf{U}\cdot\partial=\gamma(c,\mathbf{u})\cdot(\partial_t/c,-\nabla)=\gamma(\partial_t+\mathbf{u}\cdot\nabla)$
$=\gamma(\partial_t+(dx/dt)\partial_x+(dy/dt)\partial_y+(dz/dt)\partial_z)$
$= \gamma d/dt = d/d\tau$

ProperTime Differential
$d\tau=(1/\gamma)dt$
=Time Dilation

Relativity of Simultaneity
$\mathbf{U}\cdot\Delta\mathbf{R} = \gamma(c,\mathbf{u})\cdot(c\Delta t,\Delta\mathbf{r})$
$= \gamma(c^2\Delta t - \mathbf{u}\cdot\Delta\mathbf{r})$
$= c^2\Delta t_o = c^2\Delta\tau$

Continuity of 4-Velocity Flow
$\partial\cdot\mathbf{U}=0$

SRQM Diagram

4-Velocity
$\mathbf{U}=\gamma(c,\mathbf{u})$
$=d\mathbf{R}/d\tau$

Invariant Magnitude LightSpeed
$\mathbf{U}\cdot\mathbf{U}=c^2$

SR : Minkowski SpaceTime is 4D

SR 4-Tensor
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SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

$Trace[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu{}_\mu = T$
$\mathbf{V}\cdot\mathbf{V} = V^\mu\eta_{\mu\nu}V^\nu = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$
= Lorentz Scalar

SR → QM

4-Vector SRQM Interpretation of QM

SRQM Diagram:
The Basis of Classical SR Physics
The Minkowski Metric ($\eta^{\mu\nu}$), Measurement

A Tensor Study of Physical 4-Vectors

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4-Gradient $\partial=(\partial_t/c,-\nabla)$

4-Position $R=(ct,\mathbf{r})$=<Event>

The SR Minkowski Metric = "Flat" SR SpaceTime
$\partial[R] = \partial^\mu R^\nu = \eta^{\mu\nu} \rightarrow \mathrm{Diag}[1,-1,-1,-1]_{\{rect\}} = \mathrm{Diag}[1,-\delta^{jk}] = \mathrm{Diag}[1,-I_{(3)}]$

Derivation:
$\partial[R] = \partial^\mu R^\nu$
$=(\partial_t/c,-\nabla)[(ct,\mathbf{r})]$

$=[\partial_t/c*ct, -\nabla ct]$
$[\partial_t/c*\mathbf{r}, -\nabla\mathbf{r}]$

$=[\partial_t t, \mathbf{0}]$
$[\mathbf{0}, -\nabla\mathbf{r}]$

$=\mathrm{Diag}[+1,-\delta^{jk}]$
$=\eta^{\mu\nu}$

The component representation of the Minkowski Metric $\eta^{\mu\nu}$ will differ with the chosen basis, just like with 4-Vectors.

$\eta^{\mu\nu}\rightarrow\mathrm{Diag}[1,-1,-1,-1]$: Cartesian/Rectangular basis
$\eta^{\mu\nu}\rightarrow\mathrm{Diag}[1,-1,-1/r^2,-1]$: Polar/Cylindrical basis
$\eta^{\mu\nu}\rightarrow\mathrm{Diag}[1,-1,-1/r^2,-1/(r\sin[\theta])^2]$: Spherical basis

Generally, components $[\eta^{\mu\mu}] = 1/[\eta_{\mu\mu}]$ and $\eta_\mu{}^\nu = \delta_\mu{}^\nu$

SR:Minkowski Metric
$\partial[R] = \partial^\mu R^\nu = \eta^{\mu\nu} = V^{\mu\nu} + H^{\mu\nu} \rightarrow$
$\mathrm{Diag}[1,-1,-1,-1] = \mathrm{Diag}[1,-I_{(3)}] = \mathrm{Diag}[1,-\delta^{jk}]$
{in Cartesian form} "Particle Physics" Convention
$\{\eta_{\mu\mu}\} = 1/\{\eta^{\mu\mu}\} : \eta_\mu{}^\nu = \delta_\mu{}^\nu$ $\mathrm{Tr}[\eta^{\mu\nu}]=4$

4-UnitTemporal
$T=T^\mu=\gamma(1,\boldsymbol{\beta})=U/c$

⊗

SR:Temporal Projection
"Vertical" $V^{\mu\nu} = T^\mu T^\nu \rightarrow$
$\mathrm{Diag}[1,0,0,0] = \mathrm{Diag}[1,0^{jk}]$

+

SR:Spatial Projection
"Horizontal" $H^{\mu\nu} = \eta^{\mu\nu}-T^\mu T^\nu \rightarrow$
$\mathrm{Diag}[0,-1,-1,-1] = \mathrm{Diag}[0,-\delta^{jk}]$

"vertical" $V^{\mu\nu}$ temporal

$H^{\mu\nu}$ spatial "horizontal"

4-Displacement
$\Delta R=(c\Delta t,\Delta\mathbf{r})$
$dR=(cdt,d\mathbf{r})$

4-Position
$R=(ct,\mathbf{r})$

Invariant Interval
$R\cdot R=(ct)^2-\mathbf{r}\cdot\mathbf{r}=(c\tau)^2$
$\Delta R\cdot\Delta R=(c\Delta t)^2-\Delta\mathbf{r}\cdot\Delta\mathbf{r}=(c\Delta\tau)^2$
$dR\cdot dR=(cdt)^2-d\mathbf{r}\cdot d\mathbf{r}=(cd\tau)^2$

$\partial\cdot R=4$
SpaceTime Dimension

$\partial[R]=\partial^\mu R^\nu=\eta^{\mu\nu}$
$\rightarrow\mathrm{Diag}[1,-1,-1,-1]$
$=\mathrm{Diag}[1,-\delta^{jk}]$
Minkowski Metric

$\eta_{\mu\nu}$ ·

$\partial_\nu[R^{\mu'}]$
$=\partial R^{\mu'}/\partial R^\nu=\Lambda^{\mu'}{}_\nu$
Lorentz Transform

SpaceTime Dim
$\mathrm{Tr}[\eta^{\mu\nu}] = 4 = \Lambda_{\mu\nu}\Lambda^{\mu\nu}$

4-Gradient
$\partial=(\partial_t/c,-\nabla)$
$=(\partial_t/c,-\partial_x,-\partial_y,-\partial_z)$
$=(\partial/c\partial t,-\partial/\partial x,-\partial/\partial y,-\partial/\partial z)$

$U\cdot\partial$ [..]
$\gamma d/dt$ [..]
$d/d\tau$ [..]

ProperTime Derivative
$U\cdot\partial=\gamma(c,\mathbf{u})\cdot(\partial_t/c,-\nabla)=\gamma(\partial_t+\mathbf{u}\cdot\nabla)$
$=\gamma(\partial_t+(dx/dt)\partial_x+(dy/dt)\partial_y+(dz/dt)\partial_z)$
$=\gamma d/dt = d/d\tau$

Invariant d'Alembertian Wave Equation
$\partial\cdot\partial=(\partial_t/c)^2-\nabla\cdot\nabla$

Relativity of Simultaneity
$U\cdot\Delta R = \gamma(c,\mathbf{u})\cdot(c\Delta t,\Delta\mathbf{r})$
$=\gamma(c^2\Delta t - \mathbf{u}\cdot\Delta\mathbf{r})$
$=c^2\Delta t_o = c^2\Delta\tau$

Continuity of 4-Velocity Flow
$\partial\cdot U=0$

ProperTime Differential
$d\tau=(1/\gamma)dt$
=Time Dilation

SRQM Diagram

4-Velocity
$U=\gamma(c,\mathbf{u})$
$=dR/d\tau$

Invariant Magnitude LightSpeed
$U\cdot U=c^2$

The SR:Minkowski Metric $\eta^{\mu\nu}$ is the fundamental SR (2,0)-Tensor, which shows how intervals are measured in SR SpaceTime.
It is itself the low-mass = (Curvature ~ 0) limiting-case of the more general GR metric $g^{\mu\nu}$. It can be divided into temporal and spatial parts.
The Minkowski Metric can be used to raise/lower indices on other tensors and 4-Vectors.

Alt. Derivation: $\partial^\mu X^\nu = \eta^{\mu\sigma}\partial_\sigma X^\nu = \eta^{\mu\sigma}(\partial/\partial X^\sigma)X^\nu = \eta^{\mu\sigma}(\partial X^\nu/\partial X^\sigma) = \eta^{\mu\sigma}(\delta_\sigma{}^\nu) = \eta^{\mu\nu}$

The SR : Minkowski Metric $\eta^{\mu\nu}$ is the "Flat SpaceTime" low-curvature limiting-case of the more general GR Metric $g^{\mu\nu}$.

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^\mu{}_\nu$ or $T_\mu{}^\nu$
(0,2)-Tensor $T_{\mu\nu}$

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(1,0)-Tensor $V^\mu = V = (v^0,\mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0,-\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu{}_\mu = T$
$V\cdot V = V^\mu\eta_{\mu\nu}V^\nu = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$
= Lorentz Scalar

A Tensor Study of Physical 4-Vectors

SRQM Diagram:
The Basis of Classical SR Physics
The Lorentz Transform $\partial_\nu[R^{\mu'}]=\Lambda^{\mu'}{}_\nu$

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4-Gradient
$\partial=(\partial_t/c,-\nabla)$

4-Position
$\mathbf{R}=(ct,\mathbf{r})$=<Event>

Tensorial Lorentz Transform $\Lambda^{\mu'}{}_\nu$ acting on 4-Vector [$R^{\mu'} = \Lambda^{\mu'}{}_\nu R^\nu$]
$\partial_\nu[R^{\mu'}] = (\partial/\partial R^\nu)[R^{\mu'}] = (\partial/\partial R^\nu)[\Lambda^{\mu'}{}_\alpha R^\alpha]$
$= \Lambda^{\mu'}{}_\alpha (\partial/\partial R^\nu)[R^\alpha] = \Lambda^{\mu'}{}_\alpha \eta^\alpha{}_\nu = \Lambda^{\mu'}{}_\nu$

$\Lambda_{\mu\nu}\Lambda^{\mu\nu}=4$

$\mathrm{Det}[\Lambda^\mu{}_\nu]=\pm1$

4-Displacement
$\Delta\mathbf{R}=(c\Delta t,\Delta\mathbf{r})$
$\mathbf{dR}=(cdt,\mathbf{dr})$

4-Position
$\mathbf{R}=(ct,\mathbf{r})$

$\partial\cdot\mathbf{R}=4$
SpaceTime Dimension

$\partial[\mathbf{R}]=\partial^\mu R^\nu=\eta^{\mu\nu}$
$\to$Diag[1,-1,-1,-1]
=Diag[1,$-\delta^{jk}$]
Minkowski Metric

$\partial_\nu[R^{\mu'}]$
$=\partial R^{\mu'}/\partial R^\nu=\Lambda^{\mu'}{}_\nu$
Lorentz Transform

SpaceTime Dim
$\mathrm{Tr}[\eta^{\mu\nu}] = 4 = \Lambda_{\mu\nu}\Lambda^{\mu\nu}$

$\eta_{\mu\nu}$
$\cdot$

4-Gradient
$\partial=(\partial_t/c,-\nabla)$
$=(\partial_t/c,-\partial_x,-\partial_y,-\partial_z)$
$=(\partial/c\partial t,-\partial/\partial x,-\partial/\partial y,-\partial/\partial z)$

Invariant Interval
$\mathbf{R}\cdot\mathbf{R}=(ct)^2-\mathbf{r}\cdot\mathbf{r}=(c\tau)^2$
$\Delta\mathbf{R}\cdot\Delta\mathbf{R}=(c\Delta t)^2-\Delta\mathbf{r}\cdot\Delta\mathbf{r}=(c\Delta\tau)^2$
$\mathbf{dR}\cdot\mathbf{dR}=(cdt)^2-\mathbf{dr}\cdot\mathbf{dr}=(cd\tau)^2$

$\mathbf{U}\cdot\partial$ [..]
$\gamma d/dt$ [..]
$d/d\tau$ [..]

ProperTime Derivative
$\mathbf{U}\cdot\partial=\gamma(c,\mathbf{u})\cdot(\partial_t/c,-\nabla)=\gamma(\partial_t+\mathbf{u}\cdot\nabla)$
$=\gamma(\partial_t+(dx/dt)\partial_x+(dy/dt)\partial_y+(dz/dt)\partial_z)$
$= \gamma d/dt = d/d\tau$

Invariant d'Alembertian Wave Equation
$\partial\cdot\partial=(\partial_t/c)^2-\nabla\cdot\nabla$

ProperTime Differential
$d\tau =(1/\gamma)dt$
=Time Dilation

Relativity of Simultaneity
$\mathbf{U}\cdot\Delta\mathbf{R} = \gamma(c,\mathbf{u})\cdot(c\Delta t,\Delta\mathbf{r})$
$= \gamma(c^2\Delta t - \mathbf{u}\cdot\Delta\mathbf{r})$
$= c^2\Delta t_o = c^2\Delta\tau$

Continuity of 4-Velocity Flow
$\partial\cdot\mathbf{U}=0$

SRQM Diagram

4-Velocity
$\mathbf{U}=\gamma(c,\mathbf{u})$
$=\mathbf{dR}/d\tau$

Invariant Magnitude LightSpeed
$\mathbf{U}\cdot\mathbf{U}=c^2$

<u>General Lorentz Boost Transform (symmetric,continuous):</u>
for a linear-velocity time-space-mixing frame-shift (Boost) in the $\mathbf{v}/c=\boldsymbol{\beta}=(\beta^1,\beta^2,\beta^3)$-direction: $\Lambda^{\mu'}{}_\nu \to B^{\mu'}{}_\nu$ =

$$\begin{bmatrix} \gamma & -\gamma\beta^j \\ -\gamma\beta^i & (\gamma-1)\beta^i\beta^j/(\boldsymbol{\beta}\cdot\boldsymbol{\beta})+\delta^{ij} \end{bmatrix}$$

<u>General Lorentz Rotation Transform (non-symmetric,continuous):</u>
for an angular-displacement spatial-only frame-shift (Rotation) angle θ about the $\hat{\mathbf{n}}=(n^1,n^2,n^3)$-direction: $\Lambda^{\mu'}{}_\nu \to R^{\mu'}{}_\nu$ =

$$\begin{bmatrix} 1 & 0^j \\ 0^i & (\delta^{ij}-n^in^j)\cos(\theta)-(\varepsilon^{ij}{}_k n^k)\sin(\theta)+n^in^j \end{bmatrix}$$

<u>Lorentz DiscreteTransforms (symmetric):</u>

Identity $I_{(4)}$	Time-Reverse	Parity	ComboPT
$\Lambda^{\mu'}{}_\nu \to \eta^{\mu'}{}_\nu = \delta^{\mu'}{}_\nu$	$\Lambda^{\mu'}{}_\nu \to T^{\mu'}{}_\nu$	$\Lambda^{\mu'}{}_\nu \to P^{\mu'}{}_\nu$	$\Lambda^{\mu'}{}_\nu \to (PT)^{\mu'}{}_\nu$
= Diag[1,$\delta^i{}_j$]	= Diag[1-,$\delta^i{}_j$]	= Diag[1,$-\delta^i{}_j$]	= Diag[-1,$-\delta^i{}_j$]
$\begin{bmatrix} 1 & 0 \\ 0 & \delta^i{}_j \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & \delta^i{}_j \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & -\delta^i{}_j \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & -\delta^i{}_j \end{bmatrix}$

<u>Lorentz Transform Properties:</u>
$\Lambda_\mu{}^\nu = (\Lambda^{-1})^\nu{}_\mu$
$\Lambda^\mu{}_\alpha\Lambda_\nu{}^\alpha = \eta^\mu{}_\nu = \delta^\mu{}_\nu$
$\Lambda_{\mu\nu}\Lambda^{\mu\nu} = 4$: SpaceTime Dimension
$\eta_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta = \eta_{\alpha\beta}$
$\mathrm{Det}[\Lambda^\mu{}_\nu] = \pm1$: (+)=Linearity; (-)=Anti-Linearity

The Trace Invariant of the various Lorentz Transforms leads to very interesting results: CPT Symmetry and Antimatter

SR:Lorentz Transform
$\partial_\nu[R^{\mu'}] = \partial R^{\mu'}/\partial R^\nu = \Lambda^{\mu'}{}_\nu$
$\Lambda_\mu{}^\nu = (\Lambda^{-1})^\nu{}_\mu : \Lambda^\mu{}_\alpha\Lambda_\nu{}^\alpha = \eta^\mu{}_\nu = \delta^\mu{}_\nu$
$\eta_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta = \eta_{\alpha\beta}$
$\mathrm{Det}[\Lambda^\mu{}_\nu]=\pm1$ $\Lambda_{\mu\nu}\Lambda^{\mu\nu}=4$

Invariant $\mathrm{Tr}[\Lambda^{\mu'}{}_\nu] \to$
$-\infty$,..,(-4),..,-2,..,(0),..,+2,..,(+4),…,$+\infty$
Trace identifies CPT Symmetry

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= Lorentz Scalar

SR → QM

A Tensor Study
of Physical 4-Vectors

SRQM Diagram:
The Basis of Classical SR Physics
SpaceTime Dimension = 4D, again!

4-Vector SRQM Interpretation
of QM

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$\partial\cdot\mathbf{R} = Tr[\eta^{\mu\nu}] = \Lambda^{\mu\beta}\Lambda_{\mu\beta} = 4$: The SpaceTime Dimension Relations

Tensor Invariants include: Trace, InnerProduct, Determinant, etc.
The Trace of the Minkowski Metric &
the InnerProduct of any of the Lorentz Transforms
give the Dimension of SR SpaceTime = 4D.

Minkowski Metric
Trace Invariant
$Trace[\eta^{\mu\nu}]$
$= Tr[\eta^{\mu\nu}]$
$= \eta_{\mu\nu}[\eta^{\mu\nu}]$
$= \eta_{\mu}{}^{\mu}$
$= \delta_{\mu}{}^{\mu}$
$= (1+1+1+1)$
$= 4$

4-Divergence
of 4-Position
$\partial\cdot\mathbf{R}$
$= \partial^{\mu}\cdot R^{\nu}$
$= \partial^{\mu}\eta_{\mu\nu}R^{\nu}$
$= \eta_{\mu\nu}\partial^{\mu}R^{\nu}$
$= \eta_{\mu\nu}\eta^{\mu\nu}$
$= Tr[\eta^{\mu\nu}]$
$= 4$

Lorentz Transform
Inner Prod Invariant
$\eta_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta} = \eta_{\alpha\beta}$
$\eta^{\alpha\beta}\eta_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta} = \eta_{\alpha\beta}\eta^{\alpha\beta}$
$\eta^{\alpha\beta}\Lambda^{\mu}{}_{\alpha}\eta_{\mu\nu}\Lambda^{\nu}{}_{\beta} = \eta_{\alpha\beta}\eta^{\alpha\beta}$
$(\eta^{\alpha\beta}\Lambda^{\mu}{}_{\alpha})(\eta_{\mu\nu}\Lambda^{\nu}{}_{\beta}) = \eta_{\alpha\beta}\eta^{\alpha\beta}$
$\Lambda^{\mu\beta}\Lambda_{\mu\beta} = \eta_{\alpha\beta}\eta^{\alpha\beta} = Tr[\eta^{\mu\nu}]$
$\Lambda^{\mu\beta}\Lambda_{\mu\beta} = 4$
$= 4$

4-Displacement
$\Delta\mathbf{R}=(c\Delta t,\Delta\mathbf{r})$
$d\mathbf{R}=(cdt,d\mathbf{r})$

4-Position
$\mathbf{R}=(ct,\mathbf{r})$

$\partial\cdot\mathbf{R}=4$
SpaceTime
Dimension

$\partial[\mathbf{R}]=\partial^{\mu}R^{\nu}=\eta^{\mu\nu}$
$\rightarrow Diag[1,-1,-1,-1]$
$=Diag[1,-\delta^{jk}]$
Minkowski
Metric

$\partial_{\nu}[R^{\mu'}]$
$=\partial R^{\mu'}/\partial R^{\nu}=\Lambda^{\mu'}{}_{\nu}$
Lorentz
Transform

4-Gradient
$\partial=(\partial_t/c,-\nabla)$
$=(\partial_t/c,-\partial_x,-\partial_y,-\partial_z)$
$=(\partial/c\partial t,-\partial/\partial x,-\partial/\partial y,-\partial/\partial z)$

SpaceTime Dim
$Tr[\eta^{\mu\nu}] = 4 = \Lambda_{\mu\nu}\Lambda^{\mu\nu}$

$\eta_{\mu\nu}$
$\cdot$

Invariant Interval
$\mathbf{R}\cdot\mathbf{R}=(ct)^2-\mathbf{r}\cdot\mathbf{r}=(c\tau)^2$
$\Delta\mathbf{R}\cdot\Delta\mathbf{R}=(c\Delta t)^2-\Delta\mathbf{r}\cdot\Delta\mathbf{r}=(c\Delta\tau)^2$
$d\mathbf{R}\cdot d\mathbf{R}=(cdt)^2-d\mathbf{r}\cdot d\mathbf{r}=(cd\tau)^2$

Invariant
d'Alembertian
Wave Equation
$\partial\cdot\partial=(\partial_t/c)^2-\nabla\cdot\nabla$

$\mathbf{U}\cdot\partial$ [..]
$\gamma d/dt$ [..]
$d/d\tau$ [..]

ProperTime Derivative
$\mathbf{U}\cdot\partial=\gamma(c,\mathbf{u})\cdot(\partial_t/c,-\nabla)=\gamma(\partial_t+\mathbf{u}\cdot\nabla)$
$=\gamma(\partial_t+(dx/dt)\partial_x+(dy/dt)\partial_y+(dz/dt)\partial_z)$
$= \gamma d/dt = d/d\tau$

Relativity of
Simultaneity
$\mathbf{U}\cdot\Delta\mathbf{R} = \gamma(c,\mathbf{u})\cdot(c\Delta t,\Delta\mathbf{r})$
$= \gamma(c^2\Delta t - \mathbf{u}\cdot\Delta\mathbf{r})$
$= c^2\Delta t_o = c^2\Delta\tau$

ProperTime Differential
$d\tau =(1/\gamma)dt$
=Time Dilation

Continuity of
4-Velocity Flow
$\partial\cdot\mathbf{U}=0$

SRQM Diagram

Minkowski
Trace Invariant
$Tr[\eta^{\mu\nu}]=\eta_{\nu}{}^{\nu}=(1) - (-1) - (-1) - (-1)= 4$

General Tensor
Trace Invariant
$Tr[T^{\mu\nu}]=T_{\nu}{}^{\nu}=(T_0{}^0+T_1{}^1+T_2{}^2+T_3{}^3)$
$=(T^{00}-T^{11}-T^{22}-T^{33})=T$

4-Velocity
$\mathbf{U}=\gamma(c,\mathbf{u})$
$=d\mathbf{R}/d\tau$

Invariant Magnitude
LightSpeed
$\mathbf{U}\cdot\mathbf{U}=c^2$

4-Tensor
$T^{\mu\nu} = \begin{bmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{bmatrix}$

Minkowski
Metric $\eta^{\mu\nu}$
$\rightarrow \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

$\partial\cdot\eta^{\mu\nu}$
$= \partial^{\sigma}\cdot\eta^{\mu\nu}$
$= \partial^{\sigma}\eta_{\sigma\mu}\eta^{\mu\nu}$ $\quad = \partial^{\sigma}\eta_{\sigma\mu}\eta^{\mu\nu}$
$= \partial_{\mu}\eta^{\mu\nu}$ $\quad = \partial^{\sigma}\eta_{\alpha}{}^{\nu}$
$\quad = \partial^{\sigma}\delta_{\alpha}{}^{\nu}$
$= 0^{\nu}$ $\quad = 0^{\nu}$
Conservation of Minkowksi Metric

SR : Minkowski
SpaceTime is 4D

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^{\mu}{}_{\nu}$ or $T_{\mu}{}^{\nu}$
(0,2)-Tensor $T_{\mu\nu}$

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(1,0)-Tensor $V^{\mu} = \mathbf{V} = (v^0,\mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_{\mu} = (v_0,-\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

$Trace[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^{\mu}{}_{\mu} = T$
$\mathbf{V}\cdot\mathbf{V} = V^{\mu}\eta_{\mu\nu}V^{\nu} = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$
= Lorentz Scalar

SR → QM

$\eta_{\mu\nu}$
$\cdot$

A Tensor Study of Physical 4-Vectors

SRQM Diagram:
The Basis of Classical SR Physics
Lorentz Scalar (Dot) Product ($\eta_{\mu\nu} = \cdot$)

4-Vector SRQM Interpretation of QM

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The Tensor Invariant Lorentz Scalar Product is the SR 4D (Dot) Product.
It is used to make Invariant Lorentz Scalars from two 4-Vectors.

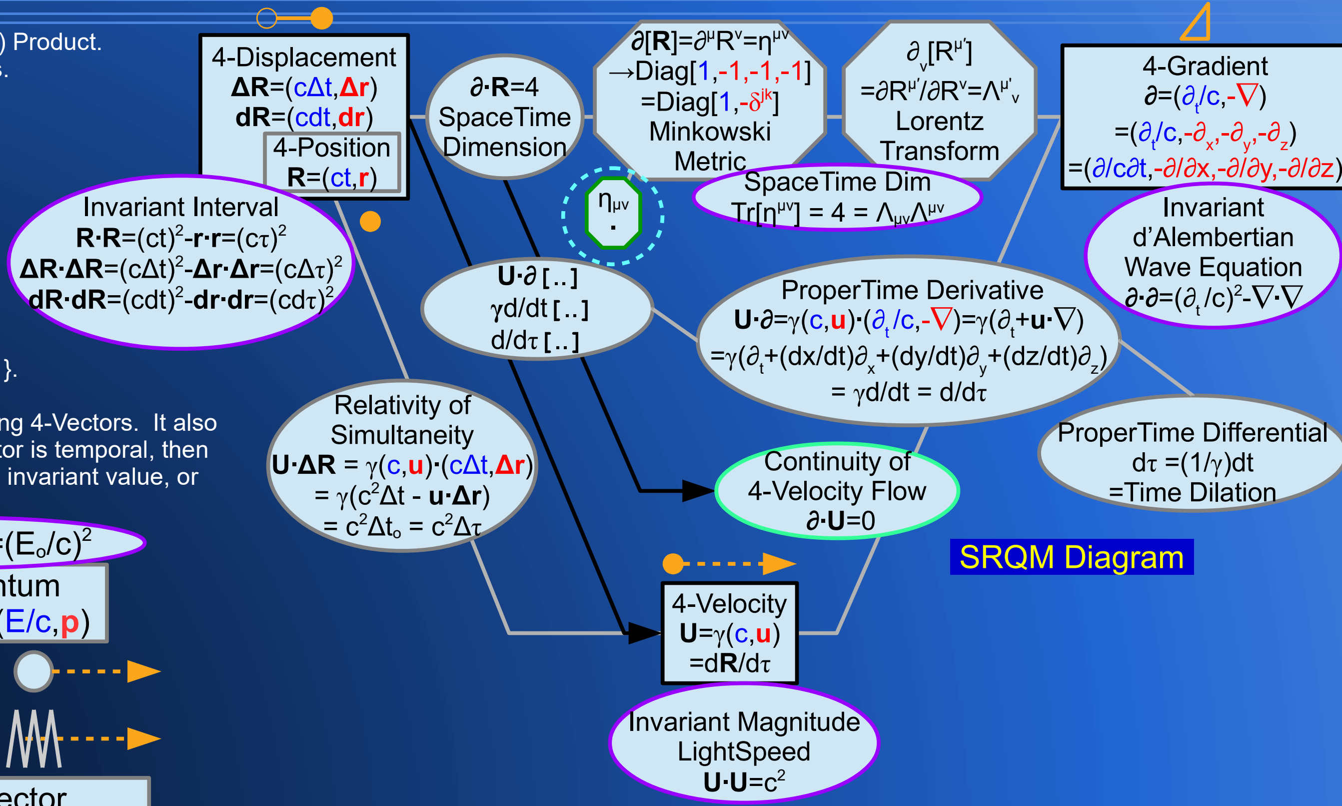
$\mathbf{A\cdot B} = A^\mu\cdot B^\nu = A^\mu\eta_{\mu\nu}B^\nu = A_\nu B^\nu = A^\mu B_\mu = (a^0b^0 - \mathbf{a\cdot b}) = (a^0{}_o b^0{}_o)$

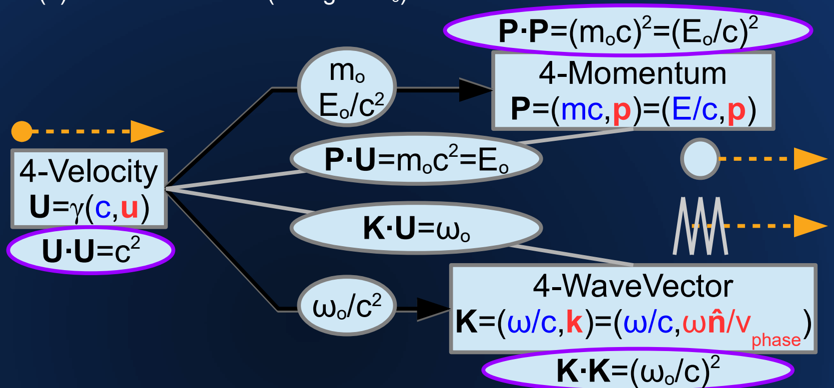
$\mathbf{A\cdot A} = A^\mu\cdot A^\nu = A^\mu\eta_{\mu\nu}A^\nu = A_\nu A^\nu = A^\mu A_\mu = (a^0a^0 - \mathbf{a\cdot a}) = (a^0{}_o)^2$

$\eta_{\mu\nu}\ \cdot = \eta_{\mu\nu}\ \cdot\ \hat{\mathbf{e}}_\mu\hat{\mathbf{e}}_\nu$ →Diag[+1,-1,-1,-1]{ Cartesian }
with $\hat{\mathbf{e}}_\mu$ and $\hat{e}_\nu$ as basis vectors
$A = A^\mu\hat{e}_\mu \rightarrow A^\mu$ { Cartesian }

($\eta_{\mu\nu}$) is itself just the lowered-index form of the SR Minkowski Metric ($\eta_{\mu\nu}$), with individual components [$\eta_{\mu\mu}$] = 1/[$\eta^{\mu\mu}$], else 0. In Cartesian basis, this gives { $\eta_{\mu\nu} = \eta^{\mu\nu}$ }.

It is used in just about every relation between any two interesting 4-Vectors. It also gives the Invariant Magnitude of a single 4-Vector. If the 4-Vector is temporal, then the spatial component can be set to zero, giving the rest-frame invariant value, or the (o)bserver rest value ("naught" = $_o$).





a^0 or a_0: $(0)^{th}$ = temporal component (can relativistically vary)
a_o: (o)bserver's rest-frame Invariant value (does not vary)

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^\mu{}_\nu$ or $T_\mu{}^\nu$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0,\mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0,-\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

$Trace[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu{}_\mu = T$
$\mathbf{V\cdot V} = V^\mu\eta_{\mu\nu}V^\nu = [(v^0)^2 - \mathbf{v\cdot v}] = (v^0{}_o)^2$
= Lorentz Scalar

SR → QM

A Tensor Study
of Physical 4-Vectors

SRQM Diagram:
The Basis of Classical SR Physics
4-Velocity U, <Event> Motion

4-Vector SRQM Interpretation
of QM

SciRealm.org
John B. Wilson

4-Velocity **U**=γ(c,**u**)=(γc,γ**u**)=(**U**·∂)**R**=(d/dτ)**R**=
=d**R**/dτ=(dt/dτ)(d**R**/dt)=γ(d**R**/dt)=γ(cṫ,**ṙ**)=γ(c,**u**)

4-Velocity **U** is the ProperTime Derivative d/dτ
of the 4-Position **R** or of the 4-Displacement **ΔR**.

It is the SR 4-Vector that describes
the motion of <Event>'s through SpaceTime.
For an un-accelerated observer,
the 4-Velocity is along the WorldLine at all points.
For an accelerated observer,
the 4-Velocity is still tangent to the WorldLine at each point,
but changes direction as the WorldLine bends.

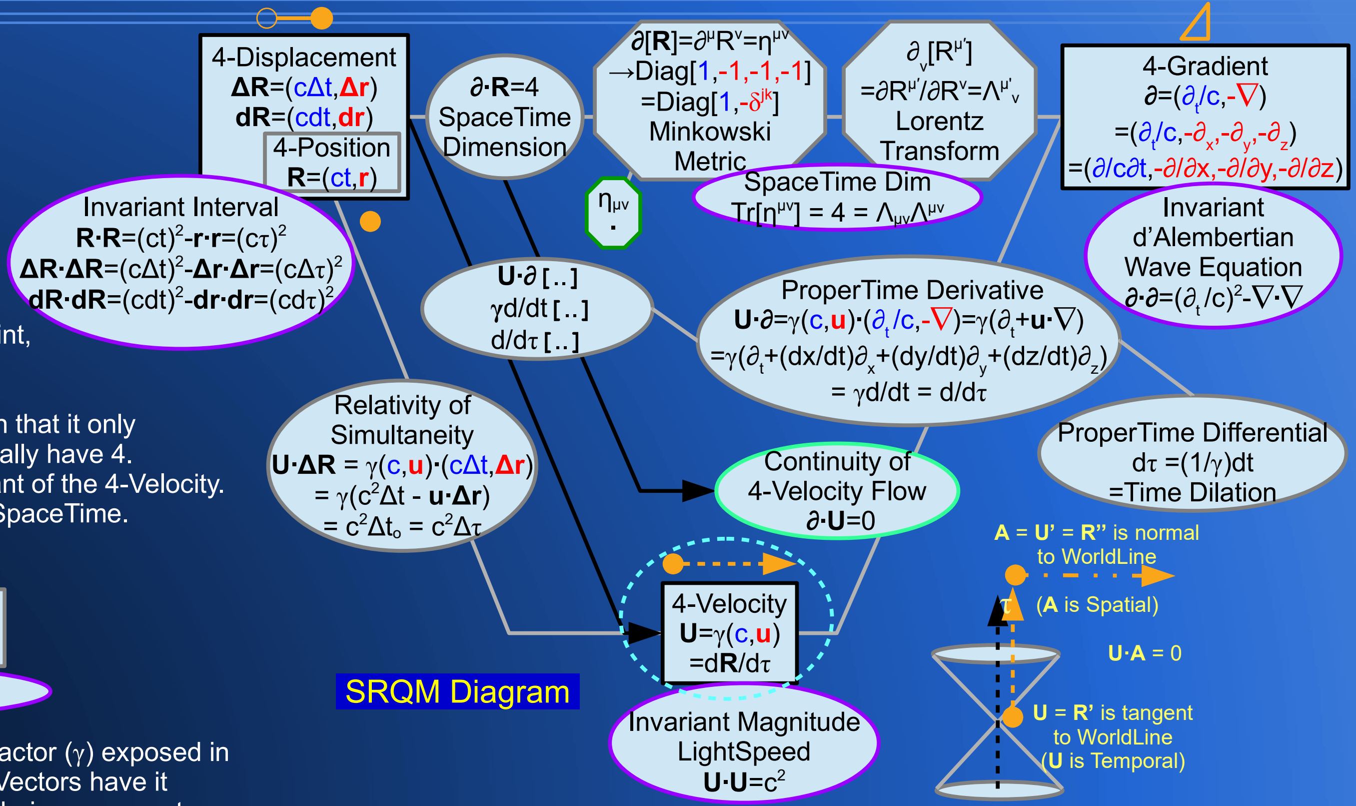
The 4-Velocity is unlike most of the other SR 4-Vectors in that it only
has 3 independent components, whereas the others usually have 4.
This is due to the constraint placed by the Tensor Invariant of the 4-Velocity.
U·**U** has a constant magnitude, the speed-of-light (c) in SpaceTime.

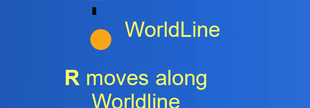
Components:
3 independent + 1 independent = 4 independent

4-Velocity **U**=γ(c,**u**) → m_o, E_o/c^2 → 4-Momentum **P**=(mc,**p**)=(E/c,**p**)

U·**U**=c^2

P·**P**=$(m_o c)^2=(E_o/c)^2$





The 4-Velocity also usually has the Relativistic Gamma factor (γ) exposed in
component form, whereas most of the other temporal 4-Vectors have it
absorbed into the Lorentz 4-Scalar factor that goes into their components.

4-Velocity **U** = γ(c,**u**) = (γc,γ**u**)
4-Momentum **P** = (mc,**p**) = m_o**U** = γm_o(c,**u**) = m(c,**u**) = (mc,m**u**) = (E/c,**p**)

P = m_o**U**
The temporal components give
Einstein's famous $E = mc^2 = \gamma m_o c^2 = \gamma E_o$
The spatial components give
p = m**u** = γm_o**u**

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(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^{\mu}{}_{\nu}$ or $T_{\mu}{}^{\nu}$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor V^{μ} = **V** = (v^0,**v**)
SR 4-CoVector
(0,1)-Tensor V_{μ} = (v_0,-**v**)

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^{\mu}{}_{\mu} = T$
V·**V** = $V^{\mu}\eta_{\mu\nu}V^{\nu} = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0_o)^2$
= Lorentz Scalar

SR → QM

SRQM Diagram:
The Basis of Classical SR Physics
4-Velocity Magnitude = Invariant Speed-of-Light (c)

4-Vector SRQM Interpretation of QM

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson

4-Velocity $\mathbf{U}=\gamma(c,\mathbf{u})=(\gamma c,\gamma\mathbf{u})=(\mathbf{U}\cdot\partial)\mathbf{R}=(d/d\tau)\mathbf{R}=$
$=d\mathbf{R}/d\tau=(dt/d\tau)(d\mathbf{R}/dt)=\gamma(d\mathbf{R}/dt)=\gamma(c\dot{t},\dot{\mathbf{r}})=\gamma(c,\mathbf{u})$

with Relativistic Gamma $\gamma = 1/\sqrt{[1 - \boldsymbol{\beta}\cdot\boldsymbol{\beta}]}$, $\boldsymbol{\beta} = \mathbf{u}/c$

The Lorentz Scalar Product of the 4-Velocity gives the Invariant Magnitude Speed-of-Light (c), one the main fundamental SR physical constants of physics. Technically, it is the maximum speed of SR causality, which any massless particles, ex. the photon, travel at.

$\mathbf{U}\cdot\mathbf{U}$
$= \gamma(c,\mathbf{u})\cdot\gamma(c,\mathbf{u})$
$= \gamma^2(c^2 - \mathbf{u}\cdot\mathbf{u})$
$= [1/(1 - \boldsymbol{\beta}\cdot\boldsymbol{\beta})](c^2 - \mathbf{u}\cdot\mathbf{u}) = [1/(1 - \boldsymbol{\beta}\cdot\boldsymbol{\beta})]c^2(1 - \boldsymbol{\beta}\cdot\boldsymbol{\beta})$
$= c^2$: Invariant Magnitude Speed-of-Light (c)

This fundamental constant Invariant (c) provides an extra constraint on the components of 4-Velocity $\mathbf{U}$, making it have only 3 independent components ($\mathbf{u}$).

This allows one to make new 4-Vectors related to 4-Velocity by multiplying by other Lorentz Scalars.
(Lorentz Scalar)*(4-Velocity) = (New 4-Vector)

$\mathbf{P} = (mc,\mathbf{p}) = m_o\mathbf{U}$
$\mathbf{K} = (\omega/c,\mathbf{k}) = (\omega_o/c^2)\mathbf{U}$

4-Velocity $\mathbf{U}=\gamma(c,\mathbf{u})$
Components: 3 independent
$\mathbf{U}\cdot\mathbf{U}=c^2$

m_o
E_o/c^2
ω_o/c^2
+1 independent

4-Momentum $\mathbf{P}=(mc,\mathbf{p})=(E/c,\mathbf{p})$
$\mathbf{P}\cdot\mathbf{P}=(m_oc)^2=(E_o/c)^2$

= 4 independent

4-WaveVector $\mathbf{K}=(\omega/c,\mathbf{k})=(\omega/c,\omega\hat{\mathbf{n}}/v_{phase})$
$\mathbf{K}\cdot\mathbf{K}=(\omega_o/c)^2$

The newly made 4-Vector thus has {3+1 = 4} independent components.

4-Displacement
$\Delta\mathbf{R}=(c\Delta t,\Delta\mathbf{r})$
$d\mathbf{R}=(cdt,d\mathbf{r})$

4-Position
$\mathbf{R}=(ct,\mathbf{r})$

Invariant Interval
$\mathbf{R}\cdot\mathbf{R}=(ct)^2-\mathbf{r}\cdot\mathbf{r}=(c\tau)^2$
$\Delta\mathbf{R}\cdot\Delta\mathbf{R}=(c\Delta t)^2-\Delta\mathbf{r}\cdot\Delta\mathbf{r}=(c\Delta\tau)^2$
$d\mathbf{R}\cdot d\mathbf{R}=(cdt)^2-d\mathbf{r}\cdot d\mathbf{r}=(cd\tau)^2$

$\partial\cdot\mathbf{R}=4$
SpaceTime Dimension

$\partial[\mathbf{R}]=\partial^\mu R^\nu=\eta^{\mu\nu}$
$\rightarrow$Diag[1,-1,-1,-1]
=Diag[1,$-\delta^{jk}$]
Minkowski Metric

$\eta_{\mu\nu}$

SpaceTime Dim
$Tr[\eta^{\mu\nu}] = 4 = \Lambda_{\mu\nu}\Lambda^{\mu\nu}$

$\partial_\nu[R^{\mu'}]$
$=\partial R^{\mu'}/\partial R^\nu=\Lambda^{\mu'}{}_\nu$
Lorentz Transform

4-Gradient
$\partial=(\partial_t/c,-\nabla)$
$=(\partial_t/c,-\partial_x,-\partial_y,-\partial_z)$
$=(\partial/c\partial t,-\partial/\partial x,-\partial/\partial y,-\partial/\partial z)$

Invariant d'Alembertian Wave Equation
$\partial\cdot\partial=(\partial_t/c)^2-\nabla\cdot\nabla$

$\mathbf{U}\cdot\partial$ [..]
$\gamma d/dt$ [..]
$d/d\tau$ [..]

ProperTime Derivative
$\mathbf{U}\cdot\partial=\gamma(c,\mathbf{u})\cdot(\partial_t/c,-\nabla)=\gamma(\partial_t+\mathbf{u}\cdot\nabla)$
$=\gamma(\partial_t+(dx/dt)\partial_x+(dy/dt)\partial_y+(dz/dt)\partial_z)$
$= \gamma d/dt = d/d\tau$

ProperTime Differential
$d\tau =(1/\gamma)dt$
=Time Dilation

Relativity of Simultaneity
$\mathbf{U}\cdot\Delta\mathbf{R} = \gamma(c,\mathbf{u})\cdot(c\Delta t,\Delta\mathbf{r})$
$= \gamma(c^2\Delta t - \mathbf{u}\cdot\Delta\mathbf{r})$
$= c^2\Delta t_o = c^2\Delta\tau$

Continuity of 4-Velocity Flow
$\partial\cdot\mathbf{U}=0$

4-Velocity
$\mathbf{U}=\gamma(c,\mathbf{u})$
$=d\mathbf{R}/d\tau$

Invariant Magnitude LightSpeed
$\mathbf{U}\cdot\mathbf{U}=c^2$

SRQM Diagram

An interesting thing to note is that all <event>'s move at the speed-of-light (c) in 4D magnitude. Massive at-rest particles simply travel at (c) temporally $\mathbf{U}_o = (c,\mathbf{0})$, while photons can move at (c) spatially as well.

If (c) was not a constant, but varied somehow, then all 4-Vectors made from the 4-Velocity would have more than 4 independent components, which is not observed. It seems a compelling argument against variable light-speed theories.

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(1,1)-Tensor $T^\mu{}_\nu$ or $T_\mu{}^\nu$
(0,2)-Tensor $T_{\mu\nu}$

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(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0,\mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0,-\mathbf{v})$

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(0,0)-Tensor S
Lorentz Scalar

$Trace[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu{}_\mu = T$
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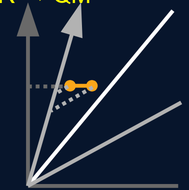
SR → QM



A Tensor Study
of Physical 4-Vectors

SRQM Diagram:
The Basis of Classical SR Physics
Relativity of Simultaneity

4-Vector SRQM Interpretation
of QM

SciRealm.org
John B. Wilson

Relativity of Simultaneity:
$\mathbf{U}\cdot\Delta\mathbf{X} = \gamma(c,\mathbf{u})\cdot(c\Delta t,\Delta\mathbf{x}) = \gamma(c^2\Delta t - \mathbf{u}\cdot\Delta\mathbf{x}) = c^2\Delta t_o = c^2\Delta\tau$

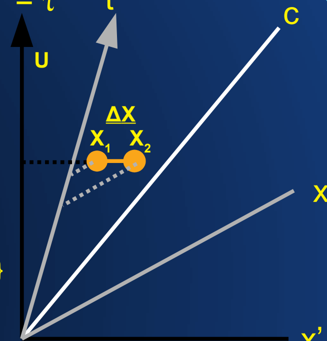
If Lorentz Scalar ($\mathbf{U}\cdot\Delta\mathbf{X} = 0 = c^2\Delta\tau$),
then the ProperTime displacement ($\Delta\tau$) is zero,
and the <Event>'s separation ($\Delta\mathbf{X} = \mathbf{X}_2 - \mathbf{X}_1$) is orthogonal
to the worldline at **U**.

<Event>'s $\mathbf{X}_1$ and $\mathbf{X}_2$ are therefore simultaneous ($\Delta\tau = 0$)
for the observer on this worldline at **U**.

Examining the equation we get $\gamma(c^2\Delta t - \mathbf{u}\cdot\Delta\mathbf{x}) = 0$.
The coordinate time difference between the events is ($\Delta t = \mathbf{u}\cdot\Delta\mathbf{x}/c^2$)
The condition for simultaneity in an alternate frame
(moving at 3-velocity **u** wrt. the worldline **U**) is $\Delta t = 0$,
which implies $(\mathbf{u}\cdot\Delta\mathbf{x}) = 0$.

This condition can be met by:
($|\mathbf{u}| = 0$), the alternate observer is not moving wrt. the events,
i.e. is on worldline **U** or on a worldline parallel to **U**.
($|\Delta\mathbf{x}| = 0$), the events are at the same spatial location (co-local).
($\mathbf{u}\cdot\Delta\mathbf{x} = 0 = |\mathbf{u}||\Delta\mathbf{x}|\cos[\theta]$), the alternate observer's motion is
perpendicular (orthogonal θ=90°) to the spatial separation $\Delta\mathbf{x}$
of the events in that frame.

If none of these conditions is met,
then the events will not be simultaneous
in the alternate reference-frame.



4-Displacement
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$d\mathbf{R}=(cdt,d\mathbf{r})$

4-Position
$\mathbf{R}=(ct,\mathbf{r})$

Invariant Interval
$\mathbf{R}\cdot\mathbf{R}=(ct)^2-\mathbf{r}\cdot\mathbf{r}=(c\tau)^2$
$\Delta\mathbf{R}\cdot\Delta\mathbf{R}=(c\Delta t)^2-\Delta\mathbf{r}\cdot\Delta\mathbf{r}=(c\Delta\tau)^2$
$d\mathbf{R}\cdot d\mathbf{R}=(cdt)^2-d\mathbf{r}\cdot d\mathbf{r}=(cd\tau)^2$

$\partial\cdot\mathbf{R}=4$
SpaceTime
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Lorentz
Transform

$\eta_{\mu\nu}$
·

SpaceTime Dim
$\mathrm{Tr}[\eta^{\mu\nu}] = 4 = \Lambda_{\mu\nu}\Lambda^{\mu\nu}$

4-Gradient
$\partial=(\partial_t/c,-\nabla)$
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Invariant
d'Alembertian
Wave Equation
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$\mathbf{U}\cdot\partial[..]$
$\gamma d/dt[..]$
$d/d\tau[..]$

ProperTime Derivative
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$=\gamma(\partial_t+(dx/dt)\partial_x+(dy/dt)\partial_y+(dz/dt)\partial_z)$
$= \gamma d/dt = d/d\tau$

ProperTime Differential
$d\tau =(1/\gamma)dt$
=Time Dilation

Relativity of
Simultaneity
$\mathbf{U}\cdot\Delta\mathbf{R} = \gamma(c,\mathbf{u})\cdot(c\Delta t,\Delta\mathbf{r})$
$= \gamma(c^2\Delta t - \mathbf{u}\cdot\Delta\mathbf{r})$
$= c^2\Delta t_o = c^2\Delta\tau$

Continuity of
4-Velocity Flow
$\partial\cdot\mathbf{U}=0$

SRQM Diagram

4-Velocity
$\mathbf{U}=\gamma(c,\mathbf{u})$
$=d\mathbf{R}/d\tau$

Invariant Magnitude
LightSpeed
$\mathbf{U}\cdot\mathbf{U}=c^2$

Realizing that simultaneity is not an invariant concept
was the breakthrough that lead Einstein to Special Relativity

SR 4-Tensor
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SR 4-Scalar
(0,0)-Tensor S
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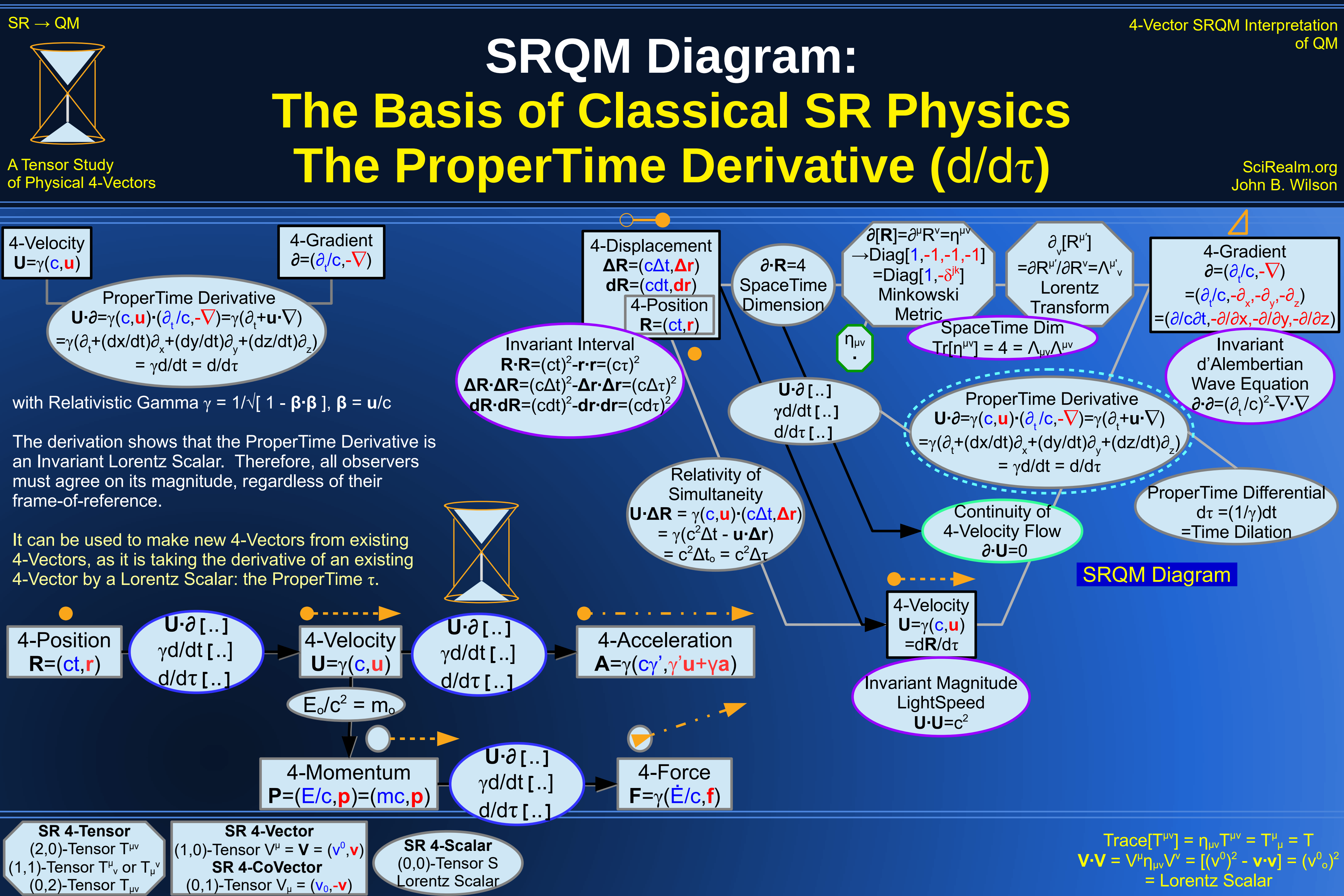
= Lorentz Scalar



SR → QM
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The ProperTime Derivative (d/dτ)
A Tensor Study of Physical 4-Vectors
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4-Gradient $\partial=(\partial_t/c,-\nabla)$
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$=\gamma(\partial_t+(dx/dt)\partial_x+(dy/dt)\partial_y+(dz/dt)\partial_z)$
$= \gamma d/dt = d/d\tau$
with Relativistic Gamma $\gamma = 1/\sqrt{[\ 1 - \boldsymbol{\beta}\cdot\boldsymbol{\beta}\ ]}$, $\boldsymbol{\beta} = \mathbf{u}/c$
The derivation shows that the ProperTime Derivative is an Invariant Lorentz Scalar. Therefore, all observers must agree on its magnitude, regardless of their frame-of-reference.
It can be used to make new 4-Vectors from existing 4-Vectors, as it is taking the derivative of an existing 4-Vector by a Lorentz Scalar: the ProperTime τ.
4-Displacement
$\Delta\mathbf{R}=(c\Delta t,\Delta\mathbf{r})$
$d\mathbf{R}=(cdt,d\mathbf{r})$
4-Position $\mathbf{R}=(ct,\mathbf{r})$
Invariant Interval
$\mathbf{R}\cdot\mathbf{R}=(ct)^2-\mathbf{r}\cdot\mathbf{r}=(c\tau)^2$
$\Delta\mathbf{R}\cdot\Delta\mathbf{R}=(c\Delta t)^2-\Delta\mathbf{r}\cdot\Delta\mathbf{r}=(c\Delta\tau)^2$
$d\mathbf{R}\cdot d\mathbf{R}=(cdt)^2-d\mathbf{r}\cdot d\mathbf{r}=(cd\tau)^2$
$\partial\cdot\mathbf{R}=4$
SpaceTime Dimension
$\partial[\mathbf{R}]=\partial^\mu R^\nu=\eta^{\mu\nu}$
$\rightarrow$Diag[1,-1,-1,-1]
=Diag[1,-δ^{jk}]
Minkowski Metric
$\partial_\nu[R^{\mu'}]$
$=\partial R^{\mu'}/\partial R^\nu=\Lambda^{\mu'}{}_\nu$
Lorentz Transform
SpaceTime Dim
$Tr[\eta^{\mu\nu}] = 4 = \Lambda_{\mu\nu}\Lambda^{\mu\nu}$
$\eta_{\mu\nu}$
4-Gradient
$\partial=(\partial_t/c,-\nabla)$
$=(\partial_t/c,-\partial_x,-\partial_y,-\partial_z)$
$=(\partial/c\partial t,-\partial/\partial x,-\partial/\partial y,-\partial/\partial z)$
Invariant d'Alembertian Wave Equation
$\partial\cdot\partial=(\partial_t/c)^2-\nabla\cdot\nabla$
U·∂ [..]
γd/dt [..]
d/dτ [..]
ProperTime Derivative
$\mathbf{U}\cdot\partial=\gamma(c,\mathbf{u})\cdot(\partial_t/c,-\nabla)=\gamma(\partial_t+\mathbf{u}\cdot\nabla)$
$=\gamma(\partial_t+(dx/dt)\partial_x+(dy/dt)\partial_y+(dz/dt)\partial_z)$
$= \gamma d/dt = d/d\tau$
Relativity of Simultaneity
$\mathbf{U}\cdot\Delta\mathbf{R} = \gamma(c,\mathbf{u})\cdot(c\Delta t,\Delta\mathbf{r})$
$= \gamma(c^2\Delta t - \mathbf{u}\cdot\Delta\mathbf{r})$
$= c^2\Delta t_o = c^2\Delta\tau$
Continuity of 4-Velocity Flow
$\partial\cdot\mathbf{U}=0$
ProperTime Differential
$d\tau=(1/\gamma)dt$
=Time Dilation
SRQM Diagram
4-Velocity
$\mathbf{U}=\gamma(c,\mathbf{u})$
$=d\mathbf{R}/d\tau$
Invariant Magnitude LightSpeed
$\mathbf{U}\cdot\mathbf{U}=c^2$
4-Position $\mathbf{R}=(ct,\mathbf{r})$
U·∂ [..]
γd/dt [..]
d/dτ [..]
4-Velocity $\mathbf{U}=\gamma(c,\mathbf{u})$
U·∂ [..]
γd/dt [..]
d/dτ [..]
4-Acceleration $\mathbf{A}=\gamma(c\gamma',\gamma'\mathbf{u}+\gamma\mathbf{a})$
$E_o/c^2 = m_o$
4-Momentum $\mathbf{P}=(E/c,\mathbf{p})=(mc,\mathbf{p})$
U·∂ [..]
γd/dt [..]
d/dτ [..]
4-Force $\mathbf{F}=\gamma(\dot{E}/c,\mathbf{f})$
SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^\mu{}_\nu$ or $T_\mu{}^\nu$
(0,2)-Tensor $T_{\mu\nu}$
SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0,\mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0,-\mathbf{v})$
SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar
$Trace[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu{}_\mu = T$
$\mathbf{V}\cdot\mathbf{V} = V^\mu\eta_{\mu\nu}V^\nu = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$
= Lorentz Scalar

SR → QM

4-Vector SRQM Interpretation of QM

SRQM Diagram:
The Basis of Classical SR Physics
ProperTime Derivative on SR 4-Vectors and Scalars

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson

The ProperTime Derivative acting on SR 4-Vectors:

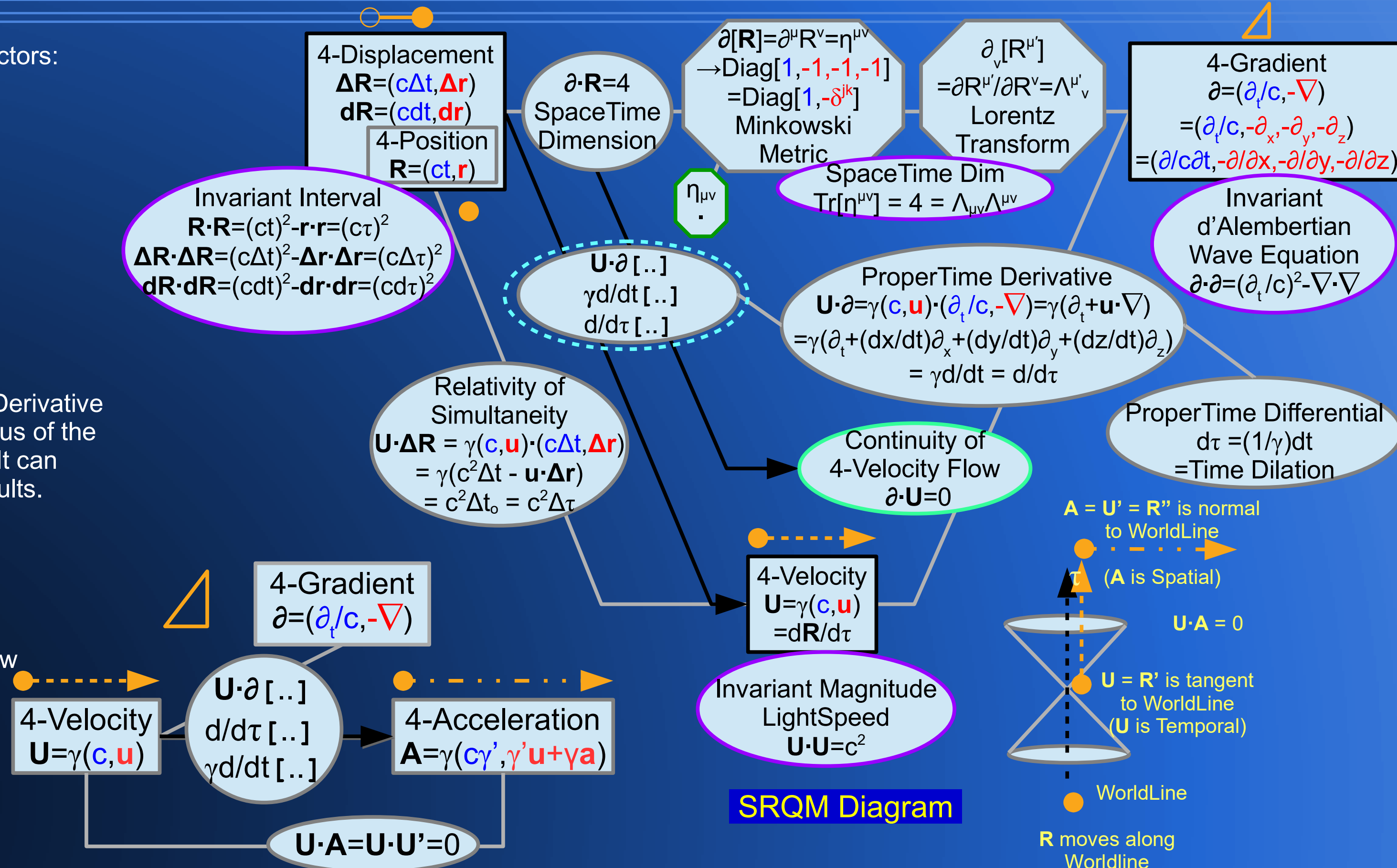
$\mathbf{U}\cdot\partial = \gamma(c,\mathbf{u})\cdot(\partial_t/c,-\nabla) = \gamma(\partial_t+\mathbf{u}\cdot\nabla) = \gamma d/dt = d/d\tau$

4-Vectors:
4-Position **R** = <Event>
4-Velocity **U** = d**R**/dτ
4-Acceleration **A** = d**U**/dτ
...
4-Momentum **P** = m_o**U**
4-Force **F** = d**P**/dτ

As one can see from the list, the ProperTime Derivative gives the 4-Vectors that are the change in status of the 4-Vector that ProperTime Derivative acts on. It can also act on Scalar Values to give deep SR results.

$\partial\cdot\mathbf{R} = 4$: SpaceTime Dimension is 4
$d/d\tau(\partial\cdot\mathbf{R}) = d/d\tau(4) = 0$
$d/d\tau(\partial\cdot\mathbf{R}) = d/d\tau[\partial]\cdot\mathbf{R} + \partial\cdot\mathbf{U} = 0$
...
$\partial\cdot\mathbf{U} = 0$: Conservation of the SR 4-Velocity Flow

$\mathbf{U}\cdot\mathbf{U} = c^2$: Tensor Invariant of 4-Velocity
$d/d\tau[\mathbf{U}\cdot\mathbf{U}] = d/d\tau[c^2] = 0$
$d/d\tau[\mathbf{U}\cdot\mathbf{U}] = d/d\tau[\mathbf{U}]\cdot\mathbf{U} + \mathbf{U}\cdot d/d\tau[\mathbf{U}] = 2(\mathbf{U}\cdot\mathbf{A}) = 0$
$\mathbf{U}\cdot\mathbf{A} = \mathbf{U}\cdot\mathbf{U}' = 0$: The 4-Velocity is SpaceTime orthogonal to it's own 4-Acceleration



SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^{\mu}{}_{\nu}$ or $T_{\mu}{}^{\nu}$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^{\mu} = \mathbf{V} = (v^0,\mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_{\mu} = (v_0,-\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^{\mu}{}_{\mu} = T$
$\mathbf{V}\cdot\mathbf{V} = V^{\mu}\eta_{\mu\nu}V^{\nu} = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$
= Lorentz Scalar

SR → QM

SRQM Diagram:
The Basis of Classical SR Physics
ProperTime Differential ($d\tau$) →
Time Dilation & Length Contraction

4-Vector SRQM Interpretation of QM

A Tensor Study of Physical 4-Vectors

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John B. Wilson

There are several ways to derive Time Dilation.

ProperTime Derivative (Lorentz 4-Scalar):
$\mathbf{U}\cdot\partial = \gamma(c,\mathbf{u})\cdot(\partial_t/c,-\nabla) = \gamma(\partial_t+\mathbf{u}\cdot\nabla) = \gamma d/dt = d/d\tau$

ProperTime Differential (Lorentz 4-Scalar): $d\tau = (1/\gamma)dt$

$\mathbf{dR}\cdot\mathbf{dR}=(cd\tau)^2$ ← $d\tau^2$ — $\mathbf{U}\cdot\mathbf{U}=c^2$

4-Differential $\mathbf{dR}=(cdt,\mathbf{dr})$ ← $d\tau$ — 4-Velocity $\mathbf{U}=\gamma(c,\mathbf{u})=\mathbf{dR}/d\tau$

Take the temporal component of the 4-Vector relation.
$dt = \gamma d\tau = \gamma dt_o$
$\Delta t = \gamma\Delta\tau = \gamma\Delta t_o$: Time Dilation!

The coordinate time Δt measured by an observer is "dilated", compared to the ProperTime as measured by a clock moving with the object. This has the effect that moving objects appear to age more slowly than at-rest objects. The effect is reciprocal as well. Since velocity is relative, each observer will see the other as ageing more slowly, similarly to the effect that each will appear smaller to the other when seen at a distance.

Now multiply both sides by the moving-frame speed [v].
$v\Delta t = \gamma v\Delta\tau$
$v\Delta t$ = distance L_o the moving clock travels wrt. frame, which is a proper (fixed-to-frame) displacement length.
$L_o = \gamma L$
$L = (1/\gamma)L_o$: Length Contraction!

4-Displacement
$\mathbf{\Delta R}=(c\Delta t,\mathbf{\Delta r})$
$\mathbf{dR}=(cdt,\mathbf{dr})$

4-Position
$\mathbf{R}=(ct,\mathbf{r})$

$\partial\cdot\mathbf{R}=4$
SpaceTime Dimension

$\partial[\mathbf{R}]=\partial^\mu R^\nu=\eta^{\mu\nu}$
$\rightarrow \mathrm{Diag}[1,-1,-1,-1]$
$=\mathrm{Diag}[1,-\delta^{jk}]$
Minkowski Metric

$\partial_\nu[R^{\mu'}]$
$=\partial R^{\mu'}/\partial R^\nu=\Lambda^{\mu'}{}_\nu$
Lorentz Transform

4-Gradient
$\partial=(\partial_t/c,-\nabla)$
$=(\partial_t/c,-\partial_x,-\partial_y,-\partial_z)$
$=(\partial/c\partial t,-\partial/\partial x,-\partial/\partial y,-\partial/\partial z)$

SpaceTime Dim
$\mathrm{Tr}[\eta^{\mu\nu}] = 4 = \Lambda_{\mu\nu}\Lambda^{\mu\nu}$

$\eta_{\mu\nu}$

Invariant Interval
$\mathbf{R}\cdot\mathbf{R}=(ct)^2-\mathbf{r}\cdot\mathbf{r}=(c\tau)^2$
$\mathbf{\Delta R}\cdot\mathbf{\Delta R}=(c\Delta t)^2-\mathbf{\Delta r}\cdot\mathbf{\Delta r}=(c\Delta\tau)^2$
$\mathbf{dR}\cdot\mathbf{dR}=(cdt)^2-\mathbf{dr}\cdot\mathbf{dr}=(cd\tau)^2$

Invariant d'Alembertian Wave Equation
$\partial\cdot\partial=(\partial_t/c)^2-\nabla\cdot\nabla$

$\mathbf{U}\cdot\partial\,[..]$
$\gamma d/dt\,[..]$
$d/d\tau\,[..]$

ProperTime Derivative
$\mathbf{U}\cdot\partial=\gamma(c,\mathbf{u})\cdot(\partial_t/c,-\nabla)=\gamma(\partial_t+\mathbf{u}\cdot\nabla)$
$=\gamma(\partial_t+(dx/dt)\partial_x+(dy/dt)\partial_y+(dz/dt)\partial_z)$
$=\gamma d/dt = d/d\tau$

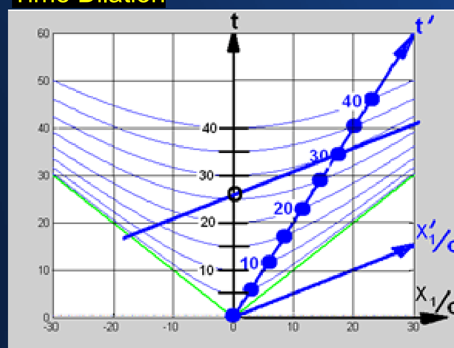
SRQM Diagram

Relativity of Simultaneity
$\mathbf{U}\cdot\mathbf{\Delta R} = \gamma(c,\mathbf{u})\cdot(c\Delta t,\mathbf{\Delta r})$
$=\gamma(c^2\Delta t - \mathbf{u}\cdot\mathbf{\Delta r})$
$=c^2\Delta t_o = c^2\Delta\tau$

Continuity of 4-Velocity Flow
$\partial\cdot\mathbf{U}=0$

ProperTime Differential
$d\tau=(1/\gamma)dt$
=Time Dilation

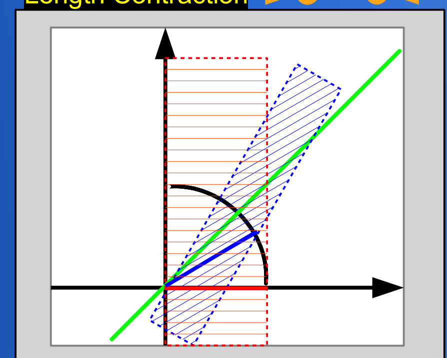
Time Dilation



4-Velocity
$\mathbf{U}=\gamma(c,\mathbf{u})$
$=\mathbf{dR}/d\tau$

Invariant Magnitude LightSpeed
$\mathbf{U}\cdot\mathbf{U}=c^2$

Length Contraction



Red and Blue lengths equal in the moving frame, blue is contracted in the ProperTime frame

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^\mu{}_\nu$ or $T_\mu{}^\nu$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0,\mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0,-\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu{}_\mu = T$
$\mathbf{V}\cdot\mathbf{V} = V^\mu\eta_{\mu\nu}V^\nu = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$
= Lorentz Scalar

SR → QM

SRQM Diagram:

The Basis of Classical SR Physics

4-Gradient ∂, SR 4-Vector Function:Operator

4-Vector SRQM Interpretation of QM

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson

4-Gradient
$\partial=\partial^\mu=(\partial_t/c,-\nabla)$
$=(\partial_t/c,-\partial_x,-\partial_y,-\partial_z)$
$=(\partial/c\partial t,-\partial/\partial x,-\partial/\partial y,-\partial/\partial z)$

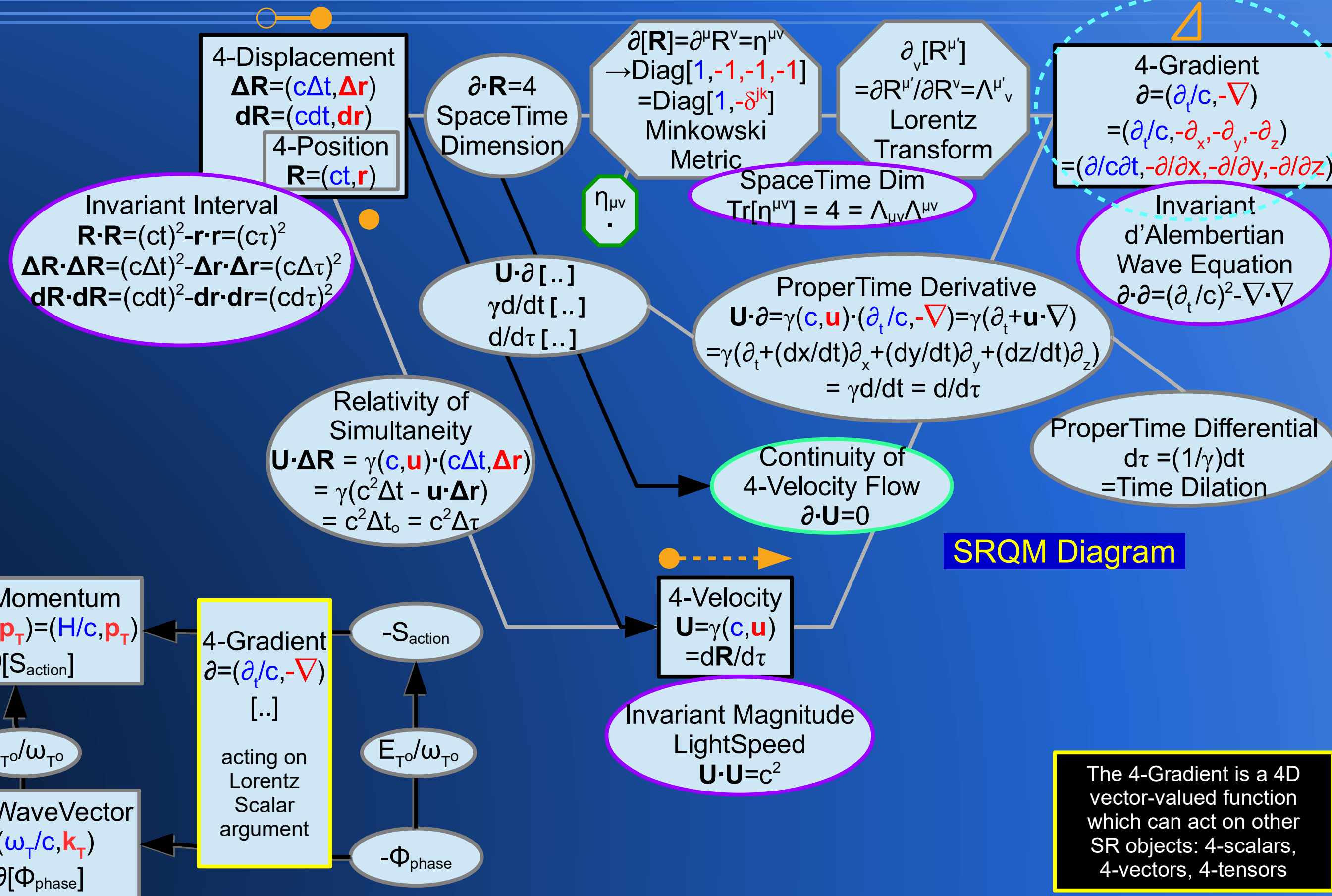
The 4-Gradient $(\partial^\mu)=(\partial_t/c,-\nabla)$ is the index-raised version of the SR Gradient One-Form $(\partial_\mu)=(\partial_t/c,\nabla)$.
It is the 4D version of the partial derivative function of calculus.

It is a 4-Vector function that can act on other 4-Vectors and 4-Scalars. The 4-Gradient tells how things change wrt. time and space.

It is instrumental in creating the ProperTime Derivative
$\mathbf{U}\cdot\partial = \gamma d/dt = d/d\tau$.

The 4-Gradient plays a major role in advanced physics, showing how SR waves are formed, creating the Hamilton-Jacobi equations, the Euler-Lagrange equations, Conservation equations, Maxwell's Equations, the Lorenz Gauge, etc. It gives the Dimension of SpaceTime, the Minkowski Metric, and the Lorentz Transformations. In QM, it provides the Schrödinger relations.

It is fundamental in connecting SR to QM.



SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^\mu{}_\nu$ or $T_\mu{}^\nu$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0,\mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0,-\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Hamilton-Jacobi Equation: $\mathbf{P_T} = -\partial[S_{action}]$
SR Plane-Wave Equation: $\mathbf{K_T} = -\partial[\Phi_{phase}]$

$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu{}_\mu = T$
$\mathbf{V}\cdot\mathbf{V} = V^\mu\eta_{\mu\nu}V^\nu = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$
= Lorentz Scalar

SR → QM

4-Vector SRQM Interpretation of QM

SRQM Diagram:
The Basis of Classical SR Physics
Invariant d'Alembertian Wave Equation ($\partial\cdot\partial$)

A Tensor Study of Physical 4-Vectors

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The Lorentz Scalar Invariant of the 4-Gradient gives the Invariant d'Alembertian Wave Equation, describing SR wave motion. It is seen in the SR Maxwell Equation for EM light waves.

$\partial\cdot\partial=(\partial_t/c)^2-\nabla\cdot\nabla$ d'Alembertian

μ_o

$(\partial\cdot\partial)\mathbf{A}-\partial(\partial\cdot\mathbf{A})=\mu_o\mathbf{J}$ Maxwell EM Wave Eqn

Lorenz Gauge Conservation of EM Potential: $\partial\cdot\mathbf{A}=0$

Importantly, the d'Alembertian is fully from basic SR rules, with no quantum axioms required. However, it will be seen again in the Klein-Gordon QM equation.

It provides for the introduction of an SR Wave 4-Vector **K**, which can also be given by the negative Gradient of a Lorentz Scalar Phase.

4-WaveVector $\mathbf{K} = (\omega_o/c^2)\mathbf{U} = (\omega/c,\mathbf{k}) = -\partial[\Phi_{phase}] = \partial[\mathbf{K}\cdot\mathbf{R}]$

The usual mathematical (complex) plane-wave solutions apply in SR:
f = (a)*e^[±i(**K·R**)], with (a)mplitude possibly {4-Scalar S, 4-Vector V^μ, 4-Tensor $T^{\mu\nu}$}
{KG wave, EM wave , Grav wave}

4-Displacement $\Delta\mathbf{R}=(c\Delta t,\Delta\mathbf{r})$ $d\mathbf{R}=(cdt,d\mathbf{r})$

4-Position $\mathbf{R}=(ct,\mathbf{r})$

$\partial\cdot\mathbf{R}=4$ SpaceTime Dimension

$\partial[\mathbf{R}]=\partial^\mu R^\nu=\eta^{\mu\nu}$ →Diag[1,-1,-1,-1] =Diag[1,-δ^{jk}] Minkowski Metric

$\partial_\nu[R^{\mu'}]$ $=\partial R^{\mu'}/\partial R^\nu=\Lambda^{\mu'}{}_\nu$ Lorentz Transform

SpaceTime Dim $Tr[\eta^{\mu\nu}] = 4 = \Lambda_{\mu\nu}\Lambda^{\mu\nu}$

$\eta_{\mu\nu}$ ·

4-Gradient $\partial=(\partial_t/c,-\nabla)$ $=(\partial_t/c,-\partial_x,-\partial_y,-\partial_z)$ $=(\partial/c\partial t,-\partial/\partial x,-\partial/\partial y,-\partial/\partial z)$

Invariant d'Alembertian Wave Equation $\partial\cdot\partial=(\partial_t/c)^2-\nabla\cdot\nabla$

Invariant Interval
$\mathbf{R}\cdot\mathbf{R}=(ct)^2-\mathbf{r}\cdot\mathbf{r}=(c\tau)^2$
$\Delta\mathbf{R}\cdot\Delta\mathbf{R}=(c\Delta t)^2-\Delta\mathbf{r}\cdot\Delta\mathbf{r}=(c\Delta\tau)^2$
$d\mathbf{R}\cdot d\mathbf{R}=(cdt)^2-d\mathbf{r}\cdot d\mathbf{r}=(cd\tau)^2$

$\mathbf{U}\cdot\partial$ [..] $\gamma d/dt$ [..] $d/d\tau$ [..]

ProperTime Derivative
$\mathbf{U}\cdot\partial=\gamma(c,\mathbf{u})\cdot(\partial_t/c,-\nabla)=\gamma(\partial_t+\mathbf{u}\cdot\nabla)$
$=\gamma(\partial_t+(dx/dt)\partial_x+(dy/dt)\partial_y+(dz/dt)\partial_z)$
$= \gamma d/dt = d/d\tau$

ProperTime Differential $d\tau =(1/\gamma)dt$ =Time Dilation

Relativity of Simultaneity
$\mathbf{U}\cdot\Delta\mathbf{R} = \gamma(c,\mathbf{u})\cdot(c\Delta t,\Delta\mathbf{r})$
$= \gamma(c^2\Delta t - \mathbf{u}\cdot\Delta\mathbf{r})$
$= c^2\Delta t_o = c^2\Delta\tau$

Continuity of 4-Velocity Flow $\partial\cdot\mathbf{U}=0$

SRQM Diagram

4-Velocity $\mathbf{U}=\gamma(c,\mathbf{u})$ $=d\mathbf{R}/d\tau$

Invariant Magnitude LightSpeed $\mathbf{U}\cdot\mathbf{U}=c^2$

Invariant Phase
$\mathbf{K}\cdot\mathbf{R} = (\omega/c,\mathbf{k})\cdot(ct,\mathbf{r})$
$=(\omega t - \mathbf{k}\cdot\mathbf{r}) = -\Phi_{phase,plane}$

4-Position $\mathbf{R}=(ct,\mathbf{r})$

$\mathbf{R}\cdot\mathbf{R}=(c\tau)^2$

$\mathbf{U}\cdot\partial$ [..] $\gamma d/dt$ [..] $d/d\tau$ [..]

4-Velocity $\mathbf{U}=\gamma(c,\mathbf{u})$

$\mathbf{U}\cdot\mathbf{U}=c^2$

ω_o/c^2

4-WaveVector $\mathbf{K}=(\omega/c,\mathbf{k})$

$\mathbf{K}\cdot\mathbf{K}=(\omega_o/c)^2$

..[**K·R**] ..∫[**K·dR**] ..[-Φ_{phase}]

4-Gradient $\partial=(\partial_t/c,-\nabla)$

$\partial\cdot\partial=(\partial_t/c)^2-\nabla\cdot\nabla$

SR is the "natural" 4D arena for the description of waves, using the d'Alembertian
$\partial\cdot\partial = (\partial_t/c)^2-\nabla\cdot\nabla$

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^\mu{}_\nu$ or $T_\mu{}^\nu$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0,\mathbf{v})$
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SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

$Trace[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu{}_\mu = T$
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= Lorentz Scalar

SR → QM

A Tensor Study of Physical 4-Vectors

4-Vector SRQM Interpretation of QM

SRQM Diagram:
The Basis of Classical SR Physics
Continuity of 4-Velocity Flow ($\partial\cdot\mathbf{U}=0$)

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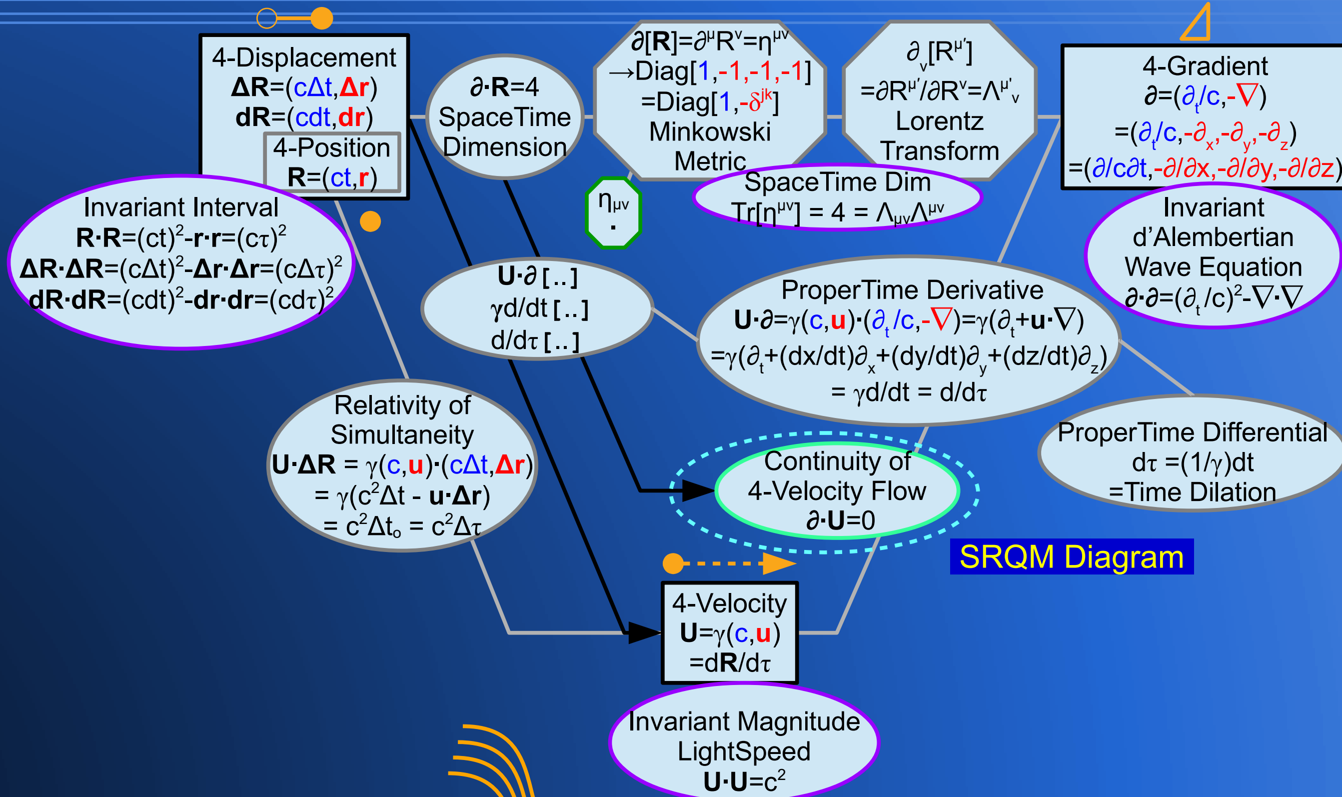
Continuity of 4-Velocity Flow $\partial\cdot\mathbf{U}=0$
This leads to all the SR Conservation Laws.

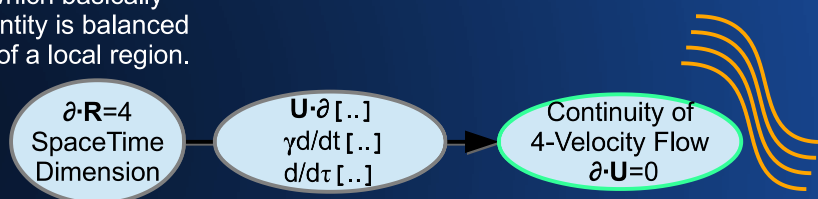
$\partial\cdot\mathbf{R} = 4$
$d/d\tau(\partial\cdot\mathbf{R}) = d/d\tau(4) = 0$
$d/d\tau(\partial\cdot\mathbf{R}) = d/d\tau(\partial)\cdot\mathbf{R} + \partial\cdot d/d\tau(\mathbf{R}) = 0$
$d/d\tau(\partial\cdot\mathbf{R}) = d/d\tau[\partial]\cdot\mathbf{R} + \partial\cdot\mathbf{U} = 0$
$\partial\cdot\mathbf{U} = -d/d\tau[\partial]\cdot\mathbf{R}$
$\partial\cdot\mathbf{U} = -(\mathbf{U}\cdot\partial)[\partial]\cdot\mathbf{R}$
$\partial\cdot\mathbf{U} = -(U_\nu\partial^\nu)[\partial_\mu]R^\mu$
$\partial\cdot\mathbf{U} = -U_\nu\partial^\nu\partial_\mu R^\mu$
$\partial\cdot\mathbf{U} = -U_\nu\partial_\mu\partial^\nu R^\mu$: I believe this is legit, partials commute
$\partial\cdot\mathbf{U} = -U_\nu\partial_\mu\eta^{\nu\mu}$
$\partial\cdot\mathbf{U} = -U_\nu(0^\nu)$
$\partial\cdot\mathbf{U} = 0$
Conservation of the 4-Velocity Flow
(4-Velocity Flow-Field)

All of the Physical Conservation Laws are in the form of a 4-Divergence, which is a Lorentz Invariant Scalar equation.

These are local continuity equations which basically say that the temporal change in a quantity is balanced by the flow of that quantity into or out of a local region.

Conservation of Charge:
$\rho_o\partial\cdot\mathbf{U} = \partial\cdot\rho_o\mathbf{U} = \partial\cdot\mathbf{J} = (\partial_t\rho + \nabla\cdot\mathbf{j}) = 0$





SR 4-Tensor
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SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0,\mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0,-\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu{}_\mu = T$
$\mathbf{V}\cdot\mathbf{V} = V^\mu\eta_{\mu\nu}V^\nu = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0_o)^2$
= Lorentz Scalar

SRQM Diagram:
The Basis of Classical SR Physics
<Event> Substantiation

A Tensor Study of Physical 4-Vectors

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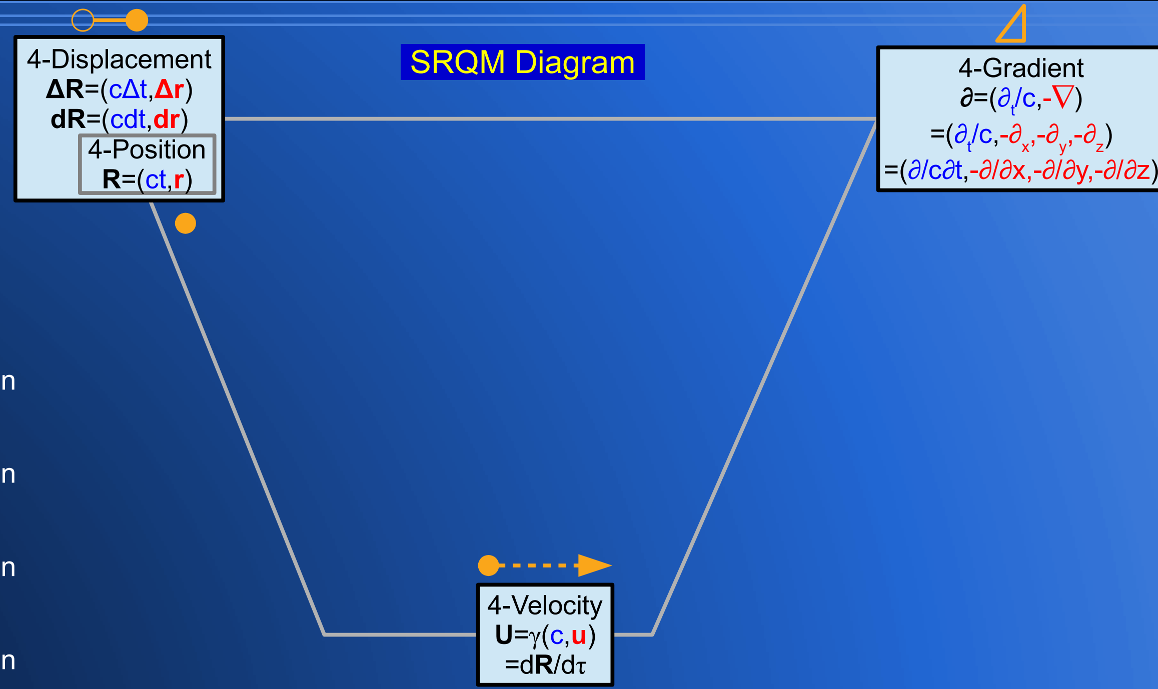
Now focus on six of the main SR 4-Vectors.

4-Position R^μ $\mathbf{R}=(ct,\mathbf{r})$=<Event>	<Event> Location
4-Velocity U^μ $\mathbf{U}=\gamma(c,\mathbf{u})$	<Event> Motion
4-Gradient ∂^μ $\partial=(\partial_t/c,-\nabla)$	<Event> Alteration
4-Momentum P^μ $\mathbf{P}=(E/c,\mathbf{p})=(mc,\mathbf{p})=(mc,m\mathbf{u})$ $=(E_o/c^2)\mathbf{U}=m_o\mathbf{U}$	<Event> Substantiation (particle:mass)
4-WaveVector K^μ $\mathbf{K}=(\omega/c,\mathbf{k})=(\omega/c,\omega\hat{\mathbf{n}}/v_{phase})$ $=(1/c\text{Ŧ},\hat{\mathbf{n}}/\text{ƛ})=(\omega_o/c^2)\mathbf{U}$	<Event> Substantiation (wave)
4-CurrentDensity:ChargeFlux J^μ $\mathbf{J}=(\rho c,\mathbf{j})=(\rho c,\rho\mathbf{u})$ $=(\rho_o)\mathbf{U}=(qn_o)\mathbf{U}=(q)\mathbf{N}$	<Event> Substantiation (charge Q)
4-(Dust)NumberFlux N^μ $\mathbf{N}=(nc,\mathbf{n})=(nc,n\mathbf{u})$ $=(n_o)\mathbf{U}$	<Event> Substantiation (dust:number N)



These seven give more of the main classical results of Special Relativity,
including SR concepts like:

SR Particles and Waves, Matter-Wave Dispersion

Einstein's $E = mc^2 = \gamma m_o c^2 = \gamma E_o$, Rest Mass, Rest Energy

Conservation of Charge (Q), Conservation of Particle Number (N), Continuity Equations

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^{\mu}{}_{\nu}$ or $T_{\mu}{}^{\nu}$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0,\mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0,-\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^{\mu}{}_{\mu} = T$

$\mathbf{V}\cdot\mathbf{V} = V^\mu\eta_{\mu\nu}V^\nu = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$
= Lorentz Scalar

SRQM Diagram:
The Basis of Classical SR Physics
4-Momentum, Einstein's $E = mc^2$

A Tensor Study of Physical 4-Vectors

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4-Position $\mathbf{R}=(ct,\mathbf{r})$
4-Gradient $\partial=(\partial_t/c,-\nabla)$
4-Velocity $\mathbf{U} = \gamma(c,\mathbf{u})$

4-Momentum $\mathbf{P} = (E/c,\mathbf{p}) = m_o\mathbf{U} = \gamma m_o(c,\mathbf{u}) = m(c,\mathbf{u})$

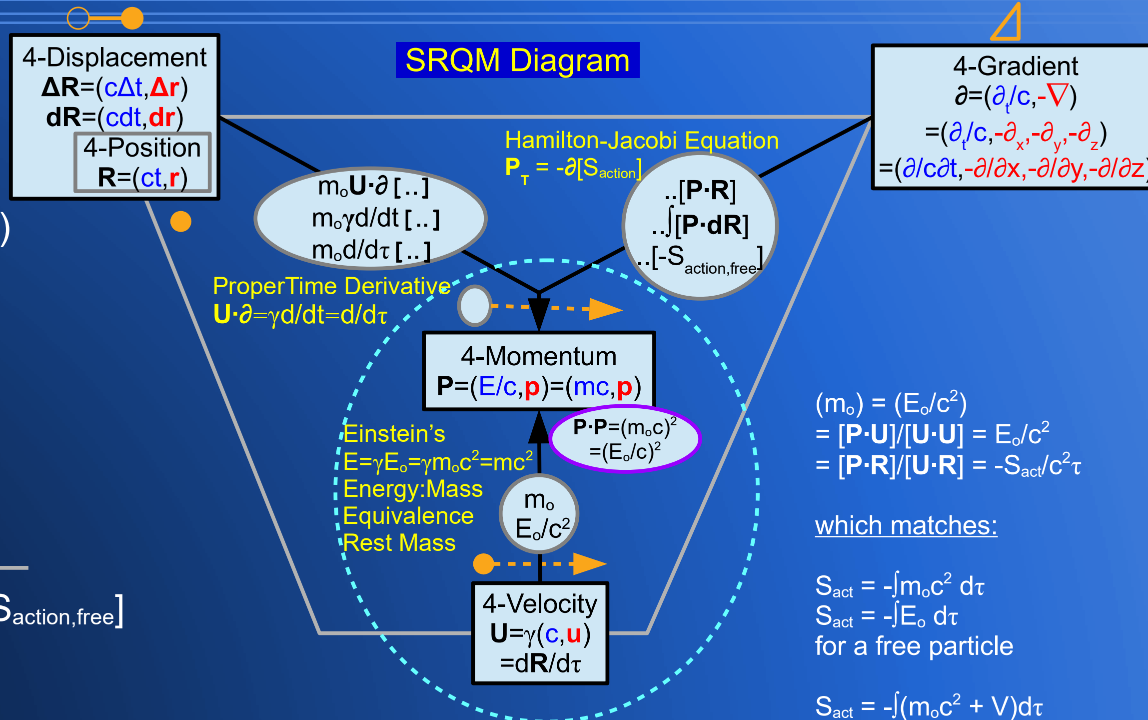
Temporal part: {energy}
$E = \gamma E_o = \gamma m_o c^2 = mc^2$
$E = m_o c^2 + (\gamma-1)m_o c^2$
(rest) + (kinetic)

Spatial part: **{3-momentum}**
$\mathbf{p} = \gamma m_o \mathbf{u} = m\mathbf{u}$

4-Momentum $\mathbf{P} = (E/c,\mathbf{p}) = -\partial[S_{action,free}] = -(\partial_t/c,-\nabla)[S_{action,free}]$

Temporal part: {energy}
$E = -\partial_t[S_{action,free}]$

Spatial part: **{3-momentum}**
$\mathbf{p} = +\nabla[S_{action,free}]$



$(m_o) = (E_o/c^2)$
$= [\mathbf{P}\cdot\mathbf{U}]/[\mathbf{U}\cdot\mathbf{U}] = E_o/c^2$
$= [\mathbf{P}\cdot\mathbf{R}]/[\mathbf{U}\cdot\mathbf{R}] = -S_{act}/c^2\tau$

which matches:

$S_{act} = -\int m_o c^2\, d\tau$
$S_{act} = -\int E_o\, d\tau$
for a free particle

$S_{act} = -\int(m_o c^2 + V)d\tau$
$S_{act} = -\int(E_o + V)d\tau$
in a potential

$(\mathbf{P}\cdot\mathbf{P}) = (E/c)^2-(\mathbf{p}\cdot\mathbf{p}) = (m_oc)^2$
$E^2 = (|\mathbf{p}|c)^2 + (m_oc^2)^2$
$E^2 = (|\mathbf{p}|c)^2 + (E_o)^2$: Einstein Mass:Energy
Relativistic Energy(E):Mass(m) vs Invariant Rest Energy(E_o):Mass(m_o)

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^{\mu}{}_{\nu}$ or $T_{\mu}{}^{\nu}$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0,\mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0,-\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu{}_\mu = T$
$\mathbf{V}\cdot\mathbf{V} = V^\mu\eta_{\mu\nu}V^\nu = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$
= Lorentz Scalar

SRQM Diagram:
The Basis of Classical SR Physics
4-WaveVector, u * $v_{phase} = c^2$

4-Position R=(ct,**r**)
4-Gradient ∂=(∂_t/c,-∇)
4-Velocity U = γ(c,**u**)

4-WaveVector K = (ω/c,**k**) = (ω_o/c^2)**U** = γ(ω_o/c^2)(c,**u**)

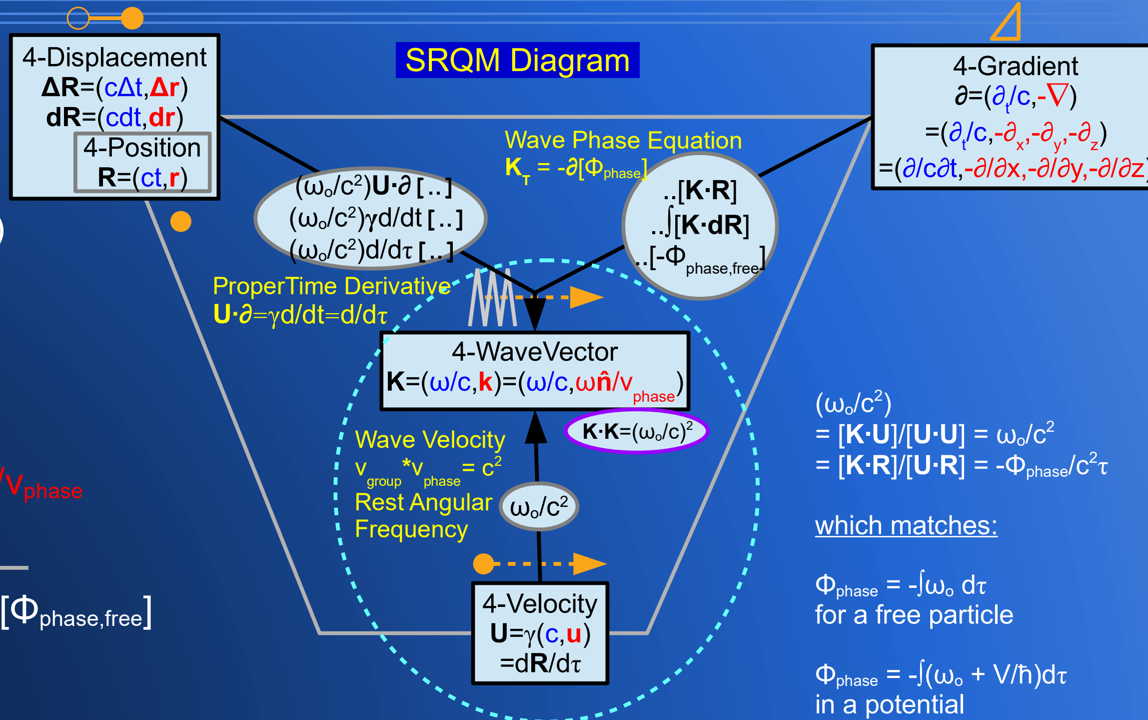
Temporal part: {angular frequency} — ω = γω_o

Spatial part: **{3-wavevector}** — **k** = γ(ω_o/c^2)**u** = (ω/c^2)**u** = ω**n̂**/v_{phase}
|u * v_{phase}| = c^2

4-WaveVector K = (ω/c,**k**) = -∂[$\Phi_{phase,free}$] = -(∂_t/c,-∇)[$\Phi_{phase,free}$]

Temporal part: {angular frequency} — ω = -∂_t[$\Phi_{phase,free}$]

Spatial part: **{3-wavevector}** — **k** = +∇[$\Phi_{phase,free}$]



(ω_o/c^2)
= [**K·U**]/[**U·U**] = ω_o/c^2
= [**K·R**]/[**U·R**] = -$\Phi_{phase}/c^2\tau$

which matches:

Φ_{phase} = -∫ω_o dτ
for a free particle

Φ_{phase} = -∫(ω_o + V/ħ)dτ
in a potential

(**K·K**) = $(\omega/c)^2$-(**k·k**) = $(\omega_o/c)^2$
ω^2 = $(|\mathbf{k}|c)^2$ + $(\omega_o)^2$: Matter-Wave Dispersion Relation
Relativistic AngFreq(ω) vs Invariant Rest AngFreq(ω_o)

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^{\mu}{}_{\nu}$ or $T_{\mu}{}^{\nu}$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor V^{μ} = **V** = (v^0,**v**)
SR 4-CoVector
(0,1)-Tensor V_{μ} = (v_0,-**v**)

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Trace[$T^{\mu\nu}$] = $\eta_{\mu\nu}T^{\mu\nu}$ = $T^{\mu}{}_{\mu}$ = T
V·V = $V^{\mu}\eta_{\mu\nu}V^{\nu}$ = [$(v^0)^2$ - **v·v**] = $(v^0{}_o)^2$
= Lorentz Scalar

SRQM Diagram:
The Basis of Classical SR Physics
4-CurrentDensity, Charge Conservation

A Tensor Study of Physical 4-Vectors

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4-Position R=(ct,**r**)
4-Gradient ∂=(∂_t/c,-∇)
4-Velocity U = γ(c,**u**)

4-CurrentDensity J = (ρc,**j**) = ρ_o**U** = γρ_o(c,**u**) = ρ(c,**u**)
4-ChargeFlux J

Temporal part: {charge-density} ρ = γρ_o

Spatial part: **{3-current-density}** **j** = γρ_o**u** = ρ**u**

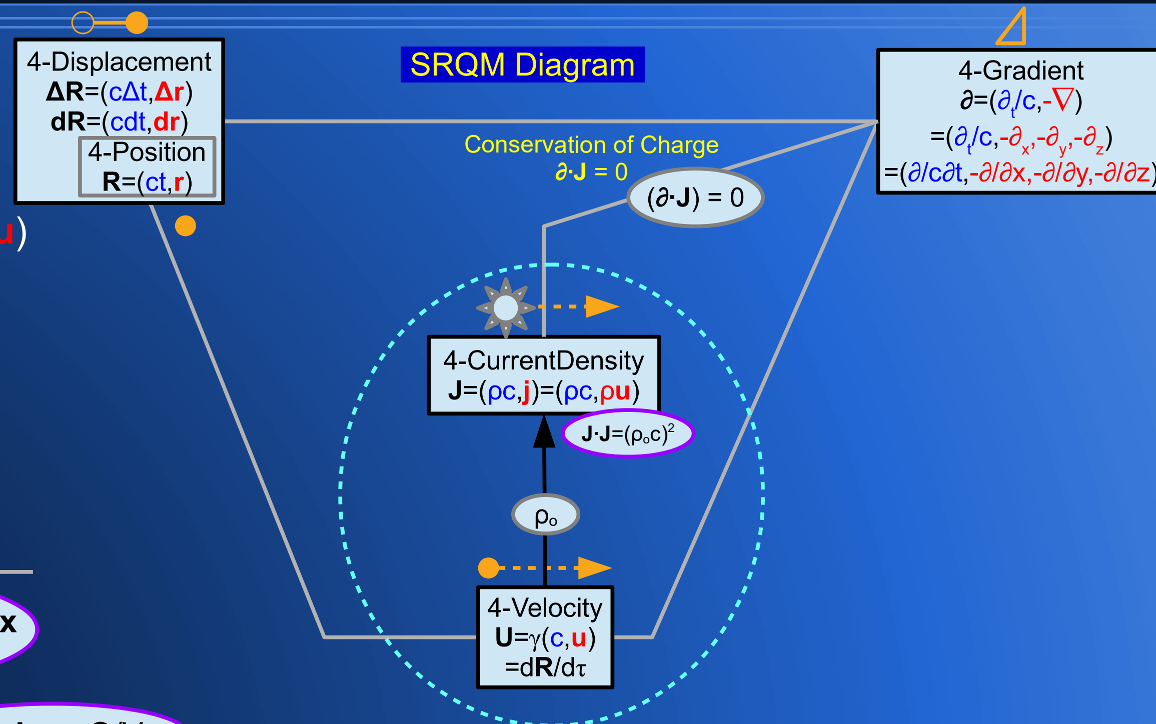
Conservation of Charge (Q)

Q = ∫ρd^3**x** = ∫γ$\rho_o d^3$**x** → $\rho_o V_o$

∂·J = (∂_t/c,-∇)·(ρc,**j**) = (∂_tρ + ∇·**j**) = 0

∫**dT·J** = -cQ/V_o

Continuity Equation:Noether's Theorem
The temporal change in charge density is balanced by the spatial change in current density.
Charge is neither created nor destroyed
It just moves around as charge currents...



(**J·J**) = $(\rho c)^2$-(**j·j**) = $(\rho_o c)^2$
ρ^2 = $(|\mathbf{j}|/c)^2$ + $(\rho_o)^2$
Relativistic ChargeDensity(ρ) vs Invariant Rest ChargeDensity(ρ_o)

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^{\mu}{}_{\nu}$ or $T_{\mu}{}^{\nu}$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor V^μ = **V** = (v^0,**v**)
SR 4-CoVector
(0,1)-Tensor V_μ = (v_0,-**v**)

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Trace[$T^{\mu\nu}$] = $\eta_{\mu\nu}T^{\mu\nu}$ = $T^{\mu}{}_{\mu}$ = T
V·V = $V^\mu \eta_{\mu\nu} V^\nu$ = [$(v^0)^2$ - **v·v**] = $(v^0{}_o)^2$
= Lorentz Scalar

SRQM Diagram:
The Basis of Classical SR Physics
4-(Dust)NumberFlux, Particle # Conservation

A Tensor Study of Physical 4-Vectors

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4-Position $\mathbf{R}=(ct,\mathbf{r})$
4-Gradient $\partial=(\partial_t/c,-\nabla)$
4-Velocity $\mathbf{U} = \gamma(c,\mathbf{u})$

4-NumberFlux $\mathbf{N} = (nc,\mathbf{n}) = n_o\mathbf{U} = \gamma n_o(c,\mathbf{u}) = n(c,\mathbf{u})$

Temporal part: {number-density} $n = \gamma n_o$

Spatial part: **{3-number-flux}** $\mathbf{n} = \gamma n_o\mathbf{u} = n\mathbf{u}$

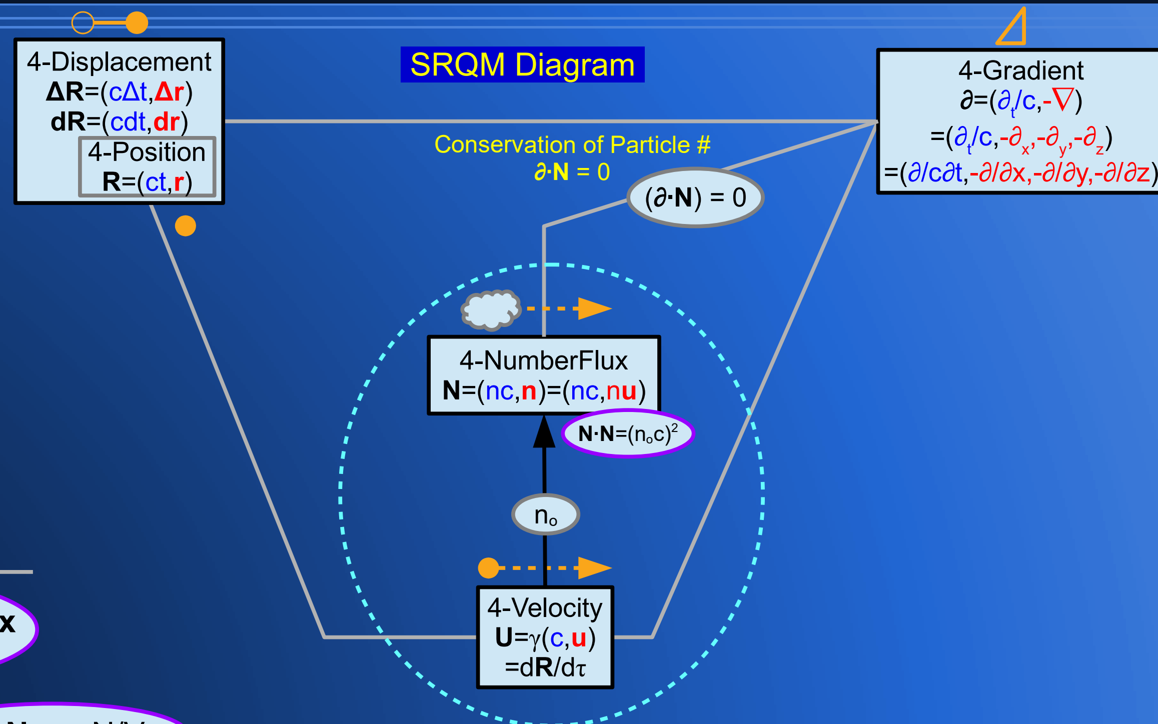
Conservation of Particle # (N)

$N = \int n d^3\mathbf{x} = \int \gamma n_o d^3\mathbf{x} \rightarrow n_o V_o$

$\partial\cdot\mathbf{N} = (\partial_t/c,-\nabla)\cdot(nc,\mathbf{n}) = (\partial_t n + \nabla\cdot\mathbf{n}) = 0$

$\int d\mathbf{T}\cdot\mathbf{N} = -cN/V_o$

Continuity Equation:Noether's Theorem
The temporal change in number density is balanced by the spatial change in number-flux.
Particle # is neither created nor destroyed
It just moves around as number currents...



$(\mathbf{N}\cdot\mathbf{N}) = (nc)^2-(\mathbf{n}\cdot\mathbf{n}) = (n_o c)^2$
$n^2 = (|\mathbf{n}|/c)^2 + (n_o)^2$
Relativistic NumberDensity(n) vs Invariant Rest NumberDensity(n_o)

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^{\mu}{}_{\nu}$ or $T_{\mu}{}^{\nu}$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0,\mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0,-\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

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$\mathbf{V}\cdot\mathbf{V} = V^\mu\eta_{\mu\nu}V^\nu = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$
= Lorentz Scalar

SR → QM

Lorentz Transforms $\Lambda^{\mu'}{}_{\nu} = \partial_{\nu}[X^{\mu'}]$
(Continuous) vs (Discrete)
(Proper Det=+1) vs (Improper Det=-1)

4-Vector SRQM Interpretation of QM

A Tensor Study of Physical 4-Vectors

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The main idea that makes a generic 4-Vector into an SR 4-Vector is that it must transform correctly according to an SR Lorentz Transformation { $\Lambda^{\mu'}{}_{\nu} = \partial X^{\mu'}/\partial X^{\nu} = \partial_{\nu}[X^{\mu'}]$}, which is basically any linear, unitary or antiunitary, transform (Determinant[$\Lambda^{\mu'}{}_{\nu}$] = ±1) which leaves the Invariant Interval unchanged.
The SR continuous transforms (variable with some parameter) have {Det = +1, Proper} and include:
"Rotation" {a mixing of space-space coordinates} and "Boost" {a mixing of time-space coordinates}.
The SR discrete transforms can be {Det = +1, Proper} or {Det = -1, Improper} and include:
"Space Parity-Inversion" {reversal of the space coordinates} , "Time-Reversal" {reversal of the temporal coordinate} ,
The "Identity" {no change}, and various single dimension Flips and their combinations.

SR:Lorentz Transform
$\partial_{\nu}[R^{\mu'}] = \partial R^{\mu'}/\partial R^{\nu} = \Lambda^{\mu'}{}_{\nu}$
$\Lambda_{\nu}{}^{\mu} = (\Lambda^{-1})_{\nu}{}^{\mu} : \Lambda^{\mu}{}_{\alpha}\Lambda^{\alpha}{}_{\nu} = \eta^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu}$
$\eta_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta} = \eta_{\alpha\beta}$
Det[$\Lambda^{\mu}{}_{\nu}$]=±1 $\Lambda_{\mu\nu}\Lambda^{\mu\nu}=4$

Continuous: ex. Boost depends on variable parameter β, with γ=1/√[1-β²]

Typical Lorentz Boost Transformation,
for a linear-velocity frame-shift (x,t)-Boost in the x̂-direction:

$A^{\nu} = (a^t, a^x, a^y, a^z)$

$A^{\mu'} = (a^t, a^x, a^y, a^z)'$
$= B^{\mu'}{}_{\nu}A^{\nu}$
$= (\gamma a^t - \gamma\beta a^x, -\gamma\beta a^t + \gamma a^x, a^y, a^z)$
{for x̂-boost Lorentz Transform}

Lorentz Boost Transform $\Lambda^{\mu'}{}_{\nu} \to B^{\mu'}{}_{\nu}$ =
$$\begin{array}{c|cccc} & t & x & y & z \\ \hline t & \gamma & -\beta\gamma & 0 & 0 \\ x & -\beta\gamma & \gamma & 0 & 0 \\ y & 0 & 0 & 1 & 0 \\ z & 0 & 0 & 0 & 1 \end{array}$$

Boosted 4-Vector
$\mathbf{A'} = A^{\mu'} = \Lambda^{\mu'}{}_{\nu}A^{\nu} \to B^{\mu'}{}_{\nu}A^{\nu} = (a^{0\prime}, \mathbf{a'})$
ex. for x̂-boost
$\to(\gamma a^t - \gamma\beta a^x, -\gamma\beta a^t + \gamma a^x, a^y, a^z)$

Det[$B^{\mu'}{}_{\nu}$]= +1, Proper
$\gamma^2 - \beta^2\gamma^2 = +1$

Proper: preserves orientation of basis

4-Vector
$\mathbf{A} = A^{\nu} = (a^0, \mathbf{a})$
$\to(a^t, a^x, a^y, a^z)$

Discrete: ex. Parity has no variable parameters

Lorentz Parity-Inversion Transformation:

$A^{\nu} = (a^t, a^x, a^y, a^z)$

$A^{\mu'} = (a^t, a^x, a^y, a^z)'$
$= P^{\mu'}{}_{\nu}A^{\nu}$
$= (a^t, -a^x, -a^y, -a^z)$
{for Parity Inverse Lorentz Transform}

Lorentz Parity Transform $\Lambda^{\mu'}{}_{\nu} \to P^{\mu'}{}_{\nu}$ =
$$\begin{array}{c|cccc} & t & x & y & z \\ \hline t & 1 & 0 & 0 & 0 \\ x & 0 & -1 & 0 & 0 \\ y & 0 & 0 & -1 & 0 \\ z & 0 & 0 & 0 & -1 \end{array}$$

Parity-Inversed 4-Vector
$\mathbf{A'} = A^{\mu'} = \Lambda^{\mu'}{}_{\nu}A^{\nu} \to P^{\mu'}{}_{\nu}A^{\nu} = (a^{0\prime}, \mathbf{a'})$
$\to(a^t, -a^x, -a^y, -a^z)$

Det[$P^{\mu'}{}_{\nu}$]= -1, Improper
$(-1)^3 = -1$

Improper: reverses orientation of basis

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^{\mu}{}_{\nu}$ or $T_{\mu}{}^{\nu}$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^{\mu} = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_{\mu} = (v_0, -\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Trace[$T^{\mu\nu}$] = $\eta_{\mu\nu}T^{\mu\nu} = T^{\mu}{}_{\mu} = T$
$\mathbf{V\cdot V} = V^{\mu}\eta_{\mu\nu}V^{\nu} = [(v^0)^2 - \mathbf{v\cdot v}] = (v^0{}_o)^2$
= Lorentz Scalar

Lorentz Transforms $\Lambda^{\mu'}{}_{\nu} = \partial_{\nu}[X^{\mu'}]$

Proper Lorentz Transforms (Det=+1): Continuous: (Boost) vs (Rotation)

$\beta = v/c$: dimensionless Velocity Beta Factor { $\beta=(0..1)$, with speed-of-light (c) at ($\beta=1$) }
$\gamma = 1/\sqrt{[1-\beta^2]} = 1/\sqrt{[1-\boldsymbol{\beta}\cdot\boldsymbol{\beta}]}$: dimensionless Lorentz Relativistic Gamma Factor { $\gamma=(1..\infty)$ }

<u>Typical Lorentz Boost Transform (symmetric):</u>
for a linear-velocity frame-shift (x,t)-Boost in the $\hat{x}$-direction:
$\Lambda^{\mu'}{}_{\nu} \rightarrow B^{\mu'}{}_{\nu}[\zeta] = e\hat{}-(\boldsymbol{\zeta}\cdot\mathbf{K}) =$

$$\begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh[\zeta] & -\sinh[\zeta] & 0 & 0 \\ -\sinh[\zeta] & \cosh[\zeta] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = e\hat{}(\zeta_x \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix})$$

$A^{\nu} = (a^t, a^x, a^y, a^z)$
$A^{\mu'} = (a^t, a^x, a^y, a^z)' = B^{\mu'}{}_{\nu}A^{\nu} = (\gamma a^t - \gamma\beta a^x, -\gamma\beta a^t + \gamma a^x, a^y, a^z)$

<u>Lorentz Transforms:</u>
Lambda(Λ) for <u>L</u>orentz
(B) for <u>B</u>oost
(R) for <u>R</u>otation

Proper Transforms
Determinant = +1

{ $\cos^2 + \sin^2 = +1$ }

{ $\gamma^2 - \beta^2\gamma^2 = +1$ }
{ $\cosh^2 - \sinh^2 = +1$ }

ζ = rapidity = hyperbolic angle
$\gamma = \cosh[\zeta] = 1/\sqrt{[1-\beta^2]}$
$\beta\gamma = \sinh[\zeta]$
$\beta = \tanh[\zeta]$

<u>Typical Lorentz Rotation Transform (non-symmetric):</u>
for an angular-displacement frame-shift (x,y)-Rotation about the $\hat{z}$-direction:
$\Lambda^{\mu'}{}_{\nu} \rightarrow R^{\mu'}{}_{\nu}[\theta] = e\hat{}(\boldsymbol{\theta}\cdot\mathbf{J}) =$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos[\theta] & -\sin[\theta] & 0 \\ 0 & \sin[\theta] & \cos[\theta] & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = e\hat{}(\theta_z \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix})$$

SR:Lorentz Transform
$\partial_{\nu}[R^{\mu'}] = \partial R^{\mu'}/\partial R^{\nu} = \Lambda^{\mu'}{}_{\nu}$
$\Lambda_{\mu}{}^{\nu} = (\Lambda^{-1})_{\nu}{}^{\mu} : \Lambda^{\mu}{}_{\alpha}\Lambda^{\alpha}{}_{\nu} = \eta^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu}$
$\eta_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta} = \eta_{\alpha\beta}$
$\mathrm{Det}[\Lambda^{\mu}{}_{\nu}]=\pm1$ $\Lambda_{\mu\nu}\Lambda^{\mu\nu}=4$

$A^{\nu} = (a^t, a^x, a^y, a^z)$
$A^{\mu'} = (a^t, a^x, a^y, a^z)' = R^{\mu'}{}_{\nu}A^{\nu} = (a^t, \cos[\theta]a^x - \sin[\theta]a^y, \sin[\theta]a^x + \cos[\theta]a^y, a^z)$

4-Vector
$\mathbf{A}=A^{\nu}=(a^0,\mathbf{a})$

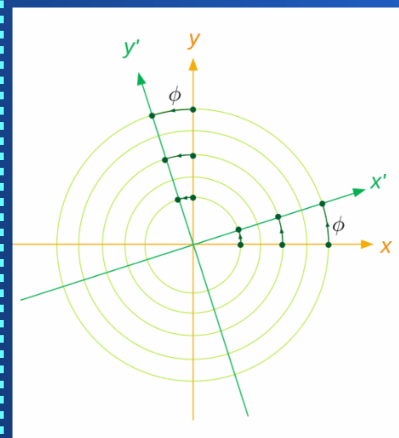
Lorentz Rotation Transform
$\Lambda^{\mu'}{}_{\nu} \rightarrow R^{\mu'}{}_{\nu}$

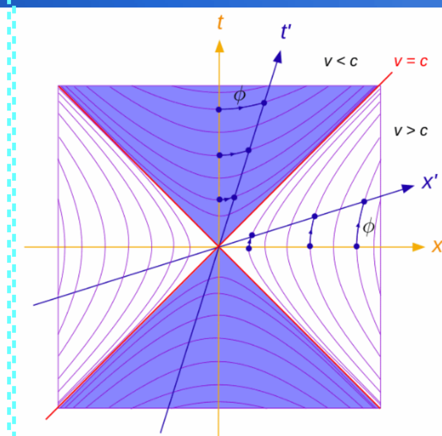
Lorentz Boost Transform
$\Lambda^{\mu'}{}_{\nu} \rightarrow B^{\mu'}{}_{\nu}$

$\mathrm{Det}[R^{\mu'}{}_{\nu}]=\mathrm{Det}[B^{\mu'}{}_{\nu}] = +1$

Rotated 4-Vector
Circularly-Rotated
$\mathbf{A'}=A^{\mu'}=R^{\mu'}{}_{\nu}A^{\nu}=(a^{0'},\mathbf{a'})$

Boosted 4-Vector
Hyperbolically-Rotated
$\mathbf{A'}=A^{\mu'}=B^{\mu'}{}_{\nu}A^{\nu}=(a^{0'},\mathbf{a'})$





SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^{\mu}{}_{\nu}$ or $T_{\mu}{}^{\nu}$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^{\mu} = \mathbf{V} = (v^0,\mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_{\mu} = (v_0,-\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^{\mu}{}_{\mu} = T$
$\mathbf{V}\cdot\mathbf{V} = V^{\mu}\eta_{\mu\nu}V^{\nu} = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$
= Lorentz Scalar

SR → QM

Lorentz Transforms $\Lambda^{\mu'}{}_{\nu} = \partial_{\nu}[X^{\mu'}]$

4-Vector SRQM Interpretation of QM

Proper Lorentz Transforms (Det=+1): (Boost) vs (Rotation) vs (Identity)

A Tensor Study of Physical 4-Vectors

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General Lorentz Boost Transform (symmetric,continuous):
for a linear-velocity frame-shift (Boost)
in the **v**/c=**β**=(β^1,β^2,β^3)-direction:
$\Lambda^{\mu'}{}_{\nu} \rightarrow B^{\mu'}{}_{\nu}$ =

$$\begin{bmatrix} \gamma & -\gamma\beta_j \\ -\gamma\beta^i & (\gamma-1)\beta^i\beta_j/(\boldsymbol{\beta}\cdot\boldsymbol{\beta})+\delta^i{}_j \end{bmatrix}$$

$\Lambda_{\mu\nu}\Lambda^{\mu\nu} = 4$

$$\Lambda^{\mu'}{}_{\nu} = \begin{bmatrix} \Lambda^{0'}{}_{0}, \Lambda^{0'}{}_{j} \\ \Lambda^{i'}{}_{0}, \Lambda^{i'}{}_{j} \end{bmatrix}$$

4-Vector
A=A^{ν}=(a^0,**a**)

No mixing

Lorentz Identity Transform
$\Lambda^{\mu'}{}_{\nu} \rightarrow \eta^{\mu'}{}_{\nu} = I_{(4)}$

Tr[$\eta^{\mu'}{}_{\nu}$]=4
Det[$\eta^{\mu'}{}_{\nu}$]=+1

Space-Space

Lorentz Rotation Transform
$\Lambda^{\mu'}{}_{\nu} \rightarrow R^{\mu'}{}_{\nu}$

Tr[$R^{\mu'}{}_{\nu}$]={0..4}
Det[$R^{\mu'}{}_{\nu}$]=+1

Time-Space

Lorentz Boost Transform
$\Lambda^{\mu'}{}_{\nu} \rightarrow B^{\mu'}{}_{\nu}$

Tr[$B^{\mu'}{}_{\nu}$]={4..Infinity}
Det[$B^{\mu'}{}_{\nu}$]=+1

Identical 4-Vector
Un-Rotated
A'=$A^{\mu'}$=$\eta^{\mu'}{}_{\nu}A^{\nu}$=($a^{0'}$,**a'**)=**A**

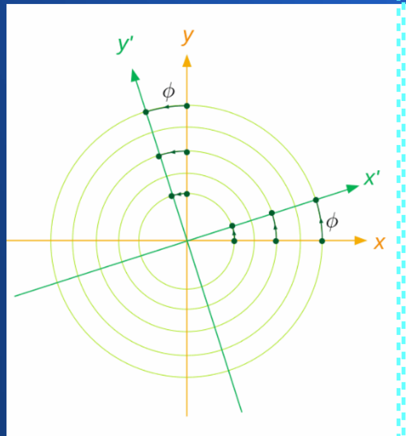
Rotated 4-Vector
Circularly-Rotated
A'=$A^{\mu'}$=$R^{\mu'}{}_{\nu}A^{\nu}$=($a^{0'}$,**a'**)

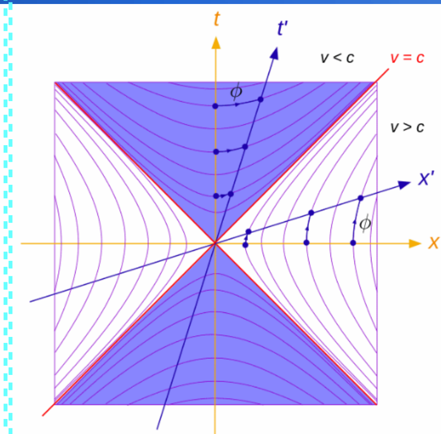
Boosted 4-Vector
Hyperbolically-Rotated
A'=$A^{\mu'}$=$B^{\mu'}{}_{\nu}A^{\nu}$=($a^{0'}$,**a'**)

General Lorentz Rotation Transform (non-symmetric,continuous):
for an angular-displacement frame-shift (Rotation)
angle θ about the **n̂**=(n^1,n^2,n^3)-direction:
$\Lambda^{\mu'}{}_{\nu} \rightarrow R^{\mu'}{}_{\nu}$ =

$$\begin{bmatrix} 1 & 0_j \\ 0^i & (\delta^i{}_j - n^i n_j)\cos(\theta) - (\varepsilon^i{}_{jk} n^k)\sin(\theta) + n^i n_j \end{bmatrix}$$

The Lorentz Identity Transform is the limit of both the Rotation and Boost Transforms when the "rotation angle" is 0





Lorentz Identity Transform (symmetric,"discrete:continuous"):
for a non-frame-shift (Identity)
in any direction
$\Lambda^{\mu'}{}_{\nu} \rightarrow \eta^{\mu'}{}_{\nu} = \delta^{\mu'}{}_{\nu}$ = Diag[1,$\delta^i{}_j$] = $I_{(4)}$ =

$$\begin{bmatrix} 1 & 0_j \\ 0^i & \delta^i{}_j \end{bmatrix}$$

SR:Lorentz Transform
$\partial_{\nu}[R^{\mu'}] = \partial R^{\mu'}/\partial R^{\nu} = \Lambda^{\mu'}{}_{\nu}$
$\Lambda_{\nu}{}^{\mu} = (\Lambda^{-1})_{\nu}{}^{\mu} : \Lambda^{\mu}{}_{\alpha}\Lambda^{\alpha}{}_{\nu} = \eta^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu}$
$\eta_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta} = \eta_{\alpha\beta}$
Det[$\Lambda^{\mu}{}_{\nu}$]=±1 $\Lambda_{\mu\nu}\Lambda^{\mu\nu}$=4

β = v/c: dimensionless Velocity Beta Factor { β=(0..1), with speed-of-light (c) at (β=1) }
γ = 1/√[1-β²] = 1/√[1-**β·β**]: dimensionless Lorentz Relativistic Gamma Factor { γ=(1..∞) }
Identity transformation for zero relative motion/rotation: B[0] = R[0] = $I_{(4)}$
Proper Transformation = positive unit determinant: det[B] = det[R] = det[η] = +1.
Inverses: $B(v)^{-1}$ = B(−v) (relative motion in the opposite direction), and $R(\theta)^{-1}$ = R(−θ) (rotation in the opposite sense about the same axis)
Matrix symmetry: B is symmetric (equals transpose, B=B^T), while R is nonsymmetric but orthogonal (transpose equals inverse, $R^T = R^{-1}$)

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^{\mu}{}_{\nu}$ or $T_{\mu}{}^{\nu}$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor V^{μ} = **V** = (v^0,**v**)
SR 4-CoVector
(0,1)-Tensor V_{μ} = (v_0,-**v**)

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Trace[$T^{\mu\nu}$] = $\eta_{\mu\nu}T^{\mu\nu} = T^{\mu}{}_{\mu} = T$
V·V = $V^{\mu}\eta_{\mu\nu}V^{\nu}$ = [$(v^0)^2$ - **v·v**] = $(v^0{}_o)^2$
= Lorentz Scalar

SR → QM

Lorentz Transforms $\Lambda^{\mu'}{}_{\nu} = \partial_{\nu}[X^{\mu'}]$

Discrete (non-continuous)

(Parity-Inversion) vs (Time-Reversal) vs (Identity)

4-Vector SRQM Interpretation of QM

A Tensor Study of Physical 4-Vectors

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<u>General Lorentz Parity-Inversion Transform:</u>
$\Lambda^{\mu'}{}_{\nu} \to P^{\mu'}{}_{\nu}$ (Improper,symmetric,discrete)
$= \begin{bmatrix} 1 & 0_j \\ 0^i & -\delta^i{}_j \end{bmatrix}$

<u>General Lorentz Time-Reversal Transform:</u>
$\Lambda^{\mu'}{}_{\nu} \to T^{\mu'}{}_{\nu}$ (Improper,symmetric,discrete)
$= \begin{bmatrix} -1 & 0_j \\ 0^i & \delta^i{}_j \end{bmatrix}$

<u>General Lorentz Identity Transform:</u>
$\Lambda^{\mu'}{}_{\nu} \to \eta^{\mu'}{}_{\nu} = \delta^{\mu'}{}_{\nu} = I_{(4)}$ (Proper,symmetric)
$= \begin{bmatrix} 1 & 0_j \\ 0^i & \delta^i{}_j \end{bmatrix}$

$\Lambda_{\mu\nu}\Lambda^{\mu\nu} = 4$

$\Lambda^{\mu'}{}_{\nu} = \begin{bmatrix} \Lambda^{0'}{}_{0}, \Lambda^{0'}{}_{j} \\ \Lambda^{i'}{}_{0}, \Lambda^{i'}{}_{j} \end{bmatrix}$

SR:Lorentz Transform
$\partial_{\nu}[R^{\mu'}] = \partial R^{\mu'}/\partial R^{\nu} = \Lambda^{\mu'}{}_{\nu}$
$\Lambda^{\mu}{}_{\nu} = (\Lambda^{-1})_{\nu}{}^{\mu} : \Lambda^{\mu}{}_{\alpha}\Lambda^{\alpha}{}_{\nu} = \eta^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu}$
$\eta_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta} = \eta_{\alpha\beta}$
$\mathrm{Det}[\Lambda^{\mu}{}_{\nu}] = \pm 1$ $\Lambda_{\mu\nu}\Lambda^{\mu\nu} = 4$

4-Vector
$\mathbf{A} = A^{\nu} = (a^0, \mathbf{a})$

No mixing	Time	Space	TimeSpace
$\mathrm{Tr}[\eta^{\mu'}{}_{\nu}] = +4$	$\mathrm{Tr}[T^{\mu'}{}_{\nu}] = +2$	$\mathrm{Tr}[P^{\mu'}{}_{\nu}] = -2$	$\mathrm{Tr}[(PT)^{\mu'}{}_{\nu}] = -4$
Lorentz Identity Transform $\Lambda^{\mu'}{}_{\nu} \to \eta^{\mu'}{}_{\nu} = I_{(4)}$ $= \begin{bmatrix} 1 & 0_j \\ 0^i & \delta^i{}_j \end{bmatrix}$	Lorentz Time-Reversal Transform $\Lambda^{\mu'}{}_{\nu} \to T^{\mu'}{}_{\nu}$ $= \begin{bmatrix} -1 & 0_j \\ 0^i & \delta^i{}_j \end{bmatrix}$	Lorentz Parity-Inversion Transform $\Lambda^{\mu'}{}_{\nu} \to P^{\mu'}{}_{\nu}$ $= \begin{bmatrix} 1 & 0_j \\ 0^i & -\delta^i{}_j \end{bmatrix}$	Lorentz ComboPT Transform $\Lambda^{\mu'}{}_{\nu} \to (PT)^{\mu'}{}_{\nu} = -I_{(4)}$ $= \begin{bmatrix} -1 & 0_j \\ 0^i & -\delta^i{}_j \end{bmatrix}$
$\mathrm{Det}[\eta^{\mu'}{}_{\nu}] = +1$	$\mathrm{Det}[T^{\mu'}{}_{\nu}] = \mathrm{Det}[P^{\mu'}{}_{\nu}] = -1$		$\mathrm{Det}[(PT)^{\mu'}{}_{\nu}] = +1$
Identical 4-Vector $\mathbf{A'} = A^{\mu'} = \eta^{\mu'}{}_{\nu}A^{\nu} = (a^{0'}, \mathbf{a'})$ $= (a^0, \mathbf{a}) = \mathbf{A}$	Time-Reversed 4-Vector $\mathbf{A'} = A^{\mu'} = T^{\mu'}{}_{\nu}A^{\nu} = (a^{0'}, \mathbf{a'})$ $= (-a^0, \mathbf{a})$	Parity-Inverted 4-Vector $\mathbf{A'} = A^{\mu'} = P^{\mu'}{}_{\nu}A^{\nu} = (a^{0'}, \mathbf{a'})$ $= (a^0, -\mathbf{a})$	Combo PT'd 4-Vector $\mathbf{A'} = A^{\mu'} = (PT)^{\mu'}{}_{\nu}A^{\nu} = (a^{0'}, \mathbf{a'})$ $= (-a^0, -\mathbf{a})$
Lorentz Identity Transform $\Lambda^{\mu'}{}_{\nu} \to \eta^{\mu'}{}_{\nu} = \delta^{\mu'}{}_{\nu}$	Lorentz Time-Reversal Transform $\Lambda^{\mu'}{}_{\nu} \to T^{\mu'}{}_{\nu}$	Lorentz Parity-Inversion Transform $\Lambda^{\mu'}{}_{\nu} \to P^{\mu'}{}_{\nu}$	Lorentz ComboPT Transform $\Lambda^{\mu'}{}_{\nu} \to (PT)^{\mu'}{}_{\nu}$

Original 4-Vector
$\mathbf{A} = A^{\nu} = (a^0, \mathbf{a})$

Both the Parity-Inversion (P) and Time-Reversal (T) have a Determinant of -1, which is an improper transform.
However, combinations (PP), (TT), (PT) have overall Determinant of +1, which is proper.
Classical SR Time Reversal neglects spin and charge. When included, there is also a Charge-Conjugation(C) transform.
Then one gets (CC),(PP),(TT),(PT)(PT) & (CPT) transforms all leading back to the Identity (I).

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^{\mu}{}_{\nu}$ or $T_{\mu}{}^{\nu}$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^{\mu} = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_{\mu} = (v_0, -\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Note that the Trace of Discrete Lorentz Transforms goes in steps from {-4,-2,2,4}. As we will see in a bit, this is a major hint for SR antimatter.

$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^{\mu}{}_{\mu} = T$
$\mathbf{V}\cdot\mathbf{V} = V^{\mu}\eta_{\mu\nu}V^{\nu} = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$
= Lorentz Scalar

Lorentz Transforms $\Lambda^{\mu'}{}_{\nu} = \partial_{\nu}[X^{\mu'}]$

Discrete & Fixed Rotation → Particle Exchange

Lorentz Coordinate-Flip Transforms

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Lorentz Flip-t Transform $\Lambda^{\mu'}{}_{\nu} \to Ft^{\mu'}{}_{\nu} = T^{\mu'}{}_{\nu}$ =

$$\begin{array}{c|cccc} & t & x & y & z \\ \hline t & -1 & 0 & 0 & 0 \\ x & 0 & 1 & 0 & 0 \\ y & 0 & 0 & 1 & 0 \\ z & 0 & 0 & 0 & 1 \end{array}$$

$Tr[Ft^{\mu'}{}_{\nu}] = 2$
$Det[Ft^{\mu'}{}_{\nu}] = -1$

$Tr[R^{\mu'}{}_{\nu}] = 2+2\cos[\theta] = \{0..4\}$
$Det[R^{\mu'}{}_{\nu}] = \cos[\theta]^2 + \sin[\theta]^2 = +1$

Lorentz z-Rotation Transform $\Lambda^{\mu'}{}_{\nu} \to R^{\mu'}{}_{\nu}$ =

$$\begin{array}{c|cccc} & t & x & y & z \\ \hline t & 1 & 0 & 0 & 0 \\ x & 0 & \cos[\theta] & -\sin[\theta] & 0 \\ y & 0 & \sin[\theta] & \cos[\theta] & 0 \\ z & 0 & 0 & 0 & 1 \end{array}$$

SR:Lorentz Transform

$\partial_{\nu}[R^{\mu'}] = \partial R^{\mu'}/\partial R^{\nu} = \Lambda^{\mu'}{}_{\nu}$

$\Lambda^{\mu}{}_{\nu} = (\Lambda^{-1})_{\nu}{}^{\mu} : \Lambda^{\mu}{}_{\alpha}\Lambda^{\alpha}{}_{\nu} = \eta^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu}$

$\eta_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta} = \eta_{\alpha\beta}$

$Det[\Lambda^{\mu}{}_{\nu}] = \pm 1$ $\quad \Lambda_{\mu\nu}\Lambda^{\mu\nu} = 4$

Lorentz Flip-x Transform $\Lambda^{\mu'}{}_{\nu} \to Fx^{\mu'}{}_{\nu}$ =

$$\begin{array}{c|cccc} & t & x & y & z \\ \hline t & 1 & 0 & 0 & 0 \\ x & 0 & -1 & 0 & 0 \\ y & 0 & 0 & 1 & 0 \\ z & 0 & 0 & 0 & 1 \end{array}$$

$Tr[Fx^{\mu'}{}_{\nu}] = 2$
$Det[Fx^{\mu'}{}_{\nu}] = -1$

Lorentz Flip-y Transform $\Lambda^{\mu'}{}_{\nu} \to Fy^{\mu'}{}_{\nu}$ =

$$\begin{array}{c|cccc} & t & x & y & z \\ \hline t & 1 & 0 & 0 & 0 \\ x & 0 & 1 & 0 & 0 \\ y & 0 & 0 & -1 & 0 \\ z & 0 & 0 & 0 & 1 \end{array}$$

$Tr[Fy^{\mu'}{}_{\nu}] = 2$
$Det[Fy^{\mu'}{}_{\nu}] = -1$

$Fx \otimes Fy$ →

$Tr[Fxy^{\mu'}{}_{\nu}] = 0$
$Det[Fxy^{\mu'}{}_{\nu}] = +1$

Lorentz Flip-xy Transform $\Lambda^{\mu'}{}_{\nu} \to Fxy^{\mu'}{}_{\nu}$ = Exchange

$$\begin{array}{c|cccc} & t & x & y & z \\ \hline t & 1 & 0 & 0 & 0 \\ x & 0 & -1 & 0 & 0 \\ y & 0 & 0 & -1 & 0 \\ z & 0 & 0 & 0 & 1 \end{array}$$

Any single Lorentz Flip Transform is Improper, with a Determinant of -1. However, pairwise combinations are Proper, with a Determinant of +1.

The combination of any two Spatial Flips is the equivalent of a Spatial Rotation by (π) about the associated rotational axis. Since this is a Proper transform, it is also the equivalent of a particle location exchange.

The combination of all three Spatial Flips, Flip-xyz, gives the Lorentz Parity Transform, which is again Improper.

The Flip-t is the standard Lorentz Time-Reversal, Improper.

$\Lambda_{\mu\nu}\Lambda^{\mu\nu} = 4$

$\Lambda^{\mu'}{}_{\nu} = \begin{bmatrix} \Lambda^{0'}{}_{0}, \Lambda^{0'}{}_{j} \\ \Lambda^{i'}{}_{0}, \Lambda^{i'}{}_{j} \end{bmatrix}$

Lorentz Flip-z Transform $\Lambda^{\mu'}{}_{\nu} \to Fz^{\mu'}{}_{\nu}$ =

$$\begin{array}{c|cccc} & t & x & y & z \\ \hline t & 1 & 0 & 0 & 0 \\ x & 0 & 1 & 0 & 0 \\ y & 0 & 0 & 1 & 0 \\ z & 0 & 0 & 0 & -1 \end{array}$$

$Tr[Fz^{\mu'}{}_{\nu}] = 2$
$Det[Fz^{\mu'}{}_{\nu}] = -1$

$Fxy \otimes Fz$ →

$Tr[Fxyz^{\mu'}{}_{\nu}] = -2$
$Det[Fxyz^{\mu'}{}_{\nu}] = -1$

Lorentz Parity Transform $\Lambda^{\mu'}{}_{\nu} \to P^{\mu'}{}_{\nu} = Fxyz^{\mu'}{}_{\nu}$

$$\begin{array}{c|cccc} & t & x & y & z \\ \hline t & 1 & 0 & 0 & 0 \\ x & 0 & -1 & 0 & 0 \\ y & 0 & 0 & -1 & 0 \\ z & 0 & 0 & 0 & -1 \end{array}$$

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
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(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^{\mu} = \mathbf{V} = (v^0, \mathbf{v})$
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= Lorentz Scalar

SR → QM

4-Vector SRQM Interpretation of QM

Lorentz Transforms $\Lambda^{\mu'}{}_{\nu} = \partial_{\nu}[X^{\mu'}]$

Lorentz Transform Connection Map

A Tensor Study of Physical 4-Vectors

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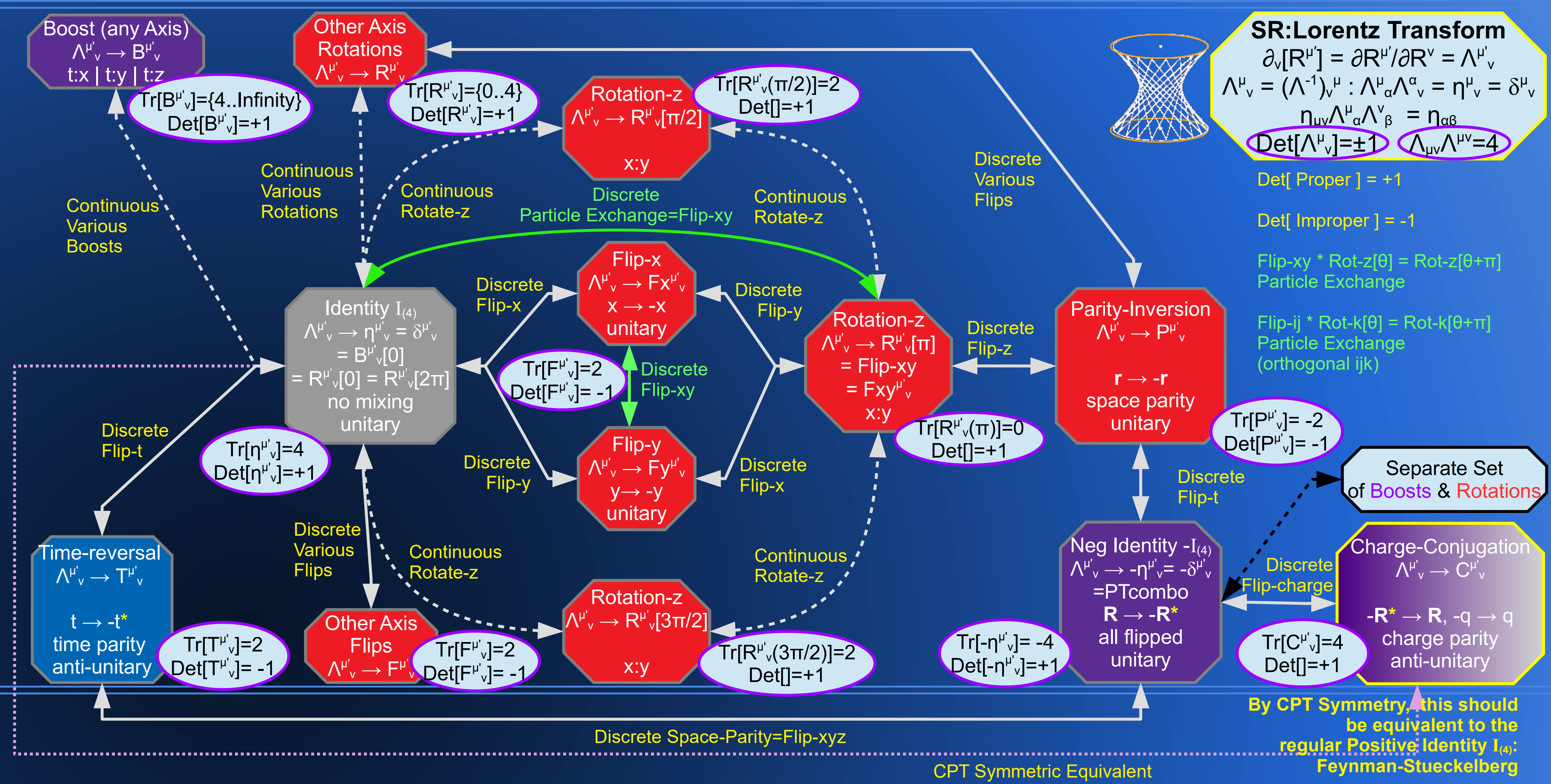

SR → QM

Lorentz Transforms $\Lambda^{\mu'}{}_{\nu} = \partial_{\nu}[X^{\mu'}]$

4-Vector SRQM Interpretation of QM

Lorentz Transform Connection Map – Discrete Transforms CPT, Big-Bang, (Matter-AntiMatter), Arrow-of-Time

A Tensor Study of Physical 4-Vectors

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Examine all possible combinations of Discrete Lorentz Transformations which are Linear (Determinant of ±1).

A lot of the standard SR texts only mention (P)arity-Inverse and (T)ime-Reversal. However, there are many others, including (F)lips and (R)otations of a fixed amount. However, the (T)imeReversal and Combo(P)arity(T)ime take one into a separate section of the chart. Taking into account all possible discrete Lorentz Transformations fills in the rest of the chart. The resulting interpretation is that there is **CPT Symmetry** (Charge:Parity:Time) and **Dual TimeSpace** (with reversed timeflow). In other words, one can go from the Identity Transform (all +1) to the Negative Identity Transform (all -1) by doing a Combo PT Lorentz Transform or by Negating the Charge (Matter→Antimatter). The Feynman-Stueckelberg Interpretation aligns with this as the AntiMatter Side.

This is similar to Dirac's prediction of AntiMatter, but without the formal need of Quantum Mechanics, or Spin. In fact, it is more general than Dirac's work, which was about the electron. This is from general Lorentz Transforms for any kind of particle.

SR:Lorentz Transform

$\partial_{\nu}[R^{\mu'}] = \partial R^{\mu'}/\partial R^{\nu} = \Lambda^{\mu'}{}_{\nu}$

$\Lambda^{\mu}{}_{\nu} = (\Lambda^{-1})_{\nu}{}^{\mu} : \Lambda^{\mu}{}_{\alpha}\Lambda^{\alpha}{}_{\nu} = \eta^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu}$

$\eta_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta} = \eta_{\alpha\beta}$

$Det[\Lambda^{\mu}{}_{\nu}]=\pm1$ $\Lambda_{\mu\nu}\Lambda^{\mu\nu}=4$

Tao – I Ching – YinYang fantastic metaphors for SR SpaceTime...
Tao: "Flow of the Universe" "way, path, route, road"
I Ching: "Book of Changes" "Transformations"
YinYang: "Positive/Negative" "complementary opposites"

Matter-AntiMatter
Dual balance along Temporal
Binary Spatial states
for 3 units:dimensions
Discrete Lorentz Transform (1,1)-Tensor
{ octagon representation }
Pair production (+ -)
in little circles (• •)

t	x	y	z	Discrete **NormalMatter** (NM) Lorentz Transform Type	Trace : Determinant
+1	+1	+1	+1	Minkowski-Identity : AM-Flip-txyz=AM-ComboPT	Tr = +4 : Det = +1 Proper
+1	+1	+1	-1	Flip-z	Tr = +2 : Det = -1 Improper
+1	+1	-1	+1	Flip-y	Tr = +2 : Det = -1 Improper
+1	+1	-1	-1	Flip-yz=Rotate-yz(π)	Tr = 0 : Det = +1 Proper
+1	-1	+1	+1	Flip-x	Tr = +2 : Det = -1 Improper
+1	-1	+1	-1	Flip-xz=Rotate-xz(π)	Tr = 0 : Det = +1 Proper
+1	-1	-1	+1	Flip-xy=Rotate-xy(π)	Tr = 0 : Det = +1 Proper
+1	-1	-1	-1	Flip-xyz=ParityInverse : AM-Flip-t=AM-TimeReversal	Tr = -2 : Det = -1 Improper
-1	+1	+1	+1	Flip-t=TimeReversal : AM-Flip-xyz=AM-ParityInverse	Tr = +2 : Det = -1 Improper
-1	+1	+1	-1	AM-Flip-xy=AM-Rotate-xy(π)	Tr = 0 : Det = +1 Proper
-1	+1	-1	+1	AM-Flip-xz=AM-Rotate-xz(π)	Tr = 0 : Det = +1 Proper
-1	+1	-1	-1	AM-Flip-x	Tr = -2 : Det = -1 Improper
-1	-1	+1	+1	AM-Flip-yz=AM-Rotate-yz(π)	Tr = 0 : Det = +1 Proper
-1	-1	+1	-1	AM-Flip-y	Tr = -2 : Det = -1 Improper
-1	-1	-1	+1	AM-Flip-z	Tr = -2 : Det = -1 Improper
-1	-1	-1	-1	AM-Minkowski-Identity : Flip-txyz=ComboPT	Tr = -4 : Det = +1 Proper
t	x	y	z	Discrete **AntiMatter** (AM) Lorentz TransformType	Trace : Determinant

Note that the (T)imeReversal and Combo (P)arityInverse & (T)imeReversal take NormalMatter ↕ AntiMatter

Lorentz Transforms $\Lambda^{\mu'}{}_{\nu} = \partial_{\nu}[X^{\mu'}]$

Lorentz Transform Connection Map – Trace Identification
CPT, Big-Bang, (Matter-AntiMatter), Arrow-of-Time

A Tensor Study of Physical 4-Vectors

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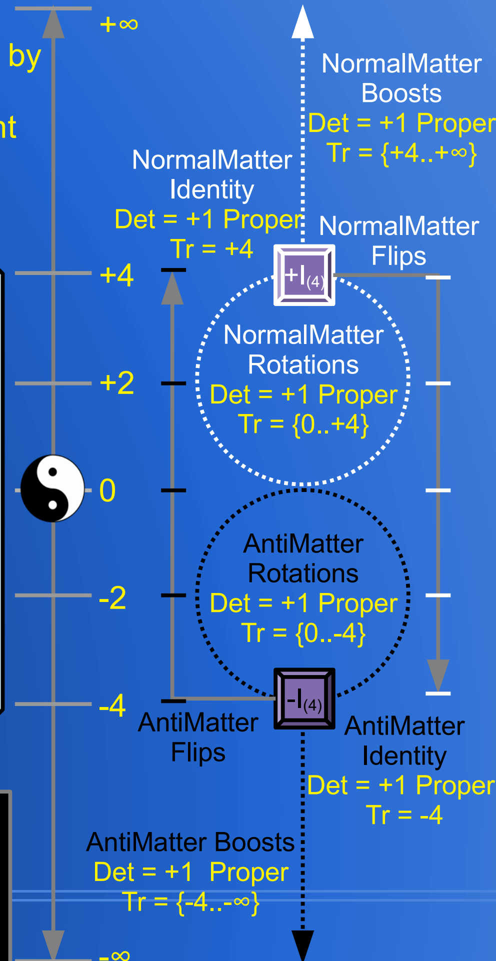
All Lorentz Transforms have Tensor Invariants: Determinant of ±1 and Inner Product of 4.
However, one can use the Tensor Invariant Trace to Identify CPT Symmetry

Tr[NM-Rotate] = {0...+4} Tr[NM-Identity] = +4 Tr[NM-Boost] = {+4...+∞}
Tr[AM-Rotate] = {0….-4} Tr[AM-Identity] = -4 Tr[AM-Boost] = {-4…..-∞}

Discrete NormalMatter (NM) Lorentz Transform Type	Trace : Determinant
Minkowski-Identity : AM-Flip-txyz=AM-ComboPT	Tr = +4 : Det = +1 Proper
Flip-t=TimeReversal, Flip-x, Flip-y, Flip-z AM-Flip-xyz=AM-ParityInverse	Tr = +2 : Det = -1 Improper
Flip-xy=Rotate-xy(π), Flip-xz=Rotate-xz(π), Flip-yz=Rotate-yz(π)	Tr = 0 : Det = +1 Proper
AM-Flip-xy=AM-Rotate-xy(π), AM-Flip-xz=AM-Rotate-xz(π), AM-Flip-yz=AM-Rotate-yz(π)	Tr = 0 : Det = +1 Proper
Flip-xyz=ParityInverse AM-Flip-t=AM-TimeReversal, AM-Flip-x, AM-Flip-y, AM-Flip-z	Tr = -2 : Det = -1 Improper
AM-Minkowski-Identity : Flip-txyz=ComboPT	Tr = -4 : Det = +1 Proper
Discrete AntiMatter (AM) Lorentz TransformType	Trace : Determinant

Line up by Trace Invariant values



SR:Lorentz Transform

$\partial_{\nu}[R^{\mu'}] = \partial R^{\mu'}/\partial R^{\nu} = \Lambda^{\mu'}{}_{\nu}$

$\Lambda^{\mu}{}_{\nu} = (\Lambda^{-1})_{\nu}{}^{\mu} : \Lambda^{\mu}{}_{\alpha}\Lambda^{\alpha}{}_{\nu} = \eta^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu}$

$\eta_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta} = \eta_{\alpha\beta}$

$\mathrm{Det}[\Lambda^{\mu}{}_{\nu}]=\pm 1$ $\Lambda_{\mu\nu}\Lambda^{\mu\nu}=4$

Two interesting properties of (1,1)-Tensors, of which the Lorentz Transform is an example:
Trace = Sum (Σ) of EigenValues : Determinant = Product (Π) of EigenValues
As Rank 4 Tensors, each Lorentz Transform has 4 EigenValues (EV's).
Create an Anti-Transform which has all EigenValue Tensor Invariants negated.
Σ[-(EV's)] = -Σ[EV's]: The Anti-Transform has negative Trace of the Transform.
Π[-(EV's)] = (-1)⁴Π[EV's] = Π[EV's]: The Anti-Transform has equal Determinant.

The Trace Invariant identifies a "Dual" Negative-Side for all Lorentz Transforms.

Lorentz Transforms $\Lambda^{\mu'}{}_{\nu} = \partial_{\nu}[X^{\mu'}]$

Lorentz Transform Connection Map - Interpretations
CPT, Big-Bang, (Matter-AntiMatter), Arrow-of-Time

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Based on the Lorentz Transform properties of the last few pages, here is interesting observation about Lorentz Transforms:
They all have Determinant of ±1, and Inner Product of 4, but the Trace varies depending on the particular Transform.

SR:Lorentz Transform

$\partial_{\nu}[R^{\mu'}] = \partial R^{\mu'}/\partial R^{\nu} = \Lambda^{\mu'}{}_{\nu}$

$\Lambda_{\mu}{}^{\nu} = (\Lambda^{-1})_{\nu}{}^{\mu} : \Lambda^{\mu}{}_{\alpha}\Lambda^{\alpha}{}_{\nu} = \eta^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu}$

$\eta_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta} = \eta_{\alpha\beta}$

$\mathrm{Det}[\Lambda^{\mu}{}_{\nu}]=\pm 1$ $\Lambda_{\mu\nu}\Lambda^{\mu\nu}=4$

The Trace of the Identity is at 4. Assume this applies to normal matter particles.
The Trace of normal matter particle Rotations varies from (0..4)
The Trace of the normal matter particle Boosts varies from (4..Infinity)
So, one can think of Trace = 4 being the connection point between normal matter Rotations and Boosts.

Now, various Flip Transforms (inc. the Time Reversal and Parity Transforms, and their combination as PT transform),
take the Trace in steps from (-4,-2,0,+2+4). Applying a bit of symmetry:

The Trace of the Negative Identity is at -4. Assume this applies to anti-matter particles.
The Trace of anti-matter particle Rotations varies from (0..-4)
The Trace of the anti-matter particle Boosts varies from (-4..-Infinity)
So, one can think of Trace = -4 being the connection point between anti-matter Rotations and Boosts.

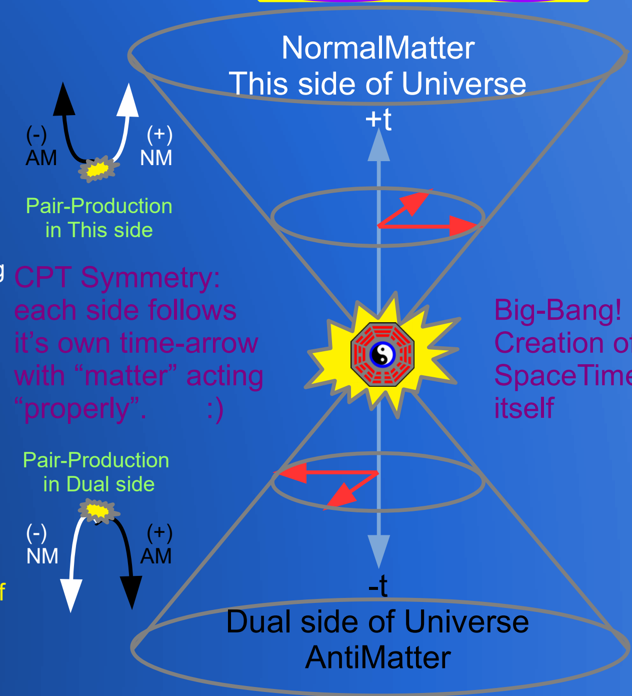
This observation would be in agreement with the CPT Theorem (Feynman-Stueckelberg) idea that normal matter particles moving backward in time are CPT symmetrically equivalent to antimatter particles moving forward in time.

Now, scale this up to Universe size: The Baryon Asymmetry problem (aka. The Matter-AntiMatter Asymmetry Problem).
If the Universe was created as a huge chunk of energy, and matter-creating energy is always transformed into matter-antimatter mirrored pairs, then where is all the antimatter??? Turns out this is directly related to the Arrow-of-Time Problem as well.

Answer: It is temporally on the "Other/Dual side" of the Big-Bang! The antimatter created at the Big-Bang is travelling in the negative time (-t) direction from the Big-Bang creation point, and the normal matter is travelling in the positive time direction (+t).
Universal CPT Symmetry. So, what happened "before" the Big-Bang? It "is" the AntiMatter Dual to our normal matter universe!
Pair-production is creation of AM-NM mirrored pairs within SpaceTime. The Big-Bang is the creation of SpaceTime itself.

This also resolves the Arrow-of-Time Problem. If all known physical microscopic processes are time-symmetric, why is the flow of Time experienced as uni-directional??? {see Wikipedia "CPT Symmetry","CP Violation","Andrei Sakharov"}

Answer: Time flow on this side of the Universe is in the (+t) direction, while time flow on the dual side of the Universe is in the (-t) direction. The math all works out. Time flow is bi-directional, but on opposite sides of the Big-Bang! **Universal CPT Symmetry**.



This gives total CPT Symmetry to all of the possible Lorentz Transforms (AM=Antimatter, NM=Normal Matter):
Various (AM_Flips) : Various (NM_Flips)
-Infinity...(AM_Boosts)...(AM_Identity=-4)...(AM_Rotations)...0...(NM_Rotations)...(+4=NM_Identity)...(NM_Boosts)...+Infinity

This solves the:
Baryon (Matter-AntiMatter) Asymmetry Problem
& Arrow(s)-of-Time Problem (+ / -)

Lorentz Transform Connection Map – Interpretations 2
CPT, Big-Bang, (Matter-AntiMatter), Arrows-of-Time

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This idea of Universal CPT Symmetry also gives a Universal Dimensional Symmetry as well.

Consider the well known "balloon" analogy of the universe expansion. The "spatial" coordinates are on the surface of the balloon, and the expansion is in the +t direction. There is symmetry in the +/- directions of the spatial coordinates, but the time flow is always uni-directional, +t, as the balloon gets bigger.

By allowing a "dual side", it provides a universal dimensional symmetry. One now has +/- symmetry for the temporal directions.

The "center" of the Universe is literally, the Big Bang Singularity. It is the "center=zero" point of both time and space directions.

The expansion gives time flow away from Big Bang singularity in both the Normal Side (+) and the Dual "Side (-).
The spatial coordinates expand in both the (+/-) directions on both sides.

Note that this gives an unusual interpretation of what came "before" the Big Bang.
The "past" on either side extends only to the BB singularity, not beyond. Time flow is always away from this creation singularity.

This is also in accord with known black hole physics, in that all matter entering a BH ends at the BH singularity.
Time and space coordinates both come to a stop at either type of singularity, from the point of view of an observer that is in the spacetime but not at the singularity.

So, the Big Bang is a "starting" singularity, and black holes are "ending" singularities.
Also provides for idea of "white holes" actually just being black holes on the alternate side. White hole=time-reversed black hole.
This way, the mass is still attractive. Time flow is simply reversed on the alternate side so stuff still goes into the hole...

So, Universal CPT Symmetry = Universal Dimensional Symmetry.

And, going even further, I suspect this is the reason there is a duality in Metric conventions.
In other words, physicists have wondered why one can use {+,-,-,-} or {-,+,+,+}.
I submit that one of these metrics applies to the Normal Matter side, while the other complementarily applies to the Dual side.
This would allow correct causality conditions to apply on either side.
Again, this is similar to the Dirac prediction of antimatter based on a duality of possible solutions.

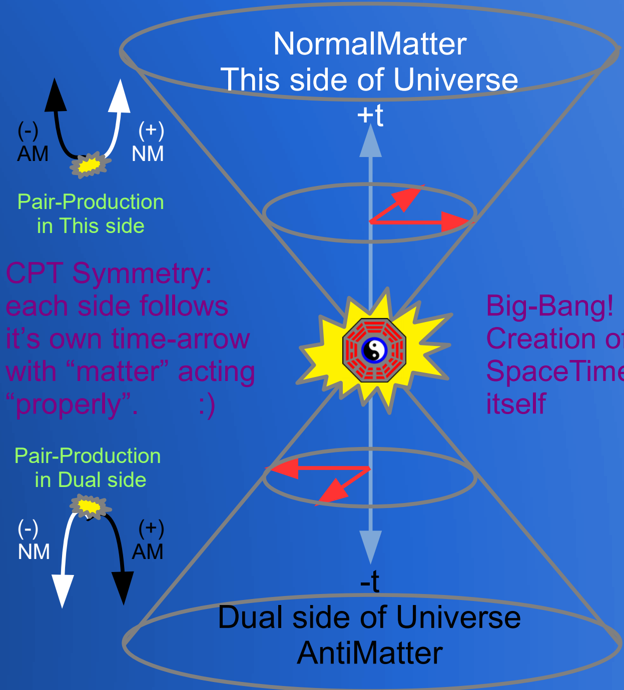
SR:Lorentz Transform

$\partial_{\nu}[R^{\mu'}] = \partial R^{\mu'}/\partial R^{\nu} = \Lambda^{\mu'}{}_{\nu}$

$\Lambda_{\mu}{}^{\nu} = (\Lambda^{-1})_{\nu}{}^{\mu} : \Lambda^{\mu}{}_{\alpha}\Lambda^{\alpha}{}_{\nu} = \eta^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu}$

$\eta_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta} = \eta_{\alpha\beta}$

$\mathrm{Det}[\Lambda^{\mu}{}_{\nu}]=\pm1$ $\Lambda_{\mu\nu}\Lambda^{\mu\nu}=4$



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This solves the:
Baryon (Matter-AntiMatter) Asymmetry Problem
& Arrow(s)-of-Time Problem (+ / -)

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SRQM Study:
Model SpaceTimes

Model SpaceTimes Klein Geometry G/H	$\Lambda < 0$	$\Lambda = 0$	$\Lambda > 0$
Lorentzian pseudo-Riemannian	Anti de Sitter SO(3,2)/SO(3,1)	Minkowski ISO(3,1)/SO(3,1) $ds^2 = (cdt)^2 - \mathbf{dx}\cdot\mathbf{dx}$	De Sitter SO(4,1)/SO(3,1)
Riemannian	Hyperbolic SO(4,1)/SO(4)	Euclidean ISO(4)/SO(4) $ds^2 = (cdt)^2 + \mathbf{dx}\cdot\mathbf{dx}$	Spherical SO(5)/SO(4)

Lie Groups

de Sitter Group SO(1,4)

de Sitter invariant relativity
(maybe?)

Poincaré Group ISO(1,3)

{ $r \ll r_{dS}$ = de Sitter Radius}

$r_{dS} = \sqrt{[3/\Lambda]} = L_H/\sqrt{[\Omega_\Lambda]}$

SR & GR Physics
(** *currently thought correct* **)

$\Lambda^{\mu'}{}_{\nu} \rightarrow B^{\mu'}{}_{\nu} =$ Boost

$$\begin{array}{c|cccc} & t & x & y & z \\ \hline t & \gamma & -\beta\gamma & 0 & 0 \\ x & -\beta\gamma & \gamma & 0 & 0 \\ y & 0 & 0 & 1 & 0 \\ z & 0 & 0 & 0 & 1 \end{array}$$

Galilei Group

{ $|\mathbf{v}| \ll c$ }
Classical Physics

$\Lambda^{\mu'}{}_{\nu} \rightarrow S^{\mu'}{}_{\nu} =$ Motion:Shear

$$\begin{array}{c|cccc} & t & x & y & z \\ \hline t & 1 & 0 & 0 & 0 \\ x & -\beta & 1 & 0 & 0 \\ y & 0 & 0 & 1 & 0 \\ z & 0 & 0 & 0 & 1 \end{array}$$

SRQM Transforms: Venn Diagram

Poincaré = Lorentz + Translations

(10) (6) (4)

A Tensor Study of Physical 4-Vectors

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Transformations

(# of independent parameters = # continuous symmetries = # Lie Dimensions)

Poincaré Transformation Group aka. Inhomogeneous Lorentz Transformation
Lie group of all affine isometries of SR:Minkowski TimeSpace (preserve quadratic form $\eta_{\mu\nu}$)
General Linear,Affine Transform $X^{\mu'} = \Lambda^{\mu'}{}_{\nu}X^{\nu} + \Delta X^{\mu'}$ with $\mathrm{Det}[\Lambda^{\mu'}{}_{\nu}] = \pm 1$
(6+4=10)

Lorentz Transform $\Lambda^{\mu'}{}_{\nu}$

(3+3=6) 4-Tensor {mixed type-(1,1)}

Discrete

Time-reversal
$\Lambda^{\mu'}{}_{\nu} \to T^{\mu'}{}_{\nu}$
(0)
t → -t*
time parity
anti-unitary

SpatialFlipCombos
$\Lambda^{\mu'}{}_{\nu} \to F^{\mu'}{}_{\nu}$
(0)
{x|y|z} → -{x|y|z}
unitary

Parity-Inversion
$\Lambda^{\mu'}{}_{\nu} \to P^{\mu'}{}_{\nu}$
(0)
r → -**r**
space parity
unitary

Identity $I_{(4)}$
$\Lambda^{\mu'}{}_{\nu} \to \eta^{\mu'}{}_{\nu} = \delta^{\mu'}{}_{\nu}$
(0)
no mixing
unitary

Charge-Conjugation
$\Lambda^{\mu'}{}_{\nu} \to C^{\mu'}{}_{\nu}$
(0)
R → -**R***, q → -q
charge parity
anti-unitary

CPT Symmetry
{Charge}
{Partiy}
{Time}

Continuous

Rotation
$\Lambda^{\mu'}{}_{\nu} \to R^{\mu'}{}_{\nu}$
(3)
x:y | x:z | y:z

Boost
$\Lambda^{\mu'}{}_{\nu} \to B^{\mu'}{}_{\nu}$
(3)
t:x | t:y | t:z

Isotropy
{same all directions}

Translation Transform $\Delta X^{\mu'}$

(1+3=4) 4-Vector

Discrete

4-Zero
$\Delta X^{\mu'} \to (0,\mathbf{0})$
(0)
no motion

Continuous

Temporal
$\Delta X^{\mu'} \to (c\Delta t,\mathbf{0})$
(1)
Δt

Spatial
$\Delta X^{\mu'} \to (0,\Delta\mathbf{x})$
(3)
Δx | Δy | Δz

Homogeneity
{same all points}

	M^{01}	M^{02}	M^{03}
M^{10}		M^{12}	M^{13}
M^{20}	M^{21}		M^{23}
M^{30}	M^{31}	M^{32}	

P^0
P^1
P^2
P^3

4-AngularMomentum $M^{\mu\nu} = X^{\mu} \wedge P^{\nu} = X^{\mu}P^{\nu} - X^{\nu}P^{\mu}$
= Generator of Lorentz Transformations (6)
= { $\Lambda^{\mu'}{}_{\nu} \to R^{\mu'}{}_{\nu}$ Rotations (3) + $\Lambda^{\mu'}{}_{\nu} \to B^{\mu'}{}_{\nu}$ Boosts (3) }

4-LinearMomentum P^{μ}
= Generator of Translation Transformations (4)
= { $\Delta X^{\mu'} \to (c\Delta t,\mathbf{0})$ Time (1) + $\Delta X^{\mu'} \to (0,\Delta\mathbf{x})$ Space (3) }

$\mathrm{Det}[\Lambda^{\mu'}{}_{\nu}] = +1$ for Proper Lorentz Transforms
$\mathrm{Det}[\Lambda^{\mu'}{}_{\nu}] = -1$ for Improper Lorentz Transforms

Lorentz Matrices can be generated by a matrix M with Tr[M]=0 which gives:
{ $\Lambda = e \wedge M = e \wedge (+\boldsymbol{\theta}\cdot\mathbf{J} - \boldsymbol{\zeta}\cdot\mathbf{K})$ }
{ $\Lambda^T = (e \wedge M)^T = e \wedge M^T$ }
{ $\Lambda^{-1} = (e \wedge M)^{-1} = e \wedge -M$ }

$M = +\boldsymbol{\theta}\cdot\mathbf{J} - \boldsymbol{\zeta}\cdot\mathbf{K}$
$B[\boldsymbol{\zeta}] = e\wedge(-\boldsymbol{\zeta}\cdot\mathbf{K})$
$R[\boldsymbol{\theta}] = e\wedge(+\boldsymbol{\theta}\cdot\mathbf{J})$
$\Lambda = e \wedge M = e \wedge (+\boldsymbol{\theta}\cdot\mathbf{J} - \boldsymbol{\zeta}\cdot\mathbf{K})$

SR:Lorentz Transform
$\partial_{\nu}[R^{\mu'}] = \partial R^{\mu'}/\partial R^{\nu} = \Lambda^{\mu'}{}_{\nu}$
$\Lambda^{\mu}{}_{\nu} = (\Lambda^{-1})_{\nu}{}^{\mu} : \Lambda^{\mu}{}_{\alpha}\Lambda^{\alpha}{}_{\nu} = \eta^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu}$
$\eta_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta} = \eta_{\alpha\beta}$
$\mathrm{Det}[\Lambda^{\mu}{}_{\nu}] = \pm 1$ $\Lambda_{\mu\nu}\Lambda^{\mu\nu} = 4$

Rotations $J_i = -\varepsilon_{imn}M^{mn}/2$, Boosts $K_i = M_{i0}$

[(**R** → -**R***)] or [(t → -t*) & (**r** → -**r**)] imply q → -q
Feynman-Stueckelberg Interpretation
Amusingly, Inhomogeneous Lorentz adds homogeneity.

Classical Transforms: Venn Diagram

Full Galilean = Galilean + Translations

(10) (6) (4)

A Tensor Study of Physical 4-Vectors

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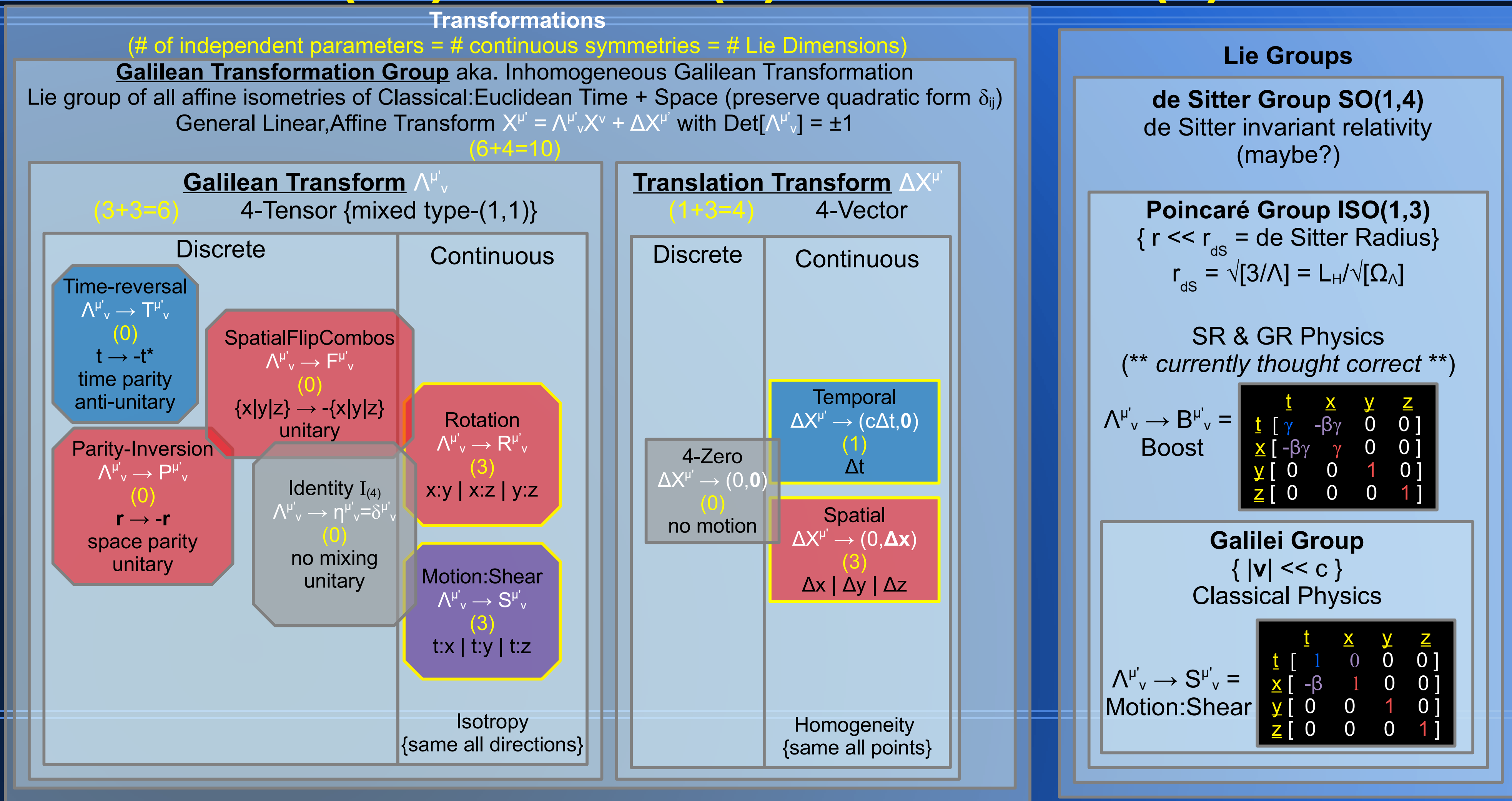

Review of SR Transforms

10 Poincaré Symmetries, 10 Conservation Laws

10 Generators : Noether's Theorem

A Tensor Study of Physical 4-Vectors

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4-Displacement
$\Delta\mathbf{X} = (c\Delta t, \Delta\mathbf{x})$

ΔTime Transform $\Delta X^{\mu'} \to (c\Delta t, \mathbf{0})$
Generated by energy $E = cp^0$

Translation Transform
Generated $\Delta X^{\mu'}(t,\mathbf{x}) = \exp[\mathbf{X}\cdot\mathbf{P}/\hbar]^{\mu'}$

ΔSpace Transform $\Delta X^{\mu'} \to (0, \Delta\mathbf{x})$
Generated by 3-momentum $\mathbf{p} = p^i$

4-Position
$\mathbf{X} = (ct, \mathbf{x})$

$\partial\cdot\mathbf{X} = 4$
SpaceTime Dimension

Minkowski
$\partial[\mathbf{X}] = \partial^\mu[X^\nu] = \eta^{\mu\nu}$
Metric

Conservation of scalar Energy (temporal)

Conservation of 4-Momentum
(1 + 3) = (4) Laws

Conservation of Linear 3-momentum (spatial)

$\mathbf{U}\cdot\partial[..]$
$\gamma d/dt[..]$
$d/d\tau[..]$

4-Velocity
$\mathbf{U} = \gamma(c, \mathbf{u})$
$= d\mathbf{X}/d\tau$

m_o
E_o/c^2

*
Generator
P^μ

4-Momentum
$P^\mu = \mathbf{P} = (mc, \mathbf{p}) = (E/c, \mathbf{p})$

*
Generator
$M^{\mu\nu}$

4-AngMomentum Tensor
$M^{\mu\nu} = X^\mu P^\nu - X^\nu P^\mu = \mathbf{X} \wedge \mathbf{P}$
$\to$

$$\begin{bmatrix} 0 & -cn^x & -cn^y & -cn^z \\ +cn^x & 0 & +l^z & -l^y \\ +cn^y & -l^z & 0 & +l^x \\ +cn^z & +l^y & -l^x & 0 \end{bmatrix} = \begin{bmatrix} 0 & -cn^j \\ +cn^i & \varepsilon^{ij}{}_k l^k \end{bmatrix}$$

4-Gradient
$\partial = (\partial_t/c, -\nabla)$

Angular $M^{\mu\nu}$ + Linear P^μ
(3 + 3) + (1 + 3)
= (6) + (4)
= 10 Symmetries = 10 Generators = 10 Conservation Laws: Noether's Theorem

Lagrange "Shift Operator" version of Taylor's Theorem: $e^{a(d/dx)} f(x) = f(x+a)$
Bloch Theorem:Translation Operator: $e^{i(\mathbf{K}\cdot\mathbf{R})}\psi(\mathbf{X}) = \psi(\mathbf{X}+\mathbf{R})$, with $\mathbf{K}$ as reciprocal lattice

Lorentz General Time-Space Boost Transform
$\Lambda^{\mu'}{}_\nu \to B^{\mu'}{}_\nu = \begin{bmatrix} \gamma & -\gamma\beta_j \\ -\gamma\beta^i & (\gamma-1)\beta^i\beta_j/(\boldsymbol{\beta}\cdot\boldsymbol{\beta}) + \delta^i_j \end{bmatrix}$
Generated by relativistic massmoment 3-vector $c\mathbf{n}$

Conservation of relativistic 3-mass-moment (temporal-spatial)

Lorentz Transform $\partial_\nu[X^{\mu'}] = \partial X^{\mu'}/\partial X^\nu = \Lambda^{\mu'}{}_\nu$
Generated $\Lambda^{\mu'}{}_\nu(\boldsymbol{\zeta},\boldsymbol{\theta}) = \exp[1/2\ \omega_{\alpha\beta}M^{\alpha\beta}]^{\mu'}{}_\nu = \exp[\boldsymbol{\zeta}\cdot\mathbf{K} + \boldsymbol{\theta}\cdot\mathbf{J}]^{\mu'}{}_\nu$

Lorentz General Space-Space Rotation Transform
$\Lambda^{\mu'}{}_\nu \to R^{\mu'}{}_\nu = \begin{bmatrix} 1 & 0_j \\ 0^i & (\delta^i_j - n^i n_j)\cos(\theta) - (\varepsilon^i{}_{jk} n^k)\sin(\theta) + n^i n_j \end{bmatrix}$
Generated by angular-momentum-3-vector $\mathbf{l} = \mathbf{r} \wedge \mathbf{p}$

Conservation of 4-AngularMomentum
(3 + 3) = (6) Laws

Conservation of Angular 3-momentum (spatial-spatial)

Poincaré Transformation Group aka. Inhomogeneous Lorentz Transformation
The group of all isometries of SR:Minkowski Spacetime (6 + 4 = 10)
(preserve quadratic form)

General Linear,Affine Transform $X^{\mu'} = \Lambda^{\mu'}{}_\nu X^\nu + \Delta X^{\mu'}$ with $\mathrm{Det}[\Lambda^{\mu'}{}_\nu] = \pm 1$

4-AngularMomentum $M^{\mu\nu} = \mathbf{X} \wedge \mathbf{P} = X^\mu P^\nu - X^\nu P^\mu$
= Generator of Lorentz Transformations (6)
= { $\Lambda^{\mu'}{}_\nu \to R^{\mu'}{}_\nu$ Rotations (3) + $\Lambda^{\mu'}{}_\nu \to B^{\mu'}{}_\nu$ Boosts (3) }

0	$-c\mathbf{n}$
$c\mathbf{n}^T$	$\mathbf{l} = \mathbf{x} \wedge \mathbf{p}$

4-LinearMomentum $P^\mu = \mathbf{P}$
= Generator of Translation Transformations (4)
= { $\Delta X^{\mu'} \to (c\Delta t, \mathbf{0})$ Time (1) + $\Delta X^{\mu'} \to (0, \Delta\mathbf{x})$ Space (3) }

$E/c = p^0$
$\mathbf{p} = p^j$

Jacobi's Formula for Complex Square Matrix A: Det(Exp[A])=Exp(Tr[A])
$\mathrm{Det}(A)_{4D} = ((\mathrm{tr}\,A)^4 - 6\,\mathrm{tr}(A^2)(\mathrm{tr}\,A)^2 + 3(\mathrm{tr}(A^2))^2 + 8\,\mathrm{tr}(A^3)\,\mathrm{tr}\,A - 6\,\mathrm{tr}(A^4))/24$

Review of SR Transforms
Poincaré Algebra & Generators
Casimir Invariants

The (10) one-parameter groups can be expressed directly as exponentials of the generators:
$U[I, (a^0,\mathbf{0})] = e^\wedge(ia^0\cdot H) = e^\wedge(ia^0\cdot P^0)$: (1) Hamiltonian = Energy = Temporal Momentum H
$U[I, (0,\lambda\hat{\mathbf{a}})] = e^\wedge(-i\lambda\hat{\mathbf{a}}\cdot\mathbf{p})$: (3) Linear Momentum p
$U[\Lambda(i\lambda\hat{\boldsymbol{\theta}}/2), 0] = e^\wedge(i\lambda\hat{\boldsymbol{\theta}}\cdot\mathbf{j})$: (3) Angular Momentum **j**
$U[\Lambda(\lambda\hat{\boldsymbol{\varphi}}/2), 0] = e^\wedge(i\lambda\hat{\boldsymbol{\varphi}}\cdot\mathbf{k})$: (3) Lorentz Boost **k**
The Poincaré Algebra is the Lie Algebra of the Poincaré Group:
Total of (1+3+3+3 = 4+6 = 10) Invariances from Poincaré Symmetry

Poincaré Algebra is the Lie Algebra of the Poincaré Group.

	$M^{01}= -cn^1$	$M^{02}= -cn^2$	$M^{03}= -cn^3$
$M^{10}= cn^1$		$M^{12}= l^3$	$M^{13}= -l^2$
$M^{20}= cn^2$	$M^{21}= -l^3$		$M^{23}= l^1$
$M^{30}= cn^3$	$M^{31}= l^2$	$M^{32}= -l^1$	

P^0
P^1
P^2
P^3

$M^{\mu\nu} = \mathbf{X} \wedge \mathbf{P} = X^\mu P^\nu - X^\nu P^\mu$
$P^\mu = \mathbf{P}$

0	-cn
cn^T	$\mathbf{l} = \mathbf{x} \wedge \mathbf{p}$

$E/c = p^0$
$\mathbf{p} = p^j$

M = Generator of Lorentz Transformations (6) = { Rotations (3) + Boosts (3) }
P = Generator of Translation Transformations (4) = { Time (1) + Space (3) }

Covariant form:
These are the commutators of the the Poincaré Algebra :
$[X^\mu, X^\nu] = 0^{\mu\nu}$
$[P^\mu, P^\nu] = -i\hbar q(F^{\mu\nu})$ if interacting with EM field; otherwise $= 0^{\mu\nu}$ for free particles
$M^{\mu\nu} = (X^\mu P^\nu - X^\nu P^\mu) = i\hbar(X^\mu\partial^\nu - X^\nu\partial^\mu)$
$[M^{\mu\nu}, P^\rho] = i\hbar(\eta^{\rho\nu}P^\mu - \eta^{\rho\mu}P^\nu)$
$[M^{\mu\nu}, M^{\rho\sigma}] = i\hbar(\eta^{\nu\rho}M^{\mu\sigma} + \eta^{\mu\sigma}M^{\nu\rho} + \eta^{\sigma\nu}M^{\rho\mu} + \eta^{\rho\mu}M^{\sigma\nu})$

Rotations $J_i = -\varepsilon_{imn}M^{mn}/2$, Boosts $K_i = M_{i0}$

The set of all Lorentz Generators $V = \{\boldsymbol{\zeta}\cdot\mathbf{K} + \boldsymbol{\theta}\cdot\mathbf{J}\}$ forms a vector space over the real numbers. The generators $\{J_x , J_y , J_z , K_x , K_y , K_z\}$ form a basis set of V. The components of the axis-angle vector and rapidity vector $\{\theta_x , \theta_y , \theta_z , \zeta_x , \zeta_y , \zeta_z\}$ are the coordinates of a Lorentz generator wrt. this basis.

Component form: Rotations $J_i = -\varepsilon_{imn}M^{mn}/2$, Boosts $K_i = M_{i0}$
$[J_m,P_n] = i\varepsilon_{mnk}P^k$
$[J_m,P_0] = 0$
$[K_j,P_k] = i\eta_{jk}P^0$
$[K_j,P_0] = -iP_j$
$[J_m,J_n] = i\varepsilon_{mnk}J^k$
$[J_m,K_n] = i\varepsilon_{mnk}K^k$
$[K_m,K_n] = -i\varepsilon_{mnk}J^k$, a Wigner Rotation resulting from consecutive boosts
$[J_m + iK_m,J_n - iK_n] = 0$

Very importantly, the Poincaré group has Casimir Invariant Eigenvalues = { Mass m, Spin j }, hence Mass *and* Spin are purely SR phenomena, no QM axioms required!

This Representation of the Poincaré Group or Representation of the Lorentz Group is known as Wigner's Classification in Representation Theory of Particle Physics

Poincaré Algebra has 2 Casimir Invariants = Operators that commute with all of the Poincaré Generators
These are $\{P^2 = P^\mu P_\mu = (m_o c)^2, W^2 = W^\mu W_\mu = -(m_o c)^2 j(j + 1)\}$, with $W^\mu = (-1/2)\varepsilon^{\mu\nu\rho\sigma}J_{\nu\rho}P_\sigma$ as the Pauli-Lubanski Pseudovector

$[P^2,P^0] = [P^2,P^i] = [P^2,J^i] = [P^2,K^i] = 0$: Hence the 4-Momentum Magnitude squared commutes with all Poincaré Generators
$[W^2,P^0] = [W^2,P^i] = [W^2,J^i] = [W^2,K^i] = 0$: Hence the 4-SpinMomentum Magnitude squared commutes with all Poincaré Generators

10 Poincaré Symmetry Invariances
Noether's Theorem: 10 SR Conservation Laws

d'Alembertian Invariant Wave Equation: $\partial\cdot\partial = (\partial_t/c)^2 - \nabla\cdot\nabla = (\partial_\tau/c)^2$

4-Gradient
$\partial = (\partial_t/c, -\nabla)$
$= (\partial_t/c, -\partial_x, -\partial_y, -\partial_z)$
$= (\partial/c\partial t, -\partial/\partial x, -\partial/\partial y, -\partial/\partial z)$

Invariant d'Alembertian Wave Equation
$\partial\cdot\partial = (\partial_t/c)^2 - \nabla\cdot\nabla$

<u>Time Translation:</u>
Let $\mathbf{X}_T = (ct+c\Delta t, \mathbf{x})$, then $\partial[\mathbf{X}_T] = (\partial_t/c, -\nabla)(ct+c\Delta t, \mathbf{x}) = \mathrm{Diag}[1, \mathbf{-1}] = \partial[\mathbf{X}] = \boldsymbol{\eta}^{\mu\nu}$
so $\partial[\mathbf{X}_T] = \partial[\mathbf{X}]$ and $\partial[\mathbf{K}] = [[\mathbf{0}]]$
$(\partial\cdot\partial)[\mathbf{K}\cdot\mathbf{X}_T] = \partial\cdot(\partial[\mathbf{K}\cdot\mathbf{X}_T]) = \partial[\mathbf{K}]\cdot\mathbf{X}_T + \mathbf{K}\cdot\partial[\mathbf{X}_T] = 0 + \mathbf{K}\cdot\partial[\mathbf{X}] = \partial[\mathbf{K}]\cdot\mathbf{X} + \mathbf{K}\cdot\partial[\mathbf{X}] = \partial\cdot(\partial[\mathbf{K}\cdot\mathbf{X}]) = (\partial\cdot\partial)[\mathbf{K}\cdot\mathbf{X}]$:

Time Translation Invariance (1)
Conservation of Energy = (Temporal) Momentum E
Temporal part of P^μ = (E/c,**p**)

<u>Space Translation:</u>
Let $\mathbf{X}_S = (ct, \mathbf{x}+\boldsymbol{\Delta}\mathbf{x})$, then $\partial[\mathbf{X}_S] = (\partial_t/c, -\nabla)(ct, \mathbf{x}+\boldsymbol{\Delta}\mathbf{x}) = \mathrm{Diag}[1, \mathbf{-1}] = \partial[\mathbf{X}] = \boldsymbol{\eta}^{\mu\nu}$
so $\partial[\mathbf{X}_S] = \partial[\mathbf{X}]$ and $\partial[\mathbf{K}] = [[\mathbf{0}]]$
$(\partial\cdot\partial)[\mathbf{K}\cdot\mathbf{X}_S] = \partial\cdot(\partial[\mathbf{K}\cdot\mathbf{X}_S]) = \partial[\mathbf{K}]\cdot\mathbf{X}_S + \mathbf{K}\cdot\partial[\mathbf{X}_S] = 0 + \mathbf{K}\cdot\partial[\mathbf{X}] = \partial[\mathbf{K}]\cdot\mathbf{X} + \mathbf{K}\cdot\partial[\mathbf{X}] = \partial\cdot(\partial[\mathbf{K}\cdot\mathbf{X}]) = (\partial\cdot\partial)[\mathbf{K}\cdot\mathbf{X}]$:

Space Translation Invariances (3)
Conservation of Linear (Spatial) Momentum **p**
Spatial part of P^μ = (E/c,**p**)

<u>Lorentz Space-Space Rotation:</u>
Let $\mathbf{X}_R = (ct, R[\mathbf{x}])$, then $\partial[\mathbf{X}_R] = (\partial_t/c, -\nabla)(ct, R[\mathbf{x}]) = \mathrm{Diag}[1, \mathbf{-1}] = \partial[\mathbf{X}] = \boldsymbol{\eta}^{\mu\nu}$
so $\partial[\mathbf{X}_R] = \partial[\mathbf{X}]$ and $\partial[\mathbf{K}] = [[\mathbf{0}]]$
$(\partial\cdot\partial)[\mathbf{K}\cdot\mathbf{X}_R] = \partial\cdot(\partial[\mathbf{K}\cdot\mathbf{X}_R]) = \partial[\mathbf{K}]\cdot\mathbf{X}_R + \mathbf{K}\cdot\partial[\mathbf{X}_R] = 0 + \mathbf{K}\cdot\partial[\mathbf{X}] = \partial[\mathbf{K}]\cdot\mathbf{X} + \mathbf{K}\cdot\partial[\mathbf{X}] = \partial\cdot(\partial[\mathbf{K}\cdot\mathbf{X}]) = (\partial\cdot\partial)[\mathbf{K}\cdot\mathbf{X}]$:

Lorentz Space-Space Rotation Invariances (3)
Conservation of Angular (Spatial) Momentum **l**
Spatial-Spatial part of $M^{\mu\nu}$ = **X^P**

<u>Lorentz Time-Space Boost:</u>
Let $\mathbf{X}_B = \gamma(ct-\boldsymbol{\beta}\cdot\mathbf{x}, -\boldsymbol{\beta}ct+\mathbf{x})$, then $\partial[\mathbf{X}_B] = (\partial_t/c, -\nabla)\gamma(ct-\boldsymbol{\beta}\cdot\mathbf{x}, -\boldsymbol{\beta}ct+\mathbf{x}) = [[\gamma, -\gamma\boldsymbol{\beta}], [-\gamma\boldsymbol{\beta}, \gamma]] = \boldsymbol{\Lambda}^{\mu\nu}$
$\partial[\mathbf{K}\cdot\mathbf{X}_B] = \partial[\mathbf{K}]\cdot\mathbf{X}_B + \mathbf{K}\cdot\partial[\mathbf{X}_B] = \boldsymbol{\Lambda}^{\mu\nu}\mathbf{K} = \mathbf{K}_B$ = a Lorentz Boosted **K**, as expected
$\partial\cdot\mathbf{K}_B = \partial\cdot\boldsymbol{\Lambda}^{\mu\nu}\mathbf{K} = \boldsymbol{\Lambda}_{\mu\nu}(\partial\cdot\mathbf{K}) = \boldsymbol{\Lambda}^{\mu\nu}(0) = 0 = \partial\cdot\mathbf{K}$ = Divergence of **K** = 0, as expected
$(\partial\cdot\partial)[\mathbf{K}\cdot\mathbf{X}_B] = \partial\cdot(\partial[\mathbf{K}\cdot\mathbf{X}_B]) = \partial\cdot\mathbf{K}_B = \partial\cdot\mathbf{K} = \partial\cdot(\partial[\mathbf{K}\cdot\mathbf{X}]) = (\partial\cdot\partial)[\mathbf{K}\cdot\mathbf{X}]$:

Lorentz Time-Space Boost Invariances (3)
Conservation of Relativistic Mass-Moment **n**
Temporal-Spatial part of $M^{\mu\nu}$ = **X^P**
see Wikipedia: Relativistic Angular Momentum

<u>SR Waves:</u>
Let $\Psi = ae^\wedge{-i(\mathbf{K}\cdot\mathbf{X})}$, $\Psi_T = ae^\wedge{-i(\mathbf{K}\cdot\mathbf{X}_T)}$, $\Psi_S = ae^\wedge{-i(\mathbf{K}\cdot\mathbf{X}_S)}$, $\Psi_R = ae^\wedge{-i(\mathbf{K}\cdot\mathbf{X}_R)}$, $\Psi_B = ae^\wedge{-i(\mathbf{K}\cdot\mathbf{X}_B)}$
$(\partial\cdot\partial)[\mathbf{K}\cdot\mathbf{X}_T] = (\partial\cdot\partial)[\mathbf{K}\cdot\mathbf{X}_S] = (\partial\cdot\partial)[\mathbf{K}\cdot\mathbf{X}_R] = (\partial\cdot\partial)[\mathbf{K}\cdot\mathbf{X}_B] = (\partial\cdot\partial)[\mathbf{K}\cdot\mathbf{X}]$: Wave Equation Invariant under all Poincaré transforms
Total of (1+3+3+3 = 10) Invariances from Poincaré Symmetry

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^\mu{}_\nu$ or $T_\mu{}^\nu$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu{}_\mu = T$
$\mathbf{V}\cdot\mathbf{V} = V^\mu\eta_{\mu\nu}V^\nu = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$
= Lorentz Scalar

SR 4-Vector Magnitudes
Dot Product, Lorentz Scalar Product
Einstein Summation Convention

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An example of the magnitude of a 3-vector is the length of a 3-displacement $\mathbf{\Delta r} = (\mathbf{r_1} - \mathbf{r_0})$.
Examine 3-position $\mathbf{r_1} \to \mathbf{r} = (x,y,z)$, which is a 3-displacement with the base at the origin $\mathbf{r_0} \to \mathbf{0} = (0,0,0)$.
The Dot Product of $\mathbf{r}$, $\{ \mathbf{r\cdot r} = r^j\delta_{jk}r^k = r_k r^k = r^j r_j = (x^*x + y^*y + z^*z) = (x^2 + y^2 + z^2) = r^2 \}$ is the Pythagorean Theorem.
The Kronecker Delta $\delta_{jk} = \mathrm{Diag}[1,1,1] = I_{(3)}$.
The magnitude is $\sqrt{[\mathbf{r\cdot r}]} = \sqrt{[r^2]} = |\mathbf{r}|$. 3D magnitudes are always positive.

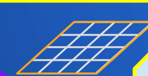
Galilean Invariant
$\mathbf{r\cdot r} = (x)^2+(y)^2+(z)^2 = (r)^2$
length r

3-position
$\mathbf{r} = r^i \to (x,y,z) =$ <location>

The magnitude of a 4-Vector is very similar to the magnitude of a 3-vector, but there are some interesting differences. One uses the Lorentz Scalar Product, a 4D Dot Product, which includes a time component, and is based on the SR:Minkowski Metric Tensor. I typically use the "Particle Physics" convention of the Minkowski Metric $\eta_{\mu\nu} \to \mathrm{Diag}[1,-1,-1,-1]$ {Cartesian form}, with the other entries zero.

SR:Minkowski Metric
$\partial[\mathbf{R}] = \partial^\mu R^\nu = \eta^{\mu\nu} = V^{\mu\nu} + H^{\mu\nu} \to$
$\mathrm{Diag}[1,-1,-1,-1] = \mathrm{Diag}[1,-I_{(3)}] = \mathrm{Diag}[1,-\delta^{jk}]$
{in Cartesian form} "Particle Physics" Convention
$\{\eta_{\mu\mu}\} = 1/\{\eta^{\mu\mu}\} : \eta_\mu{}^\nu = \delta_\mu{}^\nu$ $\mathrm{Tr}[\eta^{\mu\nu}]=4$

$\mathbf{A'\cdot A'} = \mathbf{A\cdot A} = A^\mu\eta_{\mu\nu}A^\nu = A_\nu A^\nu = A^\mu A_\mu = \Sigma_{\nu=0..3}[a_\nu a^\nu] = (a_0a^0 + a_1a^1 + a_2a^2 + a_3a^3) = \Sigma_{\mu=0..3}[a^\mu a_\mu]$
$= (a^0a^0 - a^1a^1 - a^2a^2 - a^3a^3) = (a^0a^0 - \mathbf{a\cdot a})$
using the Einstein summation convention where upper-lower paired indices are summed over.



SR:Lorentz Transform
$\partial_\nu[R^{\mu'}] = \partial R^{\mu'}/\partial R^\nu = \Lambda^{\mu'}{}_\nu$
$\Lambda_\mu{}^\nu = (\Lambda^{-1})_\nu{}^\mu : \Lambda^\mu{}_\alpha\Lambda^\alpha{}_\nu = \eta^\mu{}_\nu = \delta^\mu{}_\nu$
$\eta_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta = \eta_{\alpha\beta}$
$\mathrm{Det}[\Lambda^\mu{}_\nu]=\pm1$ $\Lambda_{\mu\nu}\Lambda^{\mu\nu}=4$

$\mathbf{R\cdot R} = (ct)^2 - \mathbf{r\cdot r} = (ct)^2 - (x^2 + y^2 + z^2) = (c\Delta\tau)^2$
for 4-Position $\mathbf{R} = (ct,\mathbf{r})$
4D magnitudes can be negative,zero,positive

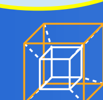
4-Position
$\mathbf{R} = R^\mu = (r^\mu) = (ct,\mathbf{r}) =$ <Event>

Lorentz Invariant
$\mathbf{R\cdot R} = (ct)^2 - \mathbf{r\cdot r} = (c\tau)^2$
Interval $c\tau$

SpaceTime
$\partial\cdot\mathbf{R} = \partial_\mu R^\mu = 4$
Dimension

The 4-Vector version has the Pythagorean element in the spatial components, the temporal component is of opposite sign.
This gives a "causality condition", where SpaceTime intervals (in the [+,-,-,-] metric) can be:

$\mathbf{\Delta R\cdot\Delta R} = [(c\Delta t)^2 - \mathbf{\Delta r\cdot\Delta r}] =$	$(c\Delta\tau)^2$	Time-like:Temporal	(+) {causal = temporally-ordered}
	(0)	Light-like:Null:Photonic	(0) {causal, maximum signal speed ($\lvert\mathbf{\Delta r}/\Delta t\rvert$=c)}
	$-(\Delta r_o)^2$	Space-like:Spatial	(–) {non-causal, spatially-extended}



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SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0,-\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Classical (scalar 3-vector)
Galilean Invariant
Not Lorentz Invariant

$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu{}_\mu = T$
$\mathbf{V\cdot V} = V^\mu\eta_{\mu\nu}V^\nu = [(v^0)^2 - \mathbf{v\cdot v}] = (v^0{}_o)^2$
= Lorentz Scalar

SRQM Study:

Lorentz Scalar Product $A \cdot B = A_\mu B^\mu$
Exterior Product $A \wedge B = A^\mu B^\nu - A^\nu B^\mu$

A Tensor Study of Physical 4-Vectors

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4-Gradient $\partial = (\partial_t/c, -\nabla)$

Minkowski $\partial[\mathbf{R}] = \partial^\mu[R^\nu] = \eta^{\mu\nu}$ Metric

Lorentz $\partial_\nu[R^{\mu'}] = \Lambda^{\mu'}{}_\nu$ Transform

SpaceTime $\partial \cdot \mathbf{R} = \partial_\mu R^\mu = 4$ Dimension

4-Position $\mathbf{R} = (ct, \mathbf{r})$

$\mathbf{U} \cdot \partial[..]$
$d/d\tau[..]$
$\gamma d/dt[..]$

$\partial \cdot \mathbf{A} = 0$ Lorenz Gauge

$\mathbf{R} \cdot \mathbf{P} = -S_{action,free}$ Action Scalar

4-Velocity $\mathbf{U} = \gamma(c, \mathbf{u})$

$F^{\mu\nu} = \partial \wedge \mathbf{A} = \partial^\mu A^\nu - \partial^\nu A^\mu$ EM Faraday 4-Tensor

$M^{\mu\nu} = \mathbf{R} \wedge \mathbf{P} = R^\mu P^\nu - R^\nu P^\mu$ 4-AngularMomentum 4-Tensor

EM

φ_o/c^2

ρ_o

m_o
E_o/c^2

Energy:Mass
$E = mc^2$

4-EMVectorPotential $\mathbf{A} = (\varphi/c, \mathbf{a})$

$(\partial \cdot \partial)\mathbf{A} - \partial(\partial \cdot \mathbf{A}) = \mu_o \mathbf{J}$ Maxwell EM Wave Eqn

4-Momentum $\mathbf{P} = (mc, \mathbf{p}) = (E/c, \mathbf{p})$

$\varepsilon_o c^2$
$1/\mu_o$

4-ChargeFlux
4-CurrentDensity
$\mathbf{J} = (\rho c, \mathbf{j}) = \rho(c, \mathbf{u})$

Electric:Magnetic
$1/(\varepsilon_o \mu_o) = c^2$

There are at least three 4-Vector relations which use the Exterior Product.

$\partial \wedge \mathbf{A} = \partial^\mu \wedge A^\nu = \partial^\mu A^\nu - \partial^\nu A^\mu = F^{\mu\nu}$: the Faraday EM 4-Tensor
$\mathbf{R} \wedge \mathbf{P} = R^\mu \wedge P^\nu = R^\mu P^\nu - R^\nu P^\mu = M^{\mu\nu}$: the 4-Angular Momentum
$\mathbf{R} \wedge \mathbf{F} = R^\mu \wedge F^\nu = R^\mu F^\nu - R^\nu F^\mu = \Gamma^{\mu\nu}$: the 4-Angular Torque

This gives the components of each remarkably similar properties.

Likewise, each of these has a physical Dot Product relation as well.

$\partial \cdot \mathbf{A} = \partial_\mu A^\mu = 0$: the Lorenz Gauge, a conservation of 4-EMVectorPotential
$\mathbf{R} \cdot \mathbf{P} = R_\mu P^\mu = -S_{action,free}$: the Action Scalar
$\mathbf{R} \cdot \mathbf{F} = R_\mu F^\mu = ???$: probably something important

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(0,0)-Tensor S
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$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu{}_\mu = T$
$\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_o)^2$
= Lorentz Scalar

SRQM Study:
4-Momentum, 4-Force
4-AngularMomentum, 4-Torque

Linear:
4-Force is the
ProperTime Derivative of 4-Momentum.

Angular:
4-Torque is the
ProperTime Derivative of 4-AngularMomentum.

$\mathrm{d}/\mathrm{d}\tau[\ M^{\mu\nu}\ ]$
$= \mathrm{d}/\mathrm{d}\tau[\ X^\mu P^\nu - X^\nu P^\mu\ ]$
$= [\ U^\mu P^\nu + X^\mu F^\nu - U^\nu P^\mu - X^\nu F^\mu\ ]$
$= [\ U^\mu m_o U^\nu + X^\mu F^\nu - U^\nu m_o U^\mu - X^\nu F^\mu\ ]$
$= [\ U^\mu m_o U^\nu - U^\nu m_o U^\mu + X^\mu F^\nu - X^\nu F^\mu\ ]$
$= [\ m_o(U^\mu U^\nu - U^\nu U^\mu) + X^\mu F^\nu - X^\nu F^\mu\ ]$
$= [\ m_o(0^{\mu\nu}) + X^\mu F^\nu - X^\nu F^\mu\ ]$
$= [\ X^\mu F^\nu - X^\nu F^\mu\ ]$

$\mathrm{d}/\mathrm{d}\tau[\ M^{\mu\nu}\ ] = \Gamma^{\mu\nu} = [\ X^\mu F^\nu - X^\nu F^\mu\ ] = \mathbf{X} \wedge \mathbf{F}$

SpaceTime
$\partial\cdot\mathbf{X} = \partial_\mu X^\mu = 4$
Dimension

Minkowski
$\partial[\mathbf{X}] = \partial^\mu[X^\nu] = \eta^{\mu\nu}$
Metric

Lorentz
$\partial_\nu[X^{\mu'}] = \Lambda^{\mu'}{}_\nu$
Transform

4-Gradient
$\partial = (\partial_t/c, -\nabla)$

4-Displacement
$\Delta\mathbf{X} = (c\Delta t, \Delta\mathbf{x})$

4-Position
$\mathbf{X} = (ct, \mathbf{x})$

$\mathbf{X}\cdot\mathbf{P} = -S_{action,free}$

$\mathbf{U}\cdot\partial[..]$
$\gamma d/dt[..]$
$d/d\tau[..]$

4-Velocity
$\mathbf{U} = \gamma(c, \mathbf{u})$
$= d\mathbf{X}/d\tau$

m_o
E_o/c^2

4-Momentum
$P^\mu = \mathbf{P} = (mc, \mathbf{p}) = (E/c, \mathbf{p})$

4-AngMomentum
Tensor
$M^{\mu\nu} = X^\mu P^\nu - X^\nu P^\mu = \mathbf{X} \wedge \mathbf{P}$
→

$$\begin{bmatrix} 0 & -cn^x & -cn^y & -cn^z \\ +cn^x & 0 & +l^z & -l^y \\ +cn^y & -l^z & 0 & +l^x \\ +cn^z & +l^y & -l^x & 0 \end{bmatrix}$$
=
$$\begin{bmatrix} 0 & -cn^j \\ +cn^i & \varepsilon^{ij}{}_k l^k \end{bmatrix}$$

$\mathbf{U}\cdot\partial[..]$
$\gamma d/dt[..]$
$d/d\tau[..]$

4-Torque
Tensor
$\Gamma^{\mu\nu} = X^\mu F^\nu - X^\nu F^\mu = \mathbf{X} \wedge \mathbf{F}$
=
$dM^{\mu\nu}/d\tau$

=

$\mathbf{U}\cdot\partial[..]$
$\gamma d/dt[..]$
$d/d\tau[..]$

4-Force
$\mathbf{F} = \gamma(\dot{E}/c, \mathbf{f})$
$= d\mathbf{P}/d\tau$

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(0,2)-Tensor $T_{\mu\nu}$

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(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu{}_\mu = T$
$\mathbf{V}\cdot\mathbf{V} = V^\mu\eta_{\mu\nu}V^\nu = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$
= Lorentz Scalar

SR → QM

4-Vector SRQM Interpretation of QM

SR Minkowski SpaceTime
4-Vectors, 4-CoVectors, Scalars, Tensors
Invariant Lorentz Scalar Product

A Tensor Study of Physical 4-Vectors

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4-Vectors are actually tensorial entities of Minkowski SpaceTime, (1,0)-Tensors, which maintain covariance for inertial observers, meaning that they may have different components for different observers, but describe the same physical object.
(like viewing a sculpture from different angles – snapshots look different but it's actually the same object)
There are also 4-CoVectors, or One-Forms, which are (0,1)-Tensors and dual to 4-Vectors.

Einstein & Lorentz "saw" the physics of SR,
Minkowski & Poincaré "saw" the mathematics of SR.
We are indebted to all of them for the simplicity, beauty, and power of how SR and 4-vectors work...

Both GR and SR use a metric tensor $g^{\mu\nu}$ to describe measurements in SpaceTime.
SR uses the "flat" Minkowski Metric $g^{\mu\nu} \to \eta^{\mu\nu} = \eta_{\mu\nu} \to \mathrm{Diag}[1,-\mathbf{I}_{(3)}] = \mathrm{Diag}[1,-\delta^{jk}] = \mathrm{Diag}[1,-1,-1,-1]$ {Cartesian form},
which is the {curvature ~ 0 limit = low-mass limit} of the GR metric $g^{\mu\nu}$.

4-Vectors = (1,0)-Tensors
$\mathbf{A} = A^\mu = (a^\mu) = (a^0,a^i) = (a^0,\mathbf{a}) = (a^0,a^1,a^2,a^3) \to (a^t,a^x,a^y,a^z)$
$\mathbf{B} = B^\mu = (b^\mu) = (b^0,b^i) = (b^0,\mathbf{b}) = (b^0,b^1,b^2,b^3) \to (b^t,b^x,b^y,b^z)$

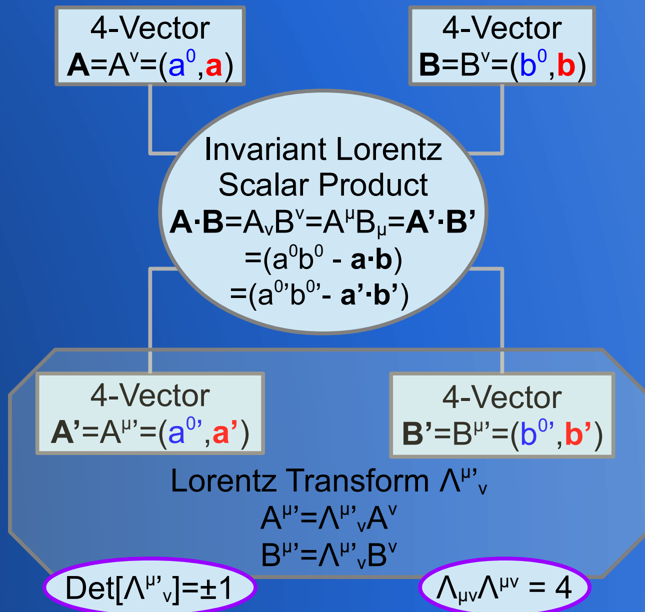
4-CoVectors = (0,1)-Tensors
$A_\mu = (a_\mu) = (a_0,a_i) = (a_0,-\mathbf{a}) = (a_0, a_1, a_2, a_3) \to (a_t, a_x, a_y, a_z)$ where $A_\mu = \eta_{\mu\nu} A^\nu$ and $A^\mu = \eta^{\mu\nu} A_\nu$
$= (a_0,a_i) = (a^0,-\mathbf{a}) = (a^0,-a^1,-a^2,-a^3) \to (a^t,-a^x,-a^y,-a^z)$
$B_\mu = (b_\mu) = (b_0,b_i) = (b_0,-\mathbf{b}) = (b_0, b_1, b_2, b_3) \to (b_t, b_x, b_y, b_z)$ where $B_\mu = \eta_{\mu\nu} B^\nu$ and $B^\mu = \eta^{\mu\nu} B_\nu$
$= (b_0,b_i) = (b^0,-\mathbf{b}) = (b^0,-b^1,-b^2,-b^3) \to (b^t,-b^x,-b^y,-b^z)$

Index raising & lowering

$\mathbf{A'}\cdot\mathbf{B'} = \mathbf{A}\cdot\mathbf{B} = A^\mu\eta_{\mu\nu}B^\nu = A_\nu B^\nu = A^\mu B_\mu = \Sigma_{\nu=0..3}[a_\nu b^\nu] = \Sigma_{u=0..3}[a^u b_u] = (a^0b^0 - \mathbf{a}\cdot\mathbf{b}) = (a^0b^0 - a^1b^1 - a^2b^2 - a^3b^3)$
using the Einstein summation convention where upper-lower paired indices are summed over

Proof that this is an invariant:
$\mathbf{A'}\cdot\mathbf{B'} = A^{\mu'}\eta_{\mu'\nu'}B^{\nu'} =$
$(\Lambda^{\mu'}{}_\alpha A^\alpha)\,\eta_{\mu'\nu'}(\Lambda^{\nu'}{}_\beta B^\beta) = (\Lambda^{\mu'}{}_\alpha \eta_{\mu'\nu'} \Lambda^{\nu'}{}_\beta)\, A^\alpha B^\beta = (\Lambda^{\nu'}{}_\alpha \Lambda^{\nu}{}_\beta)\, A^\alpha B^\beta = (\eta_{\alpha\rho}\Lambda^\rho{}_{\nu'}\Lambda^{\nu'}{}_\beta)\, A^\alpha B^\beta = (\eta_{\alpha\rho}\delta^\rho{}_\beta)\, A^\alpha B^\beta = (\eta_{\alpha\beta})\, A^\alpha B^\beta =$
$A^\alpha(\eta_{\alpha\beta})B^\beta = \mathbf{A}\cdot\mathbf{B}$

Lorentz Scalar Product → Lorentz Invariant Scalar = Same value for all inertial observers
Lorentz Invariants are also tensorial entities: (0,0)-Tensors



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= Lorentz Scalar

SR 4-Vectors & Lorentz Scalars

Rest Values (“naughts”=$_o$) are Lorentz Scalars

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson

$\mathbf{A}\cdot\mathbf{A} = (a^0a^0 - \mathbf{a}\cdot\mathbf{a}) = (a^0{}_o)^2$, where $(a^0{}_o)$ is the rest-value, the value of the temporal coordinate when the spatial coordinate is zero.
The “rest-values” of several physical properties are all Lorentz scalars.

4-Vector
$\mathbf{A}=(a^0,\mathbf{a})=(a^0,a^1,a^2,a^3)$
$\rightarrow(a^0{}_o,\mathbf{0})$ {in spatial rest frame}
$\mathbf{A}\cdot\mathbf{A}=(a^0{}_o)^2$

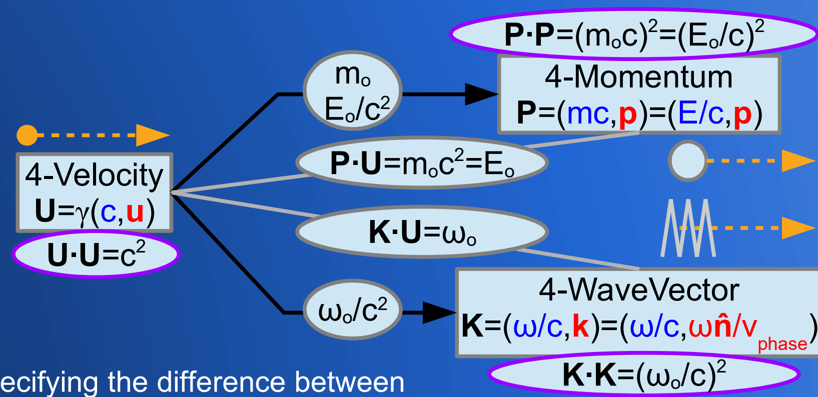
Notation:
“o” for rest values (naughts)
“0” for temporal components (0^{th} index)

$\mathbf{P} = (mc,\mathbf{p})$ | $\mathbf{K} = (\omega/c,\mathbf{k})$
$\mathbf{P}\cdot\mathbf{P} = (mc)^2 - \mathbf{p}\cdot\mathbf{p}$ | $\mathbf{K}\cdot\mathbf{K} = (\omega/c)^2 - \mathbf{k}\cdot\mathbf{k}$
($\mathbf{P}\cdot\mathbf{P}$) and ($\mathbf{K}\cdot\mathbf{K}$) are Lorentz Scalars. We can choose a frame that may simplify the expressions.

Choose a frame in which the spatial component is zero.
This is known as the “rest-frame” of the 4-Vector. It is not moving spatially.

$\mathbf{P}\cdot\mathbf{P} = (mc)^2 - \mathbf{p}\cdot\mathbf{p} = (m_oc)^2$ | $\mathbf{K}\cdot\mathbf{K} = (\omega/c)^2 - \mathbf{k}\cdot\mathbf{k} = (\omega_o/c)^2$
The resulting simpler expressions then give the “rest values”, indicated by ($_o$).
RestMass (m_o) and RestAngularFrequency (ω_o)
They are Invariant Lorentz Scalars by construction.

This leads to simple relations between 4-Vectors.
$\mathbf{P} = (m_o)\mathbf{U} = (E_o/c^2)\mathbf{U}$ | $\mathbf{K} = (\omega_o/c^2)\mathbf{U}$



And gives nice Scalar Product relations between 4-Vectors as well.
$\mathbf{P}\cdot\mathbf{U} = (m_o)\mathbf{U}\cdot\mathbf{U} = (m_o)c^2 = (E_o)$ | $\mathbf{K}\cdot\mathbf{U} = (\omega_o/c^2)\mathbf{U}\cdot\mathbf{U} = (\omega_o/c^2)c^2 = (\omega_o)$

This property of SR equations is a very good reason to use the “naught” convention for specifying the difference between relativistic component values which can vary, like (m), versus Rest Value Invariant Scalars, like (m_o), which do not vary.
They are usually related via a Lorentz Factor: { $m = \gamma m_o$ } and { $E = \gamma E_o$ }, as seen in the relation of **P** and **U**.

$\mathbf{P} = (mc,\mathbf{p}) = (m_o)\mathbf{U} = (m_o)\gamma(c,\mathbf{u}) = (\gamma m_oc,\gamma m_o\mathbf{u}) = (mc,m\mathbf{u}) = (mc,\mathbf{p})$
$\mathbf{P} = (E/c,\mathbf{p}) = (E_o/c^2)\mathbf{U} = (E_o/c^2)\gamma(c,\mathbf{u}) = (\gamma E_o/c,\gamma E_o\mathbf{u}/c^2) = (E/c,E\mathbf{u}/c^2) = (E/c,\mathbf{p})$

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(0,0)-Tensor S
Lorentz Scalar

$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^{\mu}{}_{\mu} = T$
$\mathbf{V}\cdot\mathbf{V} = V^{\mu}\eta_{\mu\nu}V^{\nu} = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$
= Lorentz Scalar

SR 4-Vectors & 4-Tensors
Lorentz Scalar Product & Tensor Trace Invariants: Similarities

All {4-Vectors:4-Tensors} have an associated {Lorentz Scalar Product:Trace}

Each 4-Vector has a "magnitude" given by taking the Lorentz Scalar Product of itself.
$\mathbf{V}\cdot\mathbf{V} = V^\mu\eta_{\mu\nu}V^\nu = V^\mu V_\mu = V_\nu V^\nu = (v_0v^0 + v_1v^1 + v_2v^2 + v_3v^3) = (v^0v^0 - \mathbf{v}\cdot\mathbf{v}) = (v^0{}_o)^2$
The absolute magnitude of $\mathbf{V}$ is $\sqrt{[|\mathbf{V}\cdot\mathbf{V}|]}$

Each 4-Tensor has a "magnitude" given by taking the Tensor Trace of itself.
$\mathrm{Trace}[T^{\mu\nu}] = \mathrm{Tr}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu{}_\mu = T_\nu{}^\nu = (T_0{}^0 + T_1{}^1 + T_2{}^2 + T_3{}^3) = (T^{00} - T^{11} - T^{22} - T^{33}) = T$
Note that the Trace runs down the diagonal of the 4-Tensor.

Notice the similarities. In both cases there is a tensor contraction with the Minkowski Metric Tensor $\eta_{\mu\nu} \rightarrow \mathrm{Diag}[1,-1,-1,-1]$ {Cartesian basis}

ex. $\mathbf{P}\cdot\mathbf{P} = (E/c)^2 - \mathbf{p}\cdot\mathbf{p} = (E_o/c)^2 = (m_oc)^2$
which says that the "magnitude" of the 4-Momentum is the RestEnergy/c = RestMass*c

ex. $\mathrm{Trace}[\eta^{\mu\nu}] = (\eta^{00} - \eta^{11} - \eta^{22} - \eta^{33}) = 1 -(-1) -(-1) -(-1) = 1+1+1+1 = 4$
which says that the "magnitude" of the Minkowski Metric = SpaceTime Dimension = 4

Lorentz Scalar Invariant

$\mathbf{V}\cdot\mathbf{V}=V^\mu V_\mu=(v^0v^0 - \mathbf{v}\cdot\mathbf{v})=(v^0{}_o)^2$

4-Vector
$\mathbf{V} = V^\mu = (v^0,\mathbf{v})$

Trace Tensor Invariant

$\mathrm{Tr}[T^{\mu\nu}]=T^\mu{}_\mu=(T^{00}-T^{11}-T^{22}-T^{33})=T$

4-Tensor
$T^{\mu\nu} = [T^{00},T^{01},T^{02},T^{03}]$
$[T^{10},T^{11},T^{12},T^{13}]$
$[T^{20},T^{21},T^{22},T^{23}]$
$[T^{30},T^{31},T^{32},T^{33}]$

$\mathbf{P}\cdot\mathbf{P}=(m_oc)^2=(E_o/c)^2$

4-Momentum
$\mathbf{P}=(mc,\mathbf{p})=(E/c,\mathbf{p})$

$\mathrm{Tr}[\eta^{\mu\nu}]= 4$

Minkowski Metric
$\partial[\mathbf{R}]=\eta^{\mu\nu}\rightarrow\mathrm{Diag}[1,-1,-1,-1]$

SR 4-Tensor
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(0,2)-Tensor $T_{\mu\nu}$

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(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0,\mathbf{v})$
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$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu{}_\mu = T$
$\mathbf{V}\cdot\mathbf{V} = V^\mu\eta_{\mu\nu}V^\nu = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$
= Lorentz Scalar

SR 4-Vectors & 4-Tensors
More 4-Vector-based Invariants
Phase Space Integration

Some 4-Vectors have an alternate form of Tensor Invariant: $d\mathbf{v}'/v^{0'} = d\mathbf{v}/v^0$
,in addition to the standard Lorentz Invariant $\mathbf{V}\cdot\mathbf{V} = V^\mu V_\mu = (v^0v^0 - \mathbf{v}\cdot\mathbf{v}) = (v^0{}_o)^2$

If $\mathbf{V}\cdot\mathbf{V}$ = (constant):, with $\mathbf{V} = (v^0,\mathbf{v})$
then $d(\mathbf{V}\cdot\mathbf{V}) = 2*(\mathbf{V}\cdot d\mathbf{V})$ = d(constant) = 0
hence $(\mathbf{V}\cdot d\mathbf{V}) = 0 = v^0dv^0 - \mathbf{v}\cdot d\mathbf{v}$
$dv^0 = \mathbf{v}\cdot d\mathbf{v}/v^0$

Generally:, with $\Lambda = \Lambda^{\mu'}{}_\nu$ = Lorentz Boost Transform in the $\boldsymbol{\beta}$-direction
$\mathbf{V}' = \Lambda\mathbf{V}$: from which the temporal component $v^{0'} = (\gamma v^0 - \gamma\boldsymbol{\beta}\cdot\mathbf{v})$
$d\mathbf{V}' = \Lambda d\mathbf{V}$: from which the spatial component $d\mathbf{v}' = (\gamma d\mathbf{v} - \gamma\boldsymbol{\beta}dv^0)$

Combining:
$d\mathbf{v}' = (\gamma d\mathbf{v} - \gamma\boldsymbol{\beta}(\mathbf{v}\cdot d\mathbf{v}/v^0))$
$d\mathbf{v}' = (1/v^0)*(\gamma v^0 d\mathbf{v} - \gamma\boldsymbol{\beta}(\mathbf{v}\cdot d\mathbf{v}))$
$d\mathbf{v}' = (1/v^0)*(\gamma v^0 - \gamma\boldsymbol{\beta}\cdot\mathbf{v})d\mathbf{v}$
$d\mathbf{v}' = (\gamma v^0 - \gamma\boldsymbol{\beta}\cdot\mathbf{v})*(1/v^0)*d\mathbf{v}$
$d\mathbf{v}' = (v^{0'}/v^0)d\mathbf{v}$
$d\mathbf{v}'/v^{0'} = d\mathbf{v}/v^0$ = Invariant of $\mathbf{V} = (v^0,\mathbf{v})$ for $\mathbf{V}\cdot\mathbf{V}$ = (constant)

So, for example:
$\mathbf{P}\cdot\mathbf{P} = (m_oc)^2$ = (constant)

An alternate approach is:
$\int d^4p\ \delta[p^2-(m_oc)^2]$
$= \int d^4p\ (1/2|m_oc|)\ (\delta[p+m_oc] + \delta[p-m_oc])$
$=cd^3p/2E$
= Invariant

Thus, $d\mathbf{p}'/(E'/c) = d\mathbf{p}/(E/c)$ = Invariant
Or: $d\mathbf{p}'/E' = d\mathbf{p}/E \rightarrow d^3p/E = dp^xdp^ydp^z/E$ = Invariant, usually seen as $\int$ F(various invariants)*d^3p/E = Invariant

Lorentz Scalar Invariant
$\mathbf{V}\cdot\mathbf{V}=V^\mu V_\mu=(v^0v^0 - \mathbf{v}\cdot\mathbf{v})=(v^0{}_o)^2$
4-Vector
$\mathbf{V} = V^\mu = (v^0,\mathbf{v})$
$d\mathbf{v}/v^0 \rightarrow d^3v/v^0$ if $\mathbf{V}\cdot\mathbf{V}$=(constant)
Phase Space Invariant

$\mathbf{P}\cdot\mathbf{P}=(m_oc)^2=(E_o/c)^2$
4-Momentum
$\mathbf{P}=(mc,\mathbf{p})=(E/c,\mathbf{p})$
d^3p/E

SR 4-Vectors & 4-Tensors

More 4-Vector-based Invariants

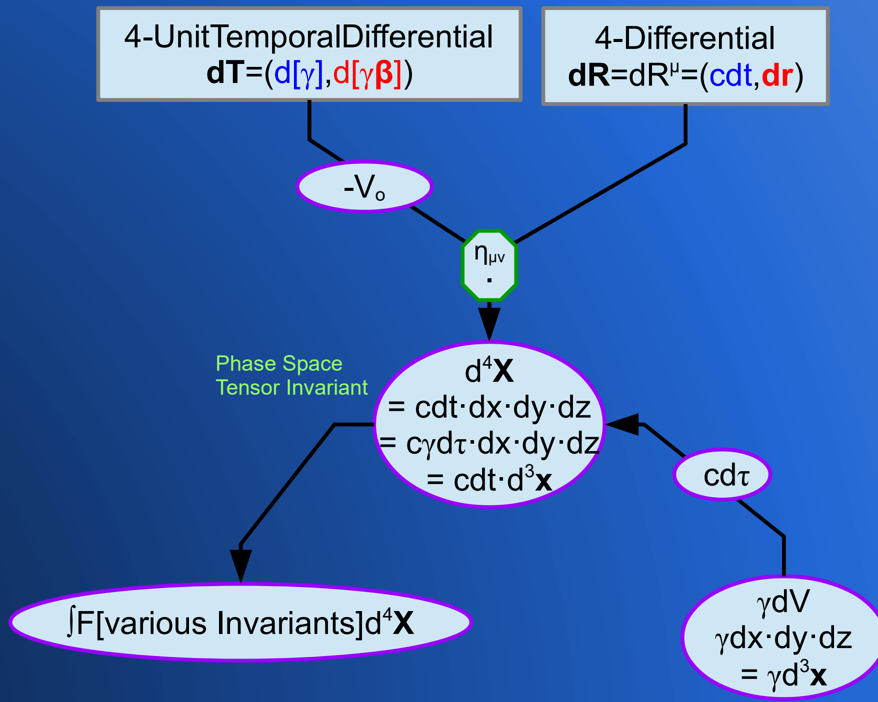
Phase Space Integration

$d^4\mathbf{X} = -(V_o)\mathbf{dT}\cdot\mathbf{dX} = -(dV_o)\mathbf{T}\cdot\mathbf{dX} = cdt\ d^3\mathbf{x} = cdt\ dx\ dy\ dz$
The 4D Position coords that are integrated to give a 4D volume: SI units $[m^4]$

4-Differential $\mathbf{dX} = (cdt,\mathbf{dx})$; $\mathbf{dR} = (cdt,\mathbf{dr})$;
4-UnitTemporal $\mathbf{T} = \gamma(1,\boldsymbol{\beta}) = (\gamma,\gamma\boldsymbol{\beta})$
4-UnitTemporalDifferential $\mathbf{dT} = d[(\gamma,\gamma\boldsymbol{\beta})] = (d[\gamma],d[\gamma\boldsymbol{\beta}])$

$V = \int dV = \int dx \int dy \int dz = \iiint dx\ dy\ dz = \int d^3\mathbf{x}$
$V = V_o/\gamma$ = 3D Spatial Volume: SI units $[m^3]$
$dV = d^3\mathbf{x}$ = 3D Spatial Volume Element
$\gamma = V_o/V$
$d\gamma = -(V_o/V^2)dV$

$-(V_o)\mathbf{dT}\cdot\mathbf{dX}$ = Invariant, because (Rest Scalar * Lorentz Scalar Product) = Invariant
$= -(V_o)(d[\gamma],d[\gamma\boldsymbol{\beta}])\cdot(cdt,\mathbf{dx})$
$= -(V_o)(d[\gamma]cdt - d[\gamma\boldsymbol{\beta}]\cdot\mathbf{dx})$
$= -(V_o)(-(V_o/V^2)dVcdt - d[\gamma\boldsymbol{\beta}]\cdot\mathbf{dx})$
$= -(V_o)(-(V_o/V_o^2)dVcdt - d[(1)(\mathbf{0})]\cdot\mathbf{dx})$ by taking the usual rest-case
$= -(V_o)(-(V_o/V_o^2)dVcdt)$
$= -(V_o)(-(1/V_o)dVcdt)$
$= dVcdt$
$= cdt\ dV$
$= cdt\ dx\ dy\ dz$
$= cdt\ d^3\mathbf{x}$
$= d^4\mathbf{X}$ = Invariant
And, this makes sense.
$\mathbf{T}$ is a temporal 4-Vector with fixed magnitude: $\mathbf{T}\cdot\mathbf{T} = 1$
Therefore, $\mathbf{dT}$ must be a spatial 4-Vector
If $\mathbf{dX}$ is also spatial, then the Lorentz scalar product { $(\mathbf{dT}\cdot\mathbf{dX})$ = -magnitude } will be negative with this choice of Minkoski Metric.
Thus, multiplying by $-(V_o)$ gives a positive volume element{ $cdt\ dx\ dy\ dz = d^4\mathbf{X}$}
It is sort of quirky though, that the temporal (cdt) comes from the $\mathbf{dX}$ part, and the spatial $(d^3\mathbf{x})$ comes from the $\mathbf{dT}$ part.



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SR 4-Vectors & 4-Tensors

More 4-Vector-based Invariants

Phase Space Integration

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson

$\rho\, d^3\mathbf{x} = \rho'\, d^3\mathbf{x}' = (-V_o/c)\mathbf{dT}\cdot\mathbf{J}$ = Lorentz Scalar Invariant
$n\, d^3\mathbf{x} = n'\, d^3\mathbf{x}' = (-V_o/c)\mathbf{dT}\cdot\mathbf{N}$ = Lorentz Scalar Invariant

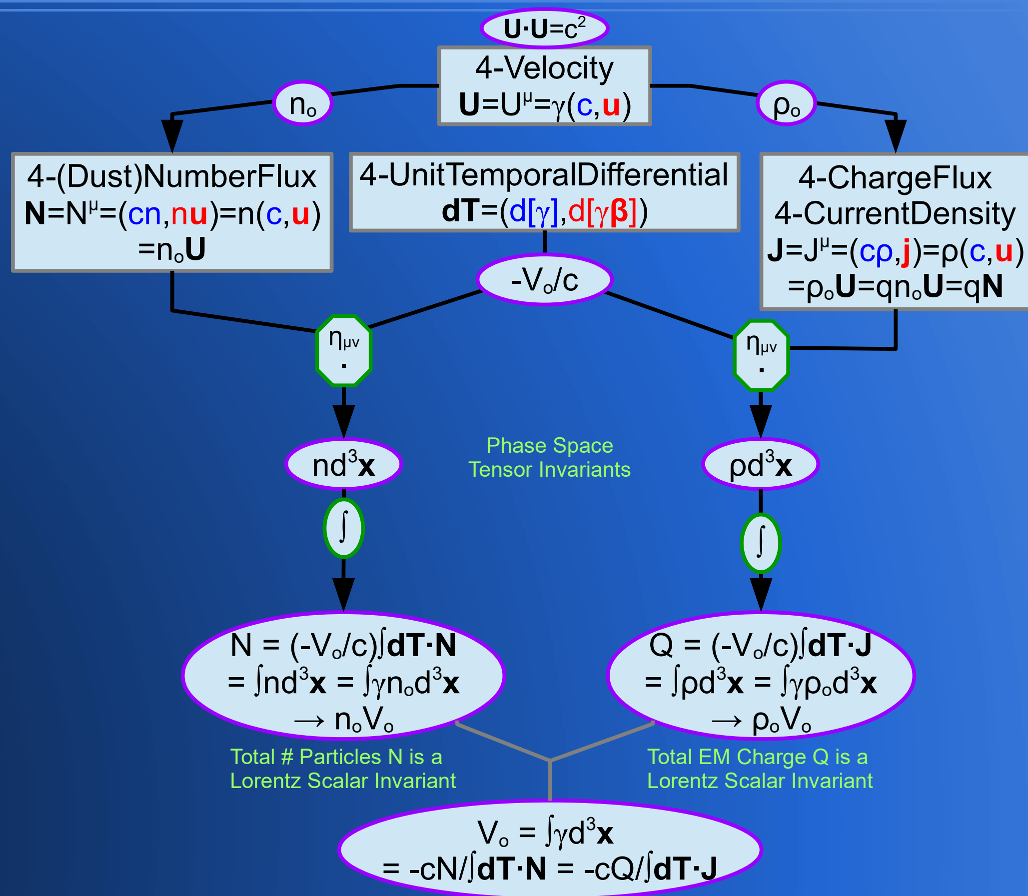
4-CurrentDensity $\mathbf{J} = (\rho c,\mathbf{j})$
4-NumberFlux $\mathbf{N} = (nc,\mathbf{n})$
4-UnitTemporal $\mathbf{T} = \gamma(1,\boldsymbol{\beta}) = (\gamma,\gamma\boldsymbol{\beta})$
4-UnitTemporalDifferential $\mathbf{dT} = d[(\gamma,\gamma\boldsymbol{\beta})] = (d[\gamma],d[\gamma\boldsymbol{\beta}])$

$V = V_o/\gamma$
$d\gamma = -(V_o/V^2)dV$

$(-V_o/c)\mathbf{dT}\cdot\mathbf{J}$ = Invariant, because (Rest Scalar * Lorentz Scalar Product) = Invariant
$= (-V_o/c)(d[\gamma],d[\gamma\boldsymbol{\beta}])\cdot(\rho c,\mathbf{j})$
$= (-V_o/c)(d[\gamma]\rho c - d[\gamma\boldsymbol{\beta}]\cdot\mathbf{j})$
$= (-V_o/c)(-(V_o/V^2)(dV)(\rho c) - d[\gamma\boldsymbol{\beta}]\cdot\mathbf{j})$
$= (-V_o/c)(-(V_o/V_o^2)(dV)(\rho c) - d[(1)\mathbf{0}]\cdot\mathbf{j})$
$= (-V_o/c)(-(V_o/V_o^2)(dV)(\rho c))$
$= (dV/c)(\rho c)$
$= (\rho c)(dV/c)$
$= (\rho)(dV)$
$= \rho\, d^3\mathbf{x}$

Total Charge $Q = \int\gamma\rho_o\, d^3\mathbf{x} = \int\rho\, d^3\mathbf{x}$ = Lorentz Scalar Invariant
Total Particle # $N = \int\gamma n_o\, d^3\mathbf{x} = \int n\, d^3\mathbf{x}$ = Lorentz Scalar Invariant
Total RestVolume $V_o = \int\gamma d^3\mathbf{x}$ = Lorentz Scalar Invariant

This also gives an alternate way to define the RestVolume Invariant V_o.
$(-V_o/c)\mathbf{dT}\cdot\mathbf{N} = nd^3\mathbf{x}$
$N = \int nd^3\mathbf{x} = \int(-V_o/c)\mathbf{dT}\cdot\mathbf{N}$
$cN/V_o = -\int\mathbf{dT}\cdot\mathbf{N}$
$V_o = -cN/\int\mathbf{dT}\cdot\mathbf{N}$



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= Lorentz Scalar

SR 4-Vectors & 4-Tensors

More 4-Vector-based Invariants

Phase Space Integration

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John B. Wilson

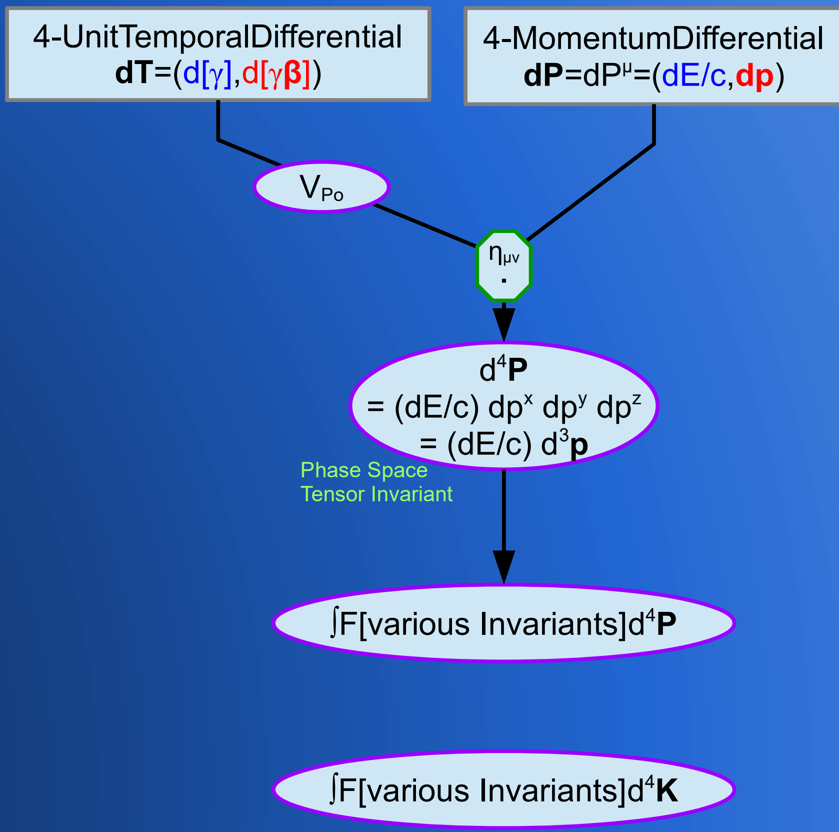
$d^4\mathbf{P} = (V_{Po})\mathbf{dT}\cdot\mathbf{dP} = (dE/c)\, d^3\mathbf{p} = (dE/c)\, dp^x\, dp^y\, dp^z$
$d^4\mathbf{K} = (V_{Ko})\mathbf{dT}\cdot\mathbf{dK} = (d\omega/c)\, d^3\mathbf{k} = (d\omega/c)\, dk^x\, dk^y\, dk^z$
The 4D Momentum coords that are integrated to give a 4D Momentum Volume: SI Units $[(\mathrm{kg}\cdot\mathrm{m/s})^4]$
The 4D WaveVector coords that are integrated to give a 4D WaveVector Volume: SI Units $[(1/\mathrm{m})^4]$

4-DifferentialMomentum $\mathbf{dP} = (dE/c,\mathbf{dp})$
4-DifferentialWaveVector $\mathbf{dK} = (d\omega/c,\mathbf{dk})$
4-UnitTemporal $\mathbf{T} = \gamma(1,\boldsymbol{\beta}) = (\gamma,\gamma\boldsymbol{\beta})$
4-UnitTemporalDifferential $\mathbf{dT} = d[(\gamma,\gamma\boldsymbol{\beta})] = (d[\gamma],d[\gamma\boldsymbol{\beta}])$

$V_P = \int dV_P = \int dp^x \int dp^y \int dp^z = \iiint dp^x\, dp^y\, dp^z = \int d^3\mathbf{p}$
$V_P = \gamma(V_{Po})$ = 3D Volume in Momentum Space: SI Units $[(\mathrm{kg}\cdot\mathrm{m/s})^3]$
$dV_P = d\gamma(V_{Po})$ = 3D Volume Element in Momentum Space
$\gamma = (V_P)/(V_{Po})$
$d\gamma = (dV_P)/(V_{Po})$

$(V_{Po})\mathbf{dT}\cdot\mathbf{dP}$ = Invariant, because Rest Scalar * Lorentz Scalar Product
$= (V_{Po}))(d[\gamma],d[\gamma\boldsymbol{\beta}])\cdot(dE/c,\mathbf{dp})$
$= (V_{Po})(d[\gamma]dE/c - d[\gamma\boldsymbol{\beta}]\cdot\mathbf{dp})$
$= (V_{Po})((dV_P/V_{Po})dE/c - d[\gamma\boldsymbol{\beta}]\cdot\mathbf{dp})$
$= (V_{Po}))((dV_P/V_{Po})dE/c - d[(1)(\mathbf{0})]\cdot\mathbf{dp})$ by taking the usual rest-case
$= (V_{Po}))((dV_P/V_{Po})dE/c)$
$= (dV_P)\,(dE/c)$
$= d^3\mathbf{p}\,(dE/c)$
$= (dE/c)\, d^3\mathbf{p}$
$= (dE/c)\, dp^x\, dp^y\, dp^z$
$= d^4\mathbf{P}$ = Invariant

Likewise, $d^4\mathbf{K}$ = Invariant



SR 4-Vectors & 4-Tensors

More 4-Vector-based Invariants

Phase Space Integration

A Tensor Study of Physical 4-Vectors

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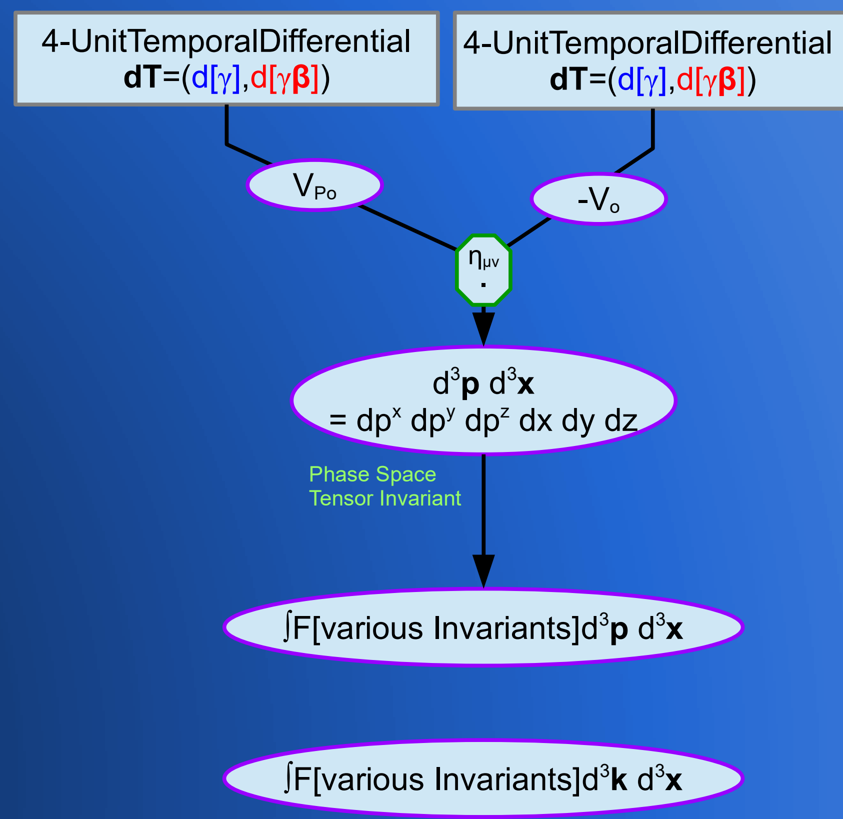
$d^3\mathbf{p}\, d^3\mathbf{x} = (V_{Po})\mathbf{dT}\cdot(-V_o)\mathbf{dT} = (-V_o)(V_{Po})\mathbf{dT}\cdot\mathbf{dT}$
$d^3\mathbf{k}\, d^3\mathbf{x} = (V_{Ko})\mathbf{dT}\cdot(-V_o)\mathbf{dT} = (-V_o)(V_{Ko})\mathbf{dT}\cdot\mathbf{dT}$

4-UnitTemporal $\mathbf{T} = \gamma(1,\boldsymbol{\beta}) = (\gamma,\gamma\boldsymbol{\beta})$
4-UnitTemporalDifferential $\mathbf{dT} = d[(\gamma,\gamma\boldsymbol{\beta})] = (d[\gamma],d[\gamma\boldsymbol{\beta}])$

$(V_{po})\mathbf{dT}\cdot(-V_o)\mathbf{dT}$ = Invariant
$= (V_{Po})(d[\gamma],d[\gamma\boldsymbol{\beta}])\cdot(-V_o)(d[\gamma],d[\gamma\boldsymbol{\beta}])$
$= (V_{Po})(-V_o)(d[\gamma]d[\gamma] - d[\gamma\boldsymbol{\beta}]\cdot d[\gamma\boldsymbol{\beta}])$
$= (V_{Po})(-V_o)(-(V_o/V^2)dV(dV_P/(V_{Po})) - d[\gamma\boldsymbol{\beta}]\cdot d[\gamma\boldsymbol{\beta}])$
$= (V_{Po})(-V_o)(-(V_o/V_o^2)dV(dV_P/(V_{Po})) - d[(1)\mathbf{0}]\cdot d[(1)\mathbf{0}])$
$= (V_{Po})(-V_o)(-(V_o/V_o^2)dV(dV_P/(V_{Po}))$
$= (V_{Po})dV(dV_P/(V_{Po}))$
$= dV\, dV_P$
$= dV_P\, dV$
$= d^3\mathbf{p}\, d^3\mathbf{x}$ = Invariant

Likewise, $d^3\mathbf{k}\, d^3\mathbf{x}$ = Invariant



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Lorentz Scalar

$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^{\mu}{}_{\mu} = T$
$\mathbf{V}\cdot\mathbf{V} = V^{\mu}\eta_{\mu\nu}V^{\nu} = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$
= Lorentz Scalar

SRQM Study: SR 4-Tensors

General → Symmetric & Anti-Symmetric

Any SR Tensor $T^{\mu\nu} = (S^{\mu\nu} + A^{\mu\nu})$ can be decomposed into parts:

Symmetric $S^{\mu\nu} = (T^{\mu\nu}+T^{\nu\mu})/2$ with $S^{\mu\nu} = +S^{\nu\mu}$

Anti-Symmetric $A^{\mu\nu} = (T^{\mu\nu}-T^{\nu\mu})/2$ with $A^{\mu\nu} = -A^{\nu\mu}$

$S^{\mu\nu} + A^{\mu\nu} = (T^{\mu\nu}+T^{\nu\mu})/2+(T^{\mu\nu}-T^{\nu\mu})/2 = T^{\mu\nu}/2 + T^{\mu\nu}/2 + T^{\nu\mu}/2 - T^{\nu\mu}/2 = T^{\mu\nu} + 0 = T^{\mu\nu}$

Independent components: { $4^2 = 16 = 10 + 6$ }

Max 16 possible

General 4-Tensor $T^{\mu\nu}$ =

$$\begin{bmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{bmatrix}$$

Max 10 possible

Symmetric 4-Tensor $S^{\mu\nu}$ =

$$\begin{bmatrix} S^{00} & S^{01} & S^{02} & S^{03} \\ S^{10} & S^{11} & S^{12} & S^{13} \\ S^{20} & S^{21} & S^{22} & S^{23} \\ S^{30} & S^{31} & S^{32} & S^{33} \end{bmatrix} = \begin{bmatrix} S^{00} & S^{01} & S^{02} & S^{03} \\ +S^{01} & S^{11} & S^{12} & S^{13} \\ +S^{02} & +S^{12} & S^{22} & S^{23} \\ +S^{03} & +S^{13} & +S^{23} & S^{33} \end{bmatrix}$$

Max 6 possible

Anti-Symmetric 4-Tensor $A^{\mu\nu}$ =

$$\begin{bmatrix} A^{00} & A^{01} & A^{02} & A^{03} \\ A^{10} & A^{11} & A^{12} & A^{13} \\ A^{20} & A^{21} & A^{22} & A^{23} \\ A^{30} & A^{31} & A^{32} & A^{33} \end{bmatrix} = \begin{bmatrix} 0 & A^{01} & A^{02} & A^{03} \\ -A^{01} & 0 & A^{12} & A^{13} \\ -A^{02} & -A^{12} & 0 & A^{23} \\ -A^{03} & -A^{13} & -A^{23} & 0 \end{bmatrix}$$

$\mathrm{Tr}[A^{\mu\nu}]=0$

aka Skew-Symmetric

Importantly, the Contraction of any Symmetric tensor with any Anti-Symmetric tensor on the same index is always 0.

Note These don't have to be composed from a single general tensor.

$S^{\mu\nu} A_{\mu\nu} = 0$

Proof:
$S^{\mu\nu} A_{\mu\nu}$
$= S^{\nu\mu} A_{\nu\mu}$: because we can switch dummy indices
$= (+S^{\mu\nu})A_{\nu\mu}$: because of symmetry
$= S^{\mu\nu}(-A_{\mu\nu})$: because of anti-symmetry
$= -S^{\mu\nu} A_{\mu\nu}$
$= 0$: because the only solution of {c = -c} is 0

Physically, the anti-symmetric part contains rotational information and the symmetric part contains information about isotropic scaling and anisotropic shear.

SRQM Study: SR 4-Tensors

Symmetric → Isotropic & Anisotropic

SciRealm.org
John B. Wilson

Any Symmetric SR Tensor $S^{\mu\nu} = (T_{iso}{}^{\mu\nu} + T_{aniso}{}^{\mu\nu})$ can be decomposed into parts:

Isotropic $T_{iso}{}^{\mu\nu} = (1/4)\mathrm{Trace}[S^{\mu\nu}]\, \eta^{\mu\nu} = (T)\, \eta^{\mu\nu}$

Anistropic $T_{aniso}{}^{\mu\nu} = S^{\mu\nu} - T_{iso}{}^{\mu\nu}$

The Anistropic part is Traceless by construction, and the Isotropic part has the same Trace as the original Symmetric Tensor. The Minkowski Metric is a symmetric, isotropic 4-tensor with T=1.

Independent components:

Max 10 possible	Max 1 possible	Max 9 possible

Symmetric 4-Tensor

$S^{\mu\nu} = $
$[S^{00}, S^{01}, S^{02}, S^{03}]$
$[S^{10}, S^{11}, S^{12}, S^{13}]$
$[S^{20}, S^{21}, S^{22}, S^{23}]$
$[S^{30}, S^{31}, S^{32}, S^{33}]$
=
$[S^{00}, S^{01}, S^{02}, S^{03}]$
$[+S^{01}, S^{11}, S^{12}, S^{13}]$
$[+S^{02}, +S^{12}, S^{22}, S^{23}]$
$[+S^{03}, +S^{13}, +S^{23}, S^{33}]$

$\mathrm{Tr}[S^{\mu\nu}]=4T$

Symmetric Isotropic 4-Tensor

$T_{iso}{}^{\mu\nu} = $
$[T, 0,0,0]$
$[0,-T,0,0]$
$[0,0,-T,0]$
$[0,0,0,-T]$

with T= (1/4)Trace$[S^{\mu\nu}]$

$\mathrm{Tr}[T_{iso}{}^{\mu\nu}]=4T$

Symmetric Anisotropic 4-Tensor

$T_{aniso}{}^{\mu\nu} = $
$[S^{00}-T, S^{01}, S^{02}, S^{03}]$
$[S^{10}, S^{11}+T, S^{12}, S^{13}]$
$[S^{20}, S^{21}, S^{22}+T, S^{23}]$
$[S^{30}, S^{31}, S^{32}, S^{33}+T]$
=
$[S^{00}-T, S^{01}, S^{02}, S^{03}]$
$[+S^{01}, S^{11}+T, S^{12}, S^{13}]$
$[+S^{02}, +S^{12}, S^{22}+T, S^{23}]$
$[+S^{03}, +S^{13}, +S^{23}, S^{33}+T]$

$\mathrm{Tr}[T_{aniso}{}^{\mu\nu}]=0$

aka Deviatoric

Importantly, the Contraction of any Symmetric tensor with any Anti-Symmetric tensor on the same index is always 0.

Note These don't have to be composed from a single general tensor.

$S^{\mu\nu} A_{\mu\nu} = 0$

Proof:
$S^{\mu\nu} A_{\mu\nu}$
$= S^{\nu\mu} A_{\nu\mu}$: because we can switch dummy indices
$= (+S^{\mu\nu})A_{\nu\mu}$: because of symmetry
$= S^{\mu\nu}(-A_{\mu\nu})$: because of anti-symmetry
$= -S^{\mu\nu} A_{\mu\nu}$
= 0: because the only solution of {c = -c} is 0

Physically, the isotropic part represents a direction independent transformation (e.g., a uniform scaling or uniform pressure); the deviatoric part represents the distortion

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^{\mu}{}_{\nu}$ or $T_{\mu}{}^{\nu}$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^{\mu} = \mathbf{V} = (v^0,\mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_{\mu} = (v_0,-\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^{\mu}{}_{\mu} = T$
$\mathbf{V}\cdot\mathbf{V} = V^{\mu}\eta_{\mu\nu}V^{\nu} = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$
= Lorentz Scalar

SRQM Study: SR 4-Tensors
SR Tensor Invariants

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson

(0,0)-Tensor = Lorentz Scalar S: Has either (0) or (1) Tensor Invariant, depending on exact meaning

(S) itself is Invariant

S

(1,0)-Tensor = 4-Vector V^μ: Has (1) Tensor Invariant = The Lorentz Scalar Product

$\mathbf{V}\cdot\mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = \eta_{\mu\nu} V^\mu V^\nu = \mathrm{Tr}[V^\mu V^\nu] = V_\nu V^\nu = (v_0 v^0 + v_1 v^1 + v_2 v^2 + v_3 v^3) = (v^0 v^0 - \mathbf{v}\cdot\mathbf{v}) = (v^0{}_o)^2$

$\mathbf{V} = V^\mu = (v^\mu) = (v^0, v^1, v^2, v^3)$

$\mathbf{V}\cdot\mathbf{V} = (v^0 v^0 - \mathbf{v}\cdot\mathbf{v}) = (v^0{}_o)^2$

(2,0)-Tensor = 4-Tensor $T^{\mu\nu}$: Has (4+) Tensor Invariants (though not all independent)

a) $T^\alpha{}_\alpha$ = Trace = Sum of EigenValues for (1,1)-Tensors (mixed)
b) $T^\alpha{}_{[\alpha} T^\beta{}_{\beta]}$ = Asymm Bi-Product → Inner Product
c) $T^\alpha{}_{[\alpha} T^\beta{}_\beta T^\gamma{}_{\gamma]}$ = Asymm Tri-Product → ?Name?
d) $T^\alpha{}_{[\alpha} T^\beta{}_\beta T^\gamma{}_\gamma T^\delta{}_{\delta]}$ = Asymm Quad-Product → 4D Determinant = Product of EigenValues for (1,1)-Tensors

eg. $T^\alpha{}_{[\alpha} T^\beta{}_{\beta]} = T^\alpha{}_\alpha T^\beta{}_\beta - T^\alpha{}_\beta T^\beta{}_\alpha = (T^\gamma{}_\gamma)^2 - T^\alpha{}_\beta T^\beta{}_\alpha\{1\} = (T^\gamma{}_\gamma)^2 - T^\alpha{}_\beta T^\beta{}_\alpha\{(1/4)\eta_{\gamma\delta}\eta^{\gamma\delta}\}$
and, bending tensor rules slightly: $= (T^\gamma{}_\gamma)^2 - T^\alpha{}_\beta T^\beta{}_\alpha\{(1/4)\eta_{\beta\delta}\eta^{\beta\delta}\} = (T^\gamma{}_\gamma)^2 - T^\alpha{}_\beta(\eta^{\beta\delta})T^\beta{}_\alpha(\eta_{\beta\delta})\{(1/4)\} = (T^\gamma{}_\gamma)^2 - T^{\alpha\delta}T_{\delta\alpha}\{(1/4)\}$
and, since linear combinations of invariants are invariant:
Examine just the $(T^{\alpha\delta}T_{\delta\alpha})$ part, which for symm|asymm is $(\pm)(T^{\alpha\delta}T_{\alpha\delta})$ ie. the InnerProduct Invariant

a): Trace$[T^{\mu\nu}]$ = $\mathrm{Tr}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T_\mu{}^\mu = T_\nu{}^\nu = (T_0{}^0 + T_1{}^1 + T_2{}^2 + T_3{}^3) = (T^{00} - T^{11} - T^{22} - T^{33}) = (T)$
for anti-symmetric: = 0

b): InnerProduct $T_{\mu\nu}T^{\mu\nu}$ = $T_{00}T^{00} + T_{i0}T^{i0} + T_{0j}T^{0j} + T_{ij}T^{ij} = (T^{00})^2 - \Sigma_i[T^{i0}]^2 - \Sigma_j[T^{0j}]^2 + \Sigma_{i,j}[T^{ij}]^2$
for symmetric | anti-symmetric: $= (T^{00})^2 - 2\Sigma_i[T^{i0}]^2 + \Sigma_{i,j}[T^{ij}]^2 = \Sigma_{\mu=\nu}[T^{\mu\nu}]^2 - 2\Sigma_i[T^{i0}]^2 + 2\Sigma_{i>j}[T^{ij}]^2$

c): Antisymmetric Triple Product $T^\alpha{}_{[\alpha} T^\beta{}_\beta T^\gamma{}_{\gamma]}$ = $\mathrm{Tr}[T^{\mu\nu}]^3 - 3(\mathrm{Tr}[T^{\mu\nu}])(T^\alpha{}_\beta T^\beta{}_\alpha) + T^\alpha{}_\beta T^\beta{}_\gamma T^\gamma{}_\alpha + T^\alpha{}_\gamma T^\beta{}_\alpha T^\gamma{}_\beta$
for anti-symmetric: = 0

If I got all the math right...

d): Determinant Det$[T^{\mu\nu}]$ =?= $-(1/2)\epsilon_{\alpha\beta\gamma\delta}T^{\alpha\beta}T^{\gamma\delta}$
for anti-symmetric: Det$[T^{\mu\nu}]$ = Pfaffian$[T^{\mu\nu}]^2$ (The Pfaffian is a special polynomial of the matrix entries)

Trace Tensor Invariant

$\mathrm{Tr}[T^{\mu\nu}] = T_\nu{}^\nu = (T^{00} - T^{11} - T^{22} - T^{33}) = T$

Set of 4 EigenValues$[T_\mu{}^\nu]$

Eigenvalues Tensor Invariants

4-Tensor

$$T^{\mu\nu} = \begin{bmatrix} T^{00}, T^{01}, T^{02}, T^{03} \\ T^{10}, T^{11}, T^{12}, T^{13} \\ T^{20}, T^{21}, T^{22}, T^{23} \\ T^{30}, T^{31}, T^{32}, T^{33} \end{bmatrix}$$

$T_{\mu\nu}T^{\mu\nu}$

Inner Product Tensor Invariant

AsymmTri$[T^{\mu\nu}]$

Asymm Tri-Product Tensor Invariant

Det$[T^{\mu\nu}]$

Determinant Tensor Invariant

The lowered-indices form of a tensor just negativizes the (time-space) and (space-time) sections of the upper-indices tensor

Lowered 4-Tensor

$$T_{\mu\nu} = \eta_{\mu\rho}\eta_{\nu\sigma}T^{\rho\sigma} = \begin{bmatrix} T_{00}, T_{01}, T_{02}, T_{03} \\ T_{10}, T_{11}, T_{12}, T_{13} \\ T_{20}, T_{21}, T_{22}, T_{23} \\ T_{30}, T_{31}, T_{32}, T_{33} \end{bmatrix} = \begin{bmatrix} +T^{00}, -T^{01}, -T^{02}, -T^{03} \\ -T^{10}, +T^{11}, +T^{12}, +T^{13} \\ -T^{20}, +T^{21}, +T^{22}, +T^{23} \\ -T^{30}, +T^{31}, +T^{32}, +T^{33} \end{bmatrix}$$

Invariants sometimes seen as
$I_1 = (1/1)\mathrm{Tr}[(T^{\mu\nu})^1]$
$I_2 = (1/2)\mathrm{Tr}[(T^{\mu\nu})^2]$
$I_3 = (1/3)\mathrm{Tr}[(T^{\mu\nu})^3]$
$I_4 = (1/4)\mathrm{Tr}[(T^{\mu\nu})^4]$

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^\mu{}_\nu$ or $T_\mu{}^\nu$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Det$[T^\alpha{}_\alpha] = \Pi_k[\lambda_k]$; with $\{\lambda_k\}$ = Eigenvalues
Characteristic Eqns: Det$[T^\alpha{}_\alpha - \lambda_k I_{(4)}] = 0$

Trace$[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu{}_\mu = T$
$\mathbf{V}\cdot\mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$
= Lorentz Scalar

SR → QM

SRQM Study: SR 4-Tensors
SR Tensor Invariants for Faraday EM Tensor

4-Vector SRQM Interpretation of QM

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson

The Faraday EM Tensor $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha = \partial \wedge \mathbf{A}$ is an anti-symmetric tensor that contains the Electric and Magnetic Fields.
The 3-electric components ($\mathbf{e} = e^i$) are in the temporal-spatial sections.
The 3-magnetic components ($\mathbf{b} = b^k$) are in the only-spatial section.

(2,0)-Tensor = 4-Tensor $T^{\mu\nu}$: Has (4+) Tensor Invariants (though not all independent)
a) $T^\alpha{}_\alpha$ = Trace = Sum of EigenValues for (1,1)-Tensors (mixed)
b) $T^\alpha{}_{[\alpha}T^\beta{}_{\beta]}$ = Asymm Bi-Product → Inner Product
c) $T^\alpha{}_{[\alpha}T^\beta{}_\beta T^\gamma{}_{\gamma]}$ = Asymm Tri-Product → ?Name?
d) $T^\alpha{}_{[\alpha}T^\beta{}_\beta T^\gamma{}_\gamma T^\delta{}_{\delta]}$ = Asymm Quad-Product → 4D Determinant = Product of EigenValues for (1,1)-Tensors

a): Faraday Trace$[F^{\mu\nu}]$ = $F_\nu{}^\nu = (F^{00}-F^{11}-F^{22}-F^{33}) = (0 -0 -0 -0) = 0$
b): Faraday Inner Product $F_{\mu\nu}F^{\mu\nu} = \Sigma_{\mu=\nu}[F^{\mu\nu}]^2 - 2\Sigma_i[F^{i0}]^2 + 2\Sigma_{i>j}[F^{ij}]^2 = (0) - 2(\mathbf{e}\cdot\mathbf{e}/c^2) + 2(\mathbf{b}\cdot\mathbf{b}) = 2\{(\mathbf{b}\cdot\mathbf{b})-(\mathbf{e}\cdot\mathbf{e}/c^2)\}$
c): Faraday AsymmTri$[F^{\mu\nu}]$ = $\mathrm{Tr}[F^{\mu\nu}]^3 - 3(\mathrm{Tr}[F^{\mu\nu}])(F^\alpha{}_\beta F^\beta{}_\alpha) + F^\alpha{}_\beta F^\beta{}_\gamma F^\gamma{}_\alpha + F^\alpha{}_\gamma F^\beta{}_\alpha F^\gamma{}_\beta = 0-3(0)+F^\alpha{}_\beta F^\beta{}_\gamma F^\gamma{}_\alpha+(-F^\alpha{}_\beta)(-F^\beta{}_\gamma)(-F^\gamma{}_\alpha) = 0$
d): Faraday Det[anti-symmetric $F^{\mu\nu}$] = Pfaffian$[F^{\mu\nu}]^2 = [(-e^x/c)(-b^x) - (-e^y/c)(b^y) + (-e^z/c)(-b^z)]^2 = [(e^xb^x/c) + (e^yb^y/c) + (e^zb^z/c)]^2 = \{(\mathbf{e}\cdot\mathbf{b})/c\}^2$

Importantly, the Faraday EM Tensor has only (2) linearly-independent invariants:
b) $2\{(\mathbf{b}\cdot\mathbf{b})-(\mathbf{e}\cdot\mathbf{e}/c^2)\}$
d) $\{(\mathbf{b}\cdot\mathbf{e})/c\}^2$
a) & c) give 0=0, and do not provide additional constraints

The 4-Gradient and 4-EMVectorPotential have (4) independent components each, for total of (8).
Subtract the (2) invariants which provide constraints to get a total of (6) independent components
= (6) independent components of a 4x4 anti-symmetric tensor
= (3) 3-electric $\mathbf{e}$ + (3) 3-magnetic $\mathbf{b}$ = (6) independent EM field components

$\partial \wedge \mathbf{A}$ is the exterior product of the 4-Gradient with the 4-EMVectorPotential.

$\varepsilon^{ij}{}_k$ is the Levi-Civita symbol, the fully anti-symmetric tensor.
with Latin indices it ranges from {1..3}, with Greek indicies it ranges from {0..3}

4-Gradient
$\partial = \partial^\mu = (\partial_t/c, -\nabla)$

$\mathrm{Tr}[F^{\mu\nu}] = F_\nu{}^\nu = 0$
Trace Tensor Invariant

$F_{\mu\nu}F^{\mu\nu} = 2\{(\mathbf{b}\cdot\mathbf{b})-(\mathbf{e}\cdot\mathbf{e}/c^2)\}$
Inner Product Tensor Invariant

AsymmTri$[F^{\mu\nu}]$ = 0
Asymm Tri-Product Tensor Invariant

$\mathrm{Det}[F^{\mu\nu}] = \{(\mathbf{e}\cdot\mathbf{b})/c\}^2$
Determinant Tensor Invariant

4-(EM)VectorPotential
$\mathbf{A} = A^\mu = (\varphi/c, \mathbf{a})$

Faraday EM Tensor
$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha = \partial \wedge \mathbf{A}$

$$\to \begin{bmatrix} F^{tt} & F^{tx} & F^{ty} & F^{tz} \\ F^{xt} & F^{xx} & F^{xy} & F^{xz} \\ F^{yt} & F^{yx} & F^{yy} & F^{yz} \\ F^{zt} & F^{zx} & F^{zy} & F^{zz} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \partial^0a^1-\partial^1a^0 & \partial^0a^2-\partial^2a^0 & \partial^0a^3-\partial^3a^0 \\ \partial^1a^0-\partial^0a^1 & 0 & \partial^1a^2-\partial^2a^1 & \partial^1a^3-\partial^3a^1 \\ \partial^2a^0-\partial^0a^2 & \partial^2a^1-\partial^1a^2 & 0 & \partial^2a^3-\partial^3a^2 \\ \partial^3a^0-\partial^0a^3 & \partial^3a^1-\partial^1a^3 & \partial^3a^2-\partial^2a^3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & (\partial^ta^x+\nabla^x\varphi)/c & (\partial^ta^y+\nabla^y\varphi)/c & (\partial^ta^z+\nabla^z\varphi)/c \\ (-\nabla^x\varphi-\partial^ta^x/c) & 0 & -\nabla^xa^y+\nabla^ya^x & -\nabla^xa^z+\nabla^za^x \\ (-\nabla^y\varphi-\partial^ta^y/c) & -\nabla^ya^x+\nabla^xa^y & 0 & -\nabla^ya^z+\nabla^za^y \\ (-\nabla^z\varphi-\partial^ta^z/c) & -\nabla^za^x+\nabla^xa^z & -\nabla^za^y+\nabla^ya^z & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -e^x/c & -e^y/c & -e^z/c \\ +e^x/c & 0 & -b^z & +b^y \\ +e^y/c & +b^z & 0 & -b^x \\ +e^z/c & -b^y & +b^x & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & , -e^j/c \\ +e^i/c, & -\varepsilon^{ij}{}_k b^k \end{bmatrix}$$

$$= \begin{bmatrix} 0 & , -\mathbf{e}/c \\ +\mathbf{e}^T/c, & -\nabla \wedge \mathbf{a} \end{bmatrix}$$

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^\mu{}_\nu$ or $T_\mu{}^\nu$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu{}_\mu = T$
$\mathbf{V}\cdot\mathbf{V} = V^\mu\eta_{\mu\nu}V^\nu = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$
= Lorentz Scalar

SRQM Study: SR 4-Tensors
SR Tensor Invariants
for 4-AngularMomentum Tensor

SciRealm.org
John B. Wilson

The 4-AngularMomentum Tensor $M^{\alpha\beta} = X^\alpha P^\beta - X^\beta P^\alpha = \mathbf{X}^\wedge\mathbf{P}$ is an anti-symmetric tensor
The 3-mass-moment components ($\mathbf{n} = n^i$) are in the temporal-spatial sections.
The 3-angular-momentum components ($\mathbf{l} = l^k$) are in the only-spatial section.

<u>(2,0)-Tensor = 4-Tensor $T^{\mu\nu}$: Has (4+) Tensor Invariants (though not all independent)</u>
a) $T^\alpha{}_\alpha$ = Trace = Sum of EigenValues for (1,1)-Tensors (mixed)
b) $T^\alpha{}_{[\alpha}T^\beta{}_{\beta]}$ = Asymm Bi-Product → Inner Product
c) $T^\alpha{}_{[\alpha}T^\beta{}_\beta T^\gamma{}_{\gamma]}$ = Asymm Tri-Product → ?Name?
d) $T^\alpha{}_{[\alpha}T^\beta{}_\beta T^\gamma{}_\gamma T^\delta{}_{\delta]}$ = Asymm Quad-Product → 4D Determinant = Product of EigenValues for (1,1)-Tensors

a): 4-AngMom Trace$[M^{\mu\nu}]$ = $M_\nu{}^\nu = (M^{00}-M^{11}-M^{22}-M^{33}) = (0 -0 -0 -0) = 0$
b): 4-AngMom Inner Product $M_{\mu\nu}M^{\mu\nu} = \Sigma_{\mu=\nu}[M^{\mu\nu}]^2 - 2\Sigma_i[M^{i0}]^2 + 2\Sigma_{i>j}[M^{ij}]^2 = (0) - 2(c^2\mathbf{n}\cdot\mathbf{n}) + 2(\mathbf{l}\cdot\mathbf{l}) = 2\{(\mathbf{l}\cdot\mathbf{l})-(c^2\mathbf{n}\cdot\mathbf{n})\}$
c): 4-AngMom AsymmTri$[M^{\mu\nu}]$ = $\mathrm{Tr}[M^{\mu\nu}]^3 - 3(\mathrm{Tr}[M^{\mu\nu}])(M^\alpha{}_\beta M^\beta{}_\alpha) + M^\alpha{}_\beta M^\beta{}_\gamma M^\gamma{}_\alpha + M^\alpha{}_\gamma M^\beta{}_\alpha M^\gamma{}_\beta = 0$
d): 4-AngMom Det[anti-symmetric $M^{\mu\nu}$] = Pfaffian$[M^{\mu\nu}]^2 = [(-cn^x)(+l^x) - (-cn^y)(-l^y) + (-cn^z)(+l^z)]^2 = [-(cn^xl^x) - (cn^yl^y) - (cn^zl^z)]^2 = \{c(\mathbf{n}\cdot\mathbf{l})\}^2$

Importantly, the 4-AngularMomentum Tensor has only (2) linearly-independent invariants:
b) $2\{(\mathbf{l}\cdot\mathbf{l})-(c^2\mathbf{n}\cdot\mathbf{n})\}$: see Wikipedia Laplace–Runge–Lenz_vector, sec. Casimir Invariants
d) $\{c(\mathbf{l}\cdot\mathbf{n})\}^2$
a) & c) give 0=0, and do not provide additional constraints

The 4-Position and 4-Momentum have (4) independent components each, for total of (8).
Subtract the (2) invariants which provide constraints to get a total of (6) independent components
= (6) independent components of a 4x4 anti-symmetric tensor
= (3) 3-mass-moment **n** + (3) 3-angular-momentum **l** = (6) independent 4-AngularMomentum components

3-massmoment $\mathbf{n} = \mathbf{x}m - t\mathbf{p} = m(\mathbf{x} - t\mathbf{u}) = m(\mathbf{r} - t\mathbf{u}) = m(\mathbf{r} - t(\boldsymbol{\omega} \times \mathbf{r}))$: Tangential velocity $\mathbf{u}_T = (\boldsymbol{\omega} \times \mathbf{r})$

$(-k/r)\mathbf{n} = -mk(\hat{\mathbf{r}} - t(\boldsymbol{\omega} \times \hat{\mathbf{r}})) = mkt(\boldsymbol{\omega} \times \hat{\mathbf{r}}) - mk\hat{\mathbf{r}} = t * d/dt(\mathbf{p}) \times \mathbf{L} - mk\hat{\mathbf{r}}$: $d/dt(\mathbf{p}) \times \mathbf{L} = mk(\boldsymbol{\omega} \times \hat{\mathbf{r}})$
n is related to the LRL = Laplace-Runge-Lenz 3-vector: $\mathbf{A} = \mathbf{p} \times \mathbf{L} - mk\hat{\mathbf{r}}$
which is another classical conserved vector. The invariance is shown here to be relativistic in origin.
Wikipedia article: Laplace-Runge-Lenz vector shows these as Casimir Invariants.
See Also: Relativistic Angular Momentum.

4-Position
$\mathbf{X} = X^\mu = (ct, \mathbf{x})$

4-Momentum
$\mathbf{P} = P^\mu = (mc, \mathbf{p}) = (E/c, \mathbf{p})$

$\mathrm{Tr}[M^{\mu\nu}] = M_\nu{}^\nu = 0$
Trace Tensor Invariant

$M_{\mu\nu}M^{\mu\nu} = 2\{(\mathbf{l}\cdot\mathbf{l})-(c^2\mathbf{n}\cdot\mathbf{n})\}$
Inner Product Tensor Invariant

$\mathrm{AsymmTri}[M^{\mu\nu}] = 0$
Asymm Tri-Product Tensor Invariant

$\mathrm{Det}[M^{\mu\nu}] = \{c(\mathbf{n}\cdot\mathbf{l})\}^2$
Determinant Tensor Invariant

4-AngularMomentum Tensor
$M^{\alpha\beta} = X^\alpha P^\beta - X^\beta P^\alpha = \mathbf{X} \wedge \mathbf{P}$
→

$$\begin{bmatrix} M^{tt} & M^{tx} & M^{ty} & M^{tz} \\ M^{xt} & M^{xx} & M^{xy} & M^{xz} \\ M^{yt} & M^{yx} & M^{yy} & M^{yz} \\ M^{zt} & M^{zx} & M^{zy} & M^{zz} \end{bmatrix}$$
=
$$\begin{bmatrix} 0 & x^0p^1-x^1p^0 & x^0p^2-x^2p^0 & x^0p^3-x^3p^0 \\ x^1p^0-x^0p^1 & 0 & x^1p^2-x^2p^1 & x^1p^3-x^3p^1 \\ x^2p^0-x^0p^2 & x^2p^1-x^1p^2 & 0 & x^2p^3-x^3p^2 \\ x^3p^0-x^0p^3 & x^3p^1-x^1p^3 & x^3p^2-x^2p^3 & 0 \end{bmatrix}$$
=
$$\begin{bmatrix} 0 & ctp^x-xE/c & ctp^y-yE/c & ctp^z-zE/c \\ xE/c-ctp^x & 0 & xp^y-yp^x & xp^z-zp^x \\ yE/c-ctp^y & yp^x-xp^y & 0 & yp^z-zp^y \\ zE/c-ctp^z & zp^x-xp^z & zp^y-yp^z & 0 \end{bmatrix}$$
=
$$\begin{bmatrix} 0 & c(tp^x-xm) & c(tp^y-ym) & c(tp^z-zm) \\ c(xm-tp^x) & 0 & xp^y-yp^x & xp^z-zp^x \\ c(ym-tp^y) & yp^x-xp^y & 0 & yp^z-zp^y \\ c(zm-tp^z) & zp^x-xp^z & zp^y-yp^z & 0 \end{bmatrix}$$
=
$$\begin{bmatrix} 0 & -cn^x & -cn^y & -cn^z \\ +cn^x & 0 & +l^z & -l^y \\ +cn^y & -l^z & 0 & +l^x \\ +cn^z & +l^y & -l^x & 0 \end{bmatrix}$$
=
$$\begin{bmatrix} 0 & , -cn^j \\ +cn^i, & \varepsilon^{ij}{}_k l^k \end{bmatrix}$$
=
$$\begin{bmatrix} 0 & , -c\mathbf{n} \\ +c\mathbf{n}^T, & \mathbf{x} \wedge \mathbf{p} \end{bmatrix}$$

$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu{}_\mu = T$
$\mathbf{V}\cdot\mathbf{V} = V^\mu\eta_{\mu\nu}V^\nu = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$
= Lorentz Scalar

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
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SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

SRQM Study: SR 4-Tensors
SR Tensor Invariants for Minkowski Metric Tensor

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson

The Minkowksi Metric Tensor $\eta^{\mu\nu}$ is the tensor all SR 4-Vectors are measured by.

(2,0)-Tensor = 4-Tensor $T^{\mu\nu}$: Has (4+) Tensor Invariants (though not all independent)

a) $T^{\alpha}{}_{\alpha}$ = Trace = Sum of EigenValues for (1,1)-Tensors (mixed)

b) $T^{\alpha}{}_{[\alpha}T^{\beta}{}_{\beta]}$ = Asymm Bi-Product → Inner Product

c) $T^{\alpha}{}_{[\alpha}T^{\beta}{}_{\beta}T^{\gamma}{}_{\gamma]}$ = Asymm Tri-Product → ?Name?

d) $T^{\alpha}{}_{[\alpha}T^{\beta}{}_{\beta}T^{\gamma}{}_{\gamma}T^{\delta}{}_{\delta]}$ = Asymm Quad-Product → 4D Determinant = Product of EigenValues for (1,1)-Tensors

a): Minkowksi Trace$[\eta^{\mu\nu}]$ = 4

b): Minkowksi Inner Product $\eta_{\mu\nu}\eta^{\mu\nu}$ = 4

c): Minkowksi AsymmTri$[\eta^{\mu\nu}]$ = 24 = 4! , if I did the math right...

d): Minkowksi Det$[\eta^{\mu\nu}]$ = -1

$\Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}{}_{\nu}\eta_{\alpha\beta} = \eta_{\mu\nu}$

Det(Exp[A])=Exp(Tr[A])

$\mathrm{Det}(A)=((\mathrm{tr}\,A)^4 - 6\,\mathrm{tr}(A^2)(\mathrm{tr}\,A)^2 + 3(\mathrm{tr}(A^2))^2 + 8\,\mathrm{tr}(A^3)\,\mathrm{tr}\,A - 6\,\mathrm{tr}(A^4))/24$

Trace Tensor Invariant

$\mathrm{Tr}[\eta^{\mu\nu}] = \eta^{\mu}{}_{\mu} = (1) -(-1) -(-1) -(-1) = 4$

4-Gradient
$\partial=\partial^{\mu}=(\partial_t/c,-\nabla)$

$\partial[\mathbf{R}] = \partial^{\mu}R^{\nu} = \eta^{\mu\nu}$
→
Diag[1,-1,-1,-1]
Diag$[1,-I_{(3)}]$
Diag$[1,-\delta^{jk}]$
=

$$\begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

{in Cartesian form}

$[\eta_{\mu\mu}] = 1/[\eta^{\mu\mu}] : \eta_{\mu}{}^{\nu} = \delta_{\mu}{}^{\nu}$

SR:Minkowski Metric

"Particle Physics" Convention

EigenValues$[\eta^{\mu'}{}_{\nu}]$ =Set{1,1,1,1}

Eigenvalues Tensor Invariants

$\eta_{\mu\nu}\eta^{\mu\nu} = 4$

Inner Product Tensor Invariant

Signature$[\eta^{\mu\nu}]$ = (+,-,-,-) = {1,3,0} = (1-3)= -2

Signature Tensor Invariant

Det$[\eta^{\mu\nu}]$ = -1
Det$[\eta^{\mu}{}_{\nu}]$ = +1

Determinant Tensor Invariant

4-Position
$\mathbf{R}=R^{\mu}=(ct,\mathbf{r})$

AsymmTri$[\eta^{\mu\nu}]$=24

Asymm Tri-Product Tensor Invariant

EigenValues not defined for the standard Minkowski Metric Tensor since it is a type (2,0)-Tensor, all upper indices. However, they are defined for the mixed form (1,1)-Tensor
EigenValues are defined for the Lorentz Transforms since they are type (1,1)-Tensors, mixed indices

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^{\mu}{}_{\nu}$ or $T_{\mu}{}^{\nu}$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor V^{μ} = **V** = $(v^0,\mathbf{v})$
SR 4-CoVector
(0,1)-Tensor V_{μ} = $(v_0,-\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

$\mathrm{Det}[T^{\alpha}{}_{\alpha}] = \Pi_k[\lambda_k]$; with $\{\lambda_k\}$ = Eigenvalues
Characteristic Eqns: $\mathrm{Det}[T^{\alpha}{}_{\alpha} - \lambda_k I_{(4)}]=0$

Trace$[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^{\mu}{}_{\mu} = T$
$\mathbf{V}\cdot\mathbf{V} = V^{\mu}\eta_{\mu\nu}V^{\nu} = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_{o})^2$
= Lorentz Scalar

SR → QM

4-Vector SRQM Interpretation of QM

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The Lorentz Transform Tensor { $\Lambda^{\mu'}{}_{\nu} = \partial x^{\mu'}/\partial x^{\nu} = \partial_{\nu}[X^{\mu'}]$ } is the tensor all SR 4-Vectors must transform by.

(2,0)-Tensor = 4-Tensor $T^{\mu\nu}$: Has (4+) Tensor Invariants (though not all independent)

a) $T^{\alpha}{}_{\alpha}$ = Trace = Sum of EigenValues for (1,1)-Tensors (mixed)

b) $T^{\alpha}{}_{[\alpha}T^{\beta}{}_{\beta]}$ = Asymm Bi-Product → Inner Product

c) $T^{\alpha}{}_{[\alpha}T^{\beta}{}_{\beta}T^{\gamma}{}_{\gamma]}$ = Asymm Tri-Product → ?Name?

d) $T^{\alpha}{}_{[\alpha}T^{\beta}{}_{\beta}T^{\gamma}{}_{\gamma}T^{\delta}{}_{\delta]}$ = Asymm Quad-Product → 4D Determinant = Product of EigenValues for (1,1)-Tensors

a): Lorentz Trace[$\Lambda^{\mu\nu}$] = {0..4..Infinitiy} Lorentz Boost meets Rotation at Identity of 4

b): Lorentz Inner Product $\Lambda_{\mu\nu}\Lambda^{\mu\nu}$ = 4 from {$\eta_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta} = \eta_{\alpha\beta}$} and {$\eta_{\mu\nu}\eta^{\mu\nu} = 4$}

c): Lorentz AsymmTri[$\Lambda^{\mu\nu}$] =

d): Lorentz Det[$\Lambda^{\mu\nu}$] = +1 for Proper Transforms, Continuous Transforms Proper

Inner Product Tensor Invariant

$\Lambda_{\mu\nu}\Lambda^{\mu\nu}=4$

Asymm Tri-Product Tensor Invariant

AsymmTri[$\Lambda^{\mu'}{}_{\nu}$]=? Not yet calc...

An even more general version would be with a & b as arbitrary complex values:

EigenValues[$\Lambda^{\mu'}{}_{\nu}$] =Set{e^{a},e^{-a},e^{b},e^{-b}}

could be 2 boosts, 2 rotations, or a boost:rotation combo

Sum of EigenValues[$\Lambda^{\mu'}{}_{\nu}$] =Tr[$\Lambda^{\mu'}{}_{\nu}$]=$\Lambda^{\mu'}{}_{\mu}$ ={e^{a}+e^{-a}+e^{b}+e^{-b}} =2(cosh[a]+cosh[b]) ={-4..Infinity}

Product of EigenValues[$\Lambda^{\mu'}{}_{\nu}$] =Det[$\Lambda^{\mu'}{}_{\nu}$] ={$e^{a}\cdot e^{-a}\cdot e^{b}\cdot e^{-b}$} =+1

Trace Tensor Invariant

Tr[Cont. $\Lambda^{\mu'}{}_{\nu}$]={0..4..Infinity} Depends on "rotation" amount

Determinant Tensor Invariant

Det[Proper $\Lambda^{\mu'}{}_{\nu}$]=+1 Proper Transform always +1

SR:Lorentz Transform

$\partial_{\nu}[R^{\mu'}] = \partial R^{\mu'}/\partial R^{\nu} = \Lambda^{\mu'}{}_{\nu}$

$\Lambda^{\mu}{}_{\nu} = (\Lambda^{-1})_{\nu}{}^{\mu} : \Lambda^{\mu}{}_{\alpha}\Lambda^{\alpha}{}_{\nu} = \eta^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu}$

$\eta_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta} = \eta_{\alpha\beta}$

Det[$\Lambda^{\mu}{}_{\nu}$]=±1 $\Lambda_{\mu\nu}\Lambda^{\mu\nu}$=4

Rotation(0) = Identity = Boost(0)

Lorentz SR Rotation Tensor $\Lambda^{\mu'}{}_{\nu} \to R^{\mu'}{}_{\nu}$ =

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos[\theta] & -\sin[\theta] & 0 \\ 0 & \sin[\theta] & \cos[\theta] & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

EigenValues[$R^{\mu'}{}_{\nu}$] =Set{1,$e^{i\theta}$,$e^{-i\theta}$,1}

Sum of EigenValues[$R^{\mu'}{}_{\nu}$] =Tr[$R^{\mu'}{}_{\nu}$]=$R^{\mu'}{}_{\mu}$ =1+$e^{i\theta}$+$e^{-i\theta}$+1 =2+2cos[θ] ={0..4}

Product of EigenValues[$R^{\mu'}{}_{\nu}$] =Det[$R^{\mu'}{}_{\nu}$] =$1\cdot e^{i\theta}\cdot e^{-i\theta}\cdot 1$ = +1

Proper

Lorentz SR Identity Tensor $\Lambda^{\mu'}{}_{\nu} \to \eta^{\mu'}{}_{\nu}$ =$R^{\mu'}{}_{\nu}$[0] = $B^{\mu'}{}_{\nu}$[0] = $\delta^{\mu'}{}_{\nu}$ =

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

= Minkowski Delta

EigenValues[$\eta^{\mu'}{}_{\nu}$] =Set{1,1,1,1}

Sum of EigenValues[$\eta^{\mu'}{}_{\nu}$] =Tr[$\eta^{\mu'}{}_{\nu}$]=$\eta^{\mu'}{}_{\mu}$ =1+1+1+1 =4 ={4}

Product of EigenValues[$\eta^{\mu'}{}_{\nu}$] =Det[$\eta^{\mu'}{}_{\nu}$] =1·1·1·1 = +1

Proper

Lorentz SR Boost Tensor $\Lambda^{\mu'}{}_{\nu} \to B^{\mu'}{}_{\nu}$ =

$$\begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

EigenValues[$B^{\mu'}{}_{\nu}$] =Set{e^{θ},$e^{-\theta}$,1,1}

Sum of EigenValues[$B^{\mu'}{}_{\nu}$] =Tr[$B^{\mu'}{}_{\nu}$]=$B^{\mu'}{}_{\mu}$ =e^{θ}+$e^{-\theta}$+1+1 =2+2cosh[θ]=2+2γ ={4..Infinity}

Product of EigenValues[$B^{\mu'}{}_{\nu}$] =Det[$B^{\mu'}{}_{\nu}$] =$e^{\theta}\cdot e^{-\theta}\cdot 1\cdot 1$ = +1

Proper

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^{\mu}{}_{\nu}$ or $T_{\mu}{}^{\nu}$
(0,2)-Tensor $T_{\mu\nu}$

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(1,0)-Tensor V^{μ} = **V** = (v^0,**v**)
SR 4-CoVector
(0,1)-Tensor V_{μ} = (v_0,-**v**)

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Det[$T^{\alpha}{}_{\alpha}$] = $\Pi_k[\lambda_k]$; with {λ_k} = Eigenvalues
Characteristic Eqns: Det[$T^{\alpha}{}_{\alpha} - \lambda_k I_{(4)}$]=0

Trace[$T^{\mu\nu}$] = $\eta_{\mu\nu}T^{\mu\nu} = T^{\mu}{}_{\mu}$ = T
V·V = $V^{\mu}\eta_{\mu\nu}V^{\nu}$ = [$(v^0)^2$ - **v·v**] = $(v^0{}_o)^2$
= Lorentz Scalar

SR → QM

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$\partial_\nu[R^{\mu'}] = \partial R^{\mu'}/\partial R^\nu = \Lambda^{\mu'}{}_\nu$

$\Lambda^\mu{}_\nu = (\Lambda^{-1})_\nu{}^\mu : \Lambda^\mu{}_\alpha \Lambda^\alpha{}_\nu = \eta^\mu{}_\nu = \delta^\mu{}_\nu$

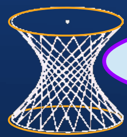
$\eta_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta = \eta_{\alpha\beta}$

$\mathrm{Det}[\Lambda^\mu{}_\nu]=\pm 1$ $\quad \Lambda_{\mu\nu}\Lambda^{\mu\nu}=4$

Inner Product Tensor Invariant

$\Lambda_{\mu\nu}\Lambda^{\mu\nu}=4$

Asymm Tri-Product Tensor Invariant



AsymmTri$[\Lambda^{\mu'}{}_\nu]$=? Not yet calc...

Trace Tensor Invariant

$\mathrm{Tr}[\text{Discrete } \Lambda^{\mu'}{}_\nu]=\{-4,-2,0,2,4\}$ Depends on transform

Determinant Tensor Invariant

$\mathrm{Det}[\Lambda^{\mu'}{}_\nu]=\pm 1$
Proper Transform = +1
Improper Transform = -1

The Trace of various discrete Lorentz transforms varies in steps from {-4,-2,0,2,4}

This includes Mirror Flips, Time Reversal, and Parity Inverse – essentially taking all combinations of ±1 on the diagonal of the transform.

Lorentz SR TPcombo Tensor $\Lambda^{\mu'}{}_\nu \to TP^{\mu'}{}_\nu = -\eta^{\mu'}{}_\nu = -\delta^{\mu'}{}_\nu =$
$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$
= Negative Identity

EigenValues$[TP^{\mu'}{}_\nu]$ =Set{-1,-1,-1,-1}

Sum of EigenValues$[TP^{\mu'}{}_\nu]$ $=\mathrm{Tr}[TP^{\mu'}{}_\nu]=TP^{\mu'}{}_\mu$ = -1-1-1-1 = -4

Product of EigenValues$[TP^{\mu'}{}_\nu]$ $=\mathrm{Det}[TP^{\mu'}{}_\nu]$ = -1·-1·-1·-1 = +1

Proper

Lorentz SR Parity-Inversion Tensor $\Lambda^{\mu'}{}_\nu \to P^{\mu'}{}_\nu$ =
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$
= Flip-xyz

EigenValues$[P^{\mu'}{}_\nu]$ =Set{1,-1,-1,-1}

Sum of EigenValues$[P^{\mu'}{}_\nu]$ $=\mathrm{Tr}[P^{\mu'}{}_\nu]=P^{\mu'}{}_\mu$ = 1-1-1-1 = -2

Product of EigenValues$[P^{\mu'}{}_\nu]$ $=\mathrm{Det}[P^{\mu'}{}_\nu]$ = 1·-1·-1·-1 = -1

Improper

Lorentz SR Flip-xy-Combo Tensor $\Lambda^{\mu'}{}_\nu \to Fxy^{\mu'}{}_\nu = -\eta^{\mu'}{}_\nu = -\delta^{\mu'}{}_\nu =$
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
= Rotation-z (π)

EigenValues$[Fxy^{\mu'}{}_\nu]$ =Set{1,-1,-1,1}

Sum of EigenValues$[Fxy^{\mu'}{}_\nu]$ $=\mathrm{Tr}[Fxy^{\mu'}{}_\nu]=Fxy^{\mu'}{}_\mu$ = 1-1-1+1 = 0

Product of EigenValues$[Fxy^{\mu'}{}_\nu]$ $=\mathrm{Det}[Fxy^{\mu'}{}_\nu]$ = -1·-1·-1·1 = +1

Proper

Lorentz SR Time-Reversal Tensor $\Lambda^{\mu'}{}_\nu \to T^{\mu'}{}_\nu$ =
$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
= Flip-t

EigenValues$[T^{\mu'}{}_\nu]$ =Set{-1,1,1,1}

Sum of EigenValues$[T^{\mu'}{}_\nu]$ $=\mathrm{Tr}[T^{\mu'}{}_\nu]=T^{\mu'}{}_\mu$ = -1+1+1+1 = 2

Product of EigenValues$[T^{\mu'}{}_\nu]$ $=\mathrm{Det}[T^{\mu'}{}_\nu]$ = -1·1·1·1 = -1

Improper

Lorentz SR Identity Tensor $\Lambda^{\mu'}{}_\nu \to \eta^{\mu'}{}_\nu = \delta^{\mu'}{}_\nu =$
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
= Minkowski Delta

EigenValues$[\eta^{\mu'}{}_\nu]$ =Set{1,1,1,1}

Sum of EigenValues$[\eta^{\mu'}{}_\nu]$ $=\mathrm{Tr}[\eta^{\mu'}{}_\nu]=\eta^{\mu'}{}_\mu$ = 1+1+1+1 = 4

Product of EigenValues$[\eta^{\mu'}{}_\nu]$ $=\mathrm{Det}[\eta^{\mu'}{}_\nu]$ = 1·1·1·1 = +1

Proper

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^\mu{}_\nu$ or $T_\mu{}^\nu$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0,\mathbf{v})$
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(0,1)-Tensor $V_\mu = (v_0,-\mathbf{v})$

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$\mathrm{Det}[T^\alpha{}_\alpha] = \Pi_k[\lambda_k]$; with $\{\lambda_k\}$ = Eigenvalues
Characteristic Eqns: $\mathrm{Det}[T^\alpha{}_\alpha - \lambda_k I_{(4)}]=0$

$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu{}_\mu = T$
$\mathbf{V}\cdot\mathbf{V} = V^\mu\eta_{\mu\nu}V^\nu = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$
= Lorentz Scalar

SR → QM

4-Vector SRQM Interpretation of QM

SRQM Study: SR 4-Tensors

More SR Tensor Invariants for Discrete Lorentz Transform Tensors

A Tensor Study of Physical 4-Vectors

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SR:Lorentz Transform

$\partial_\nu[R^{\mu'}] = \partial R^{\mu'}/\partial R^\nu = \Lambda^{\mu'}{}_\nu$

$\Lambda^\mu{}_\nu = (\Lambda^{-1})_\nu{}^\mu : \Lambda^\mu{}_\alpha \Lambda^\alpha{}_\nu = \eta^\mu{}_\nu = \delta^\mu{}_\nu$

$\eta_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta = \eta_{\alpha\beta}$

$\mathrm{Det}[\Lambda^\mu{}_\nu]=\pm1$ $\Lambda_{\mu\nu}\Lambda^{\mu\nu}=4$

Note:

The Flip-xy-Combo is the equivalent of a π-Rotation-z.

I suspect that this may be related to exchange symmetry and the Spin-Statistics idea that a particle-exchange is the equivalent of a spin-rotation.

A single Flip would not be an exchange because it leaves a mirror-inversion of <right-|-left>.

But the extra Flip along an orthogonal axis corrects the mirror-inversion, and would be an overall exchange because the particle is in a different location.

Lorentz SR 0-Rotation-z Tensor $\Lambda^{\mu'}{}_\nu \to R^{\mu'}{}_\nu$ =

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos[0] & -\sin[0] & 0 \\ 0 & \sin[0] & \cos[0] & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

EigenValues$[R^{\mu'}{}_\nu]$ =Set$\{1,e^{i0},e^{-i0},1\}$

Sum of EigenValues$[R^{\mu'}{}_\nu]$ =Tr$[R^{\mu'}{}_\nu]=R^{\mu'}{}_\mu$ $=1+e^{i0}+e^{-i0}+1$ =2+2cos[0] =4

Product of EigenValues$[R^{\mu'}{}_\nu]$ =Det$[R^{\mu'}{}_\nu]$ $=1\cdot e^{i0}\cdot e^{-i0}\cdot 1$ = +1

Proper

Lorentz SR Identity Tensor $\Lambda^{\mu'}{}_\nu \to \eta^{\mu'}{}_\nu$ $= \delta^{\mu'}{}_\nu$ =

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

= Minkowski Delta

EigenValues$[\eta^{\mu'}{}_\nu]$ =Set$\{1,1,1,1\}$

Sum of EigenValues$[\eta^{\mu'}{}_\nu]$ =Tr$[\eta^{\mu'}{}_\nu]=\eta^{\mu'}{}_\mu$ = 1+1+1+1 =2+2cos[0] = 4

Product of EigenValues$[\eta^{\mu'}{}_\nu]$ =Det$[\eta^{\mu'}{}_\nu]$ $= 1\cdot1\cdot1\cdot1$ = +1

Proper

Lorentz SR Flip-x Tensor $\Lambda^{\mu'}{}_\nu \to Fx^{\mu'}{}_\nu$ =

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

EigenValues$[Fx^{\mu'}{}_\nu]$ =Set$\{1,-1,1,1\}$

Sum of EigenValues$[Fx^{\mu'}{}_\nu]$ =Tr$[Fx^{\mu'}{}_\nu]=Fx^{\mu'}{}_\mu$ = 1-1+1+1 = 2

Product of EigenValues$[Fx^{\mu'}{}_\nu]$ =Det$[Fx^{\mu'}{}_\nu]$ $= 1\cdot{-1}\cdot1\cdot1$ = -1

Improper

Lorentz SR Flip-y Tensor $\Lambda^{\mu'}{}_\nu \to Fy^{\mu'}{}_\nu$ =

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

EigenValues$[Fy^{\mu'}{}_\nu]$ =Set$\{1,1,-1,1\}$

Sum of EigenValues$[Fy^{\mu'}{}_\nu]$ =Tr$[Fy^{\mu'}{}_\nu]=Fy^{\mu'}{}_\mu$ = 1+1-1+1 = 2

Product of EigenValues$[Fy^{\mu'}{}_\nu]$ =Det$[Fy^{\mu'}{}_\nu]$ $= 1\cdot1\cdot{-1}\cdot1$ = -1

Improper

Lorentz SR Flip-xy-Combo Tensor $\Lambda^{\mu'}{}_\nu \to Fxy^{\mu'}{}_\nu$ $= -\eta^{\mu'}{}_\nu = -\delta^{\mu'}{}_\nu$ =

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

= Rotation-z (π)

EigenValues$[Fxy^{\mu'}{}_\nu]$ =Set$\{1,-1,-1,1\}$

Sum of EigenValues$[Fxy^{\mu'}{}_\nu]$ =Tr$[Fxy^{\mu'}{}_\nu]=Fxy^{\mu'}{}_\mu$ = 1-1-1+1 =2+2cos[π] = 0

Product of EigenValues$[Fxy^{\mu'}{}_\nu]$ =Det$[Fxy^{\mu'}{}_\nu]$ $= -1\cdot{-1}\cdot{-1}\cdot1$ = +1

Proper

Lorentz SR π-Rotation-z Tensor $\Lambda^{\mu'}{}_\nu \to R^{\mu'}{}_\nu$ =

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos[\pi] & -\sin[\pi] & 0 \\ 0 & \sin[\pi] & \cos[\pi] & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

EigenValues$[R^{\mu'}{}_\nu]$ =Set$\{1,e^{i\pi},e^{-i\pi},1\}$

Sum of EigenValues$[R^{\mu'}{}_\nu]$ =Tr$[R^{\mu'}{}_\nu]=R^{\mu'}{}_\mu$ $=1+e^{i\pi}+e^{-i\pi}+1$ =2+2cos[π] =0

Product of EigenValues$[R^{\mu'}{}_\nu]$ =Det$[R^{\mu'}{}_\nu]$ $=1\cdot e^{i\pi}\cdot e^{-i\pi}\cdot 1$ = +1

Proper

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= Lorentz Scalar

SR 4-Scalars, 4-Vectors, 4-Tensors

Elegantly join many dual physical properties and relations

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson

SR 4-Scalars, 4-Vectors, and 4-Tensors beautifully and elegantly display the relations between lots of different physical properties and relations.
Their notation makes navigation through the physics very simple.

They also devolve very nicely into the limiting/approximate Newtonian cases of { $|\mathbf{v}| \ll c$ }
by letting { $\gamma \to 1$ and $\gamma' = d\gamma/dt \to 0$ }.

SR tells us that several different physical properties are actually dual aspects of the same thing,
with the only real difference being one's point of view, or reference frame.

Examples of 4-Vectors = (1,0)-Tensors include:
(Time , Space), (Energy , Momentum), (Power , Force), (Frequency , WaveNumber),
(Time Differential , Spatial Gradient),
(ChargeDensity , CurrentDensity), (EM-ScalarPotential , EM-VectorPotential), etc.

One can also examine 4-Tensors, which are type (2,0)-Tensors.
The Faraday EM Tensor similarly combines EM fields:
Electric { $\mathbf{e} = e^i = (e^x,e^y,e^z)$ } and Magnetic { $\mathbf{b} = b^k = (b^x,b^y,b^z)$ }

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -e^j/c \\ +e^i/c & -(\varepsilon^{ij}{}_k\, b^k) \end{bmatrix}$$

Also, things are even more related than that.
The 4-Momentum is just a constant times 4-Velocity.
The 4-WaveVector is just a constant times 4-Velocity.

In addition, the very important conservation/continuity equations seem to just fall out of the notation.
The universe apparently has some simple laws which can be easy to write down by using a little math and a super notation.

4-Scalar
S

SR 4-Vector $\mathbf{V} = V^\alpha$
$=(v^t,\mathbf{v})=(v^t,v^x,v^y,v^z)$
$=(\text{temporal} * c^{\pm 1},\textbf{spatial})$

4-Tensor $T^{\alpha\beta}$
$$= \begin{bmatrix} T^{tt} & T^{tx} & T^{ty} & T^{tz} \\ T^{xt} & T^{xx} & T^{xy} & T^{xz} \\ T^{yt} & T^{yx} & T^{yy} & T^{yz} \\ T^{zt} & T^{zx} & T^{zy} & T^{zz} \end{bmatrix} = \begin{bmatrix} \text{temporal} & \text{mixed} \\ \text{mixed} & \text{spatial} \end{bmatrix}$$

4-Velocity
$\mathbf{U}=\gamma(c,\mathbf{u})$

m_o
E_o/c^2

4-Momentum
$\mathbf{P}=(mc,\mathbf{p})=(E/c,\mathbf{p})$

ω_o/c^2

4-WaveVector
$\mathbf{K}=(\omega/c,\mathbf{k})=(\omega/c,\omega\hat{\mathbf{n}}/v_{phase})$

Faraday EM Tensor $F^{\alpha\beta}$
$$= \begin{bmatrix} 0 & -e^x/c & -e^y/c & -e^z/c \\ +e^x/c & 0 & -b^z & +b^y \\ +e^y/c & +b^z & 0 & -b^x \\ +e^z/c & -b^y & +b^x & 0 \end{bmatrix} = \begin{bmatrix} 0 & ,-e^j/c \\ +e^i/c, & -\varepsilon^{ij}{}_k b^k \end{bmatrix}$$

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= Lorentz Scalar

SRQM Diagram: SR 4-Vectors and Lorentz Scalars / Physical Constants

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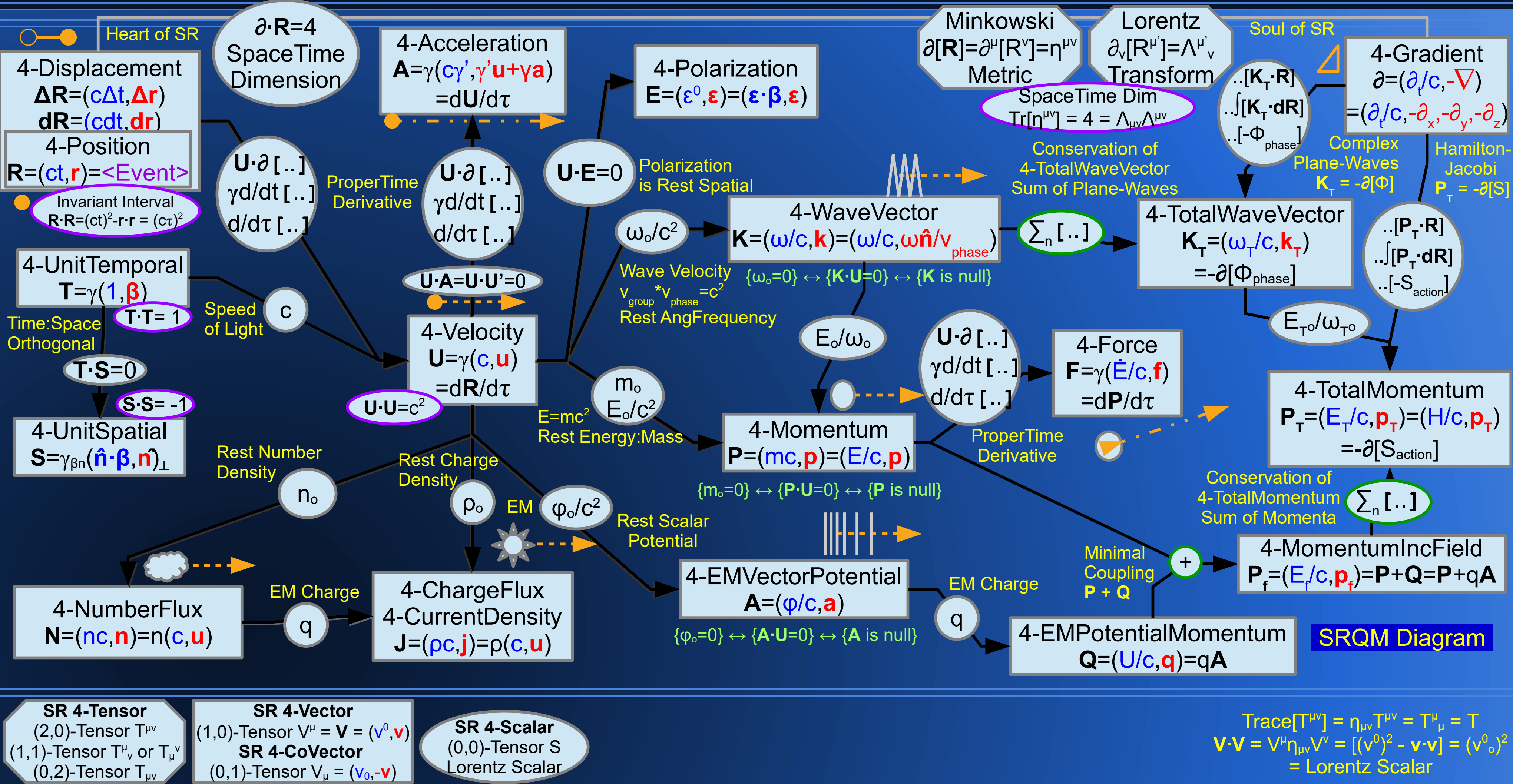

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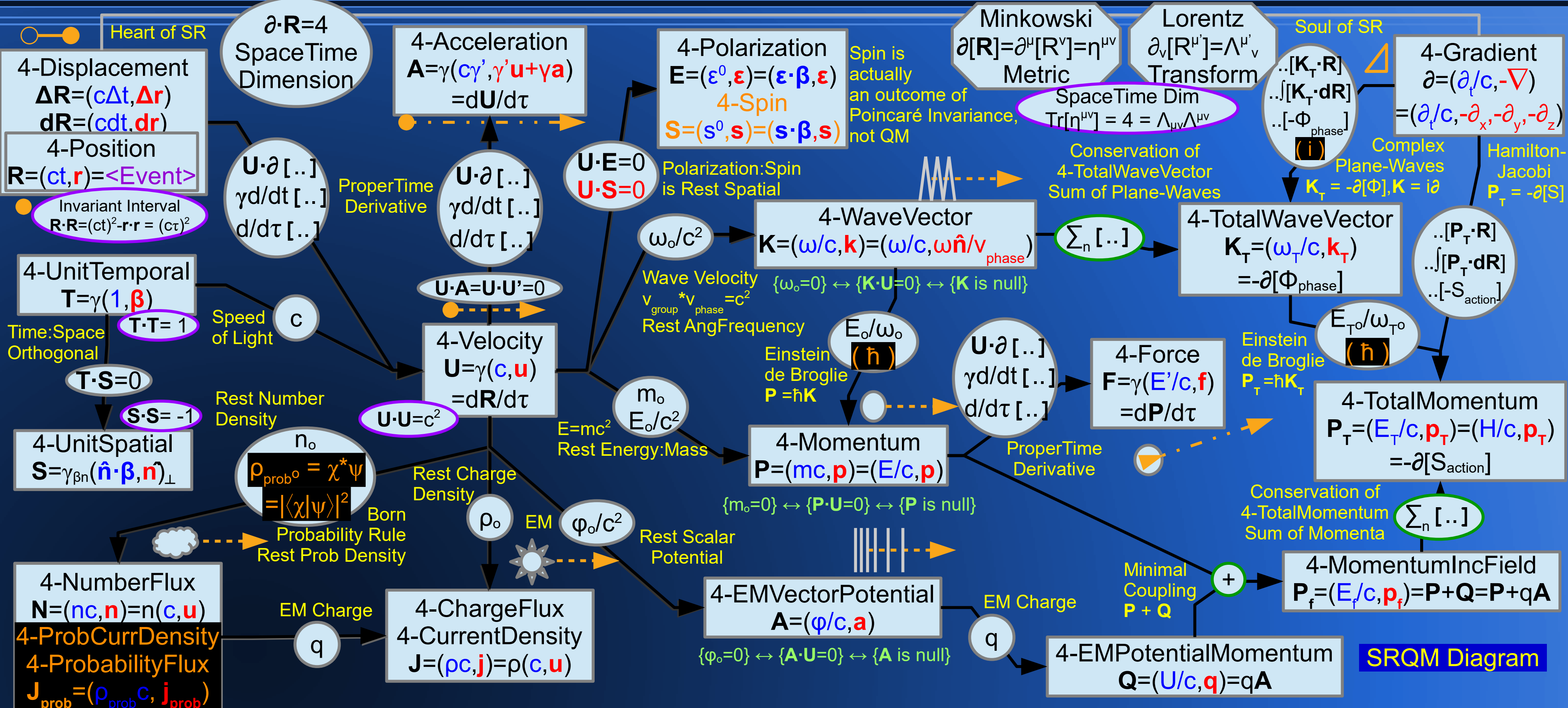

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^{\mu}{}_{\nu}$ or $T_{\mu}{}^{\nu}$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^{\mu} = \mathbf{V} = (v^0,\mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_{\mu} = (v_0,-\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Existing SR Rules
Quantum Principles

$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^{\mu}{}_{\mu} = T$
$\mathbf{V}\cdot\mathbf{V} = V^{\mu}\eta_{\mu\nu}V^{\nu} = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$
= Lorentz Scalar

SR Gradient 4-Vectors = (1,0)-Tensors
SR Gradient One-Forms = (0,1)-Tensors

4-Vector = Type (1,0)-Tensor	**[Temporal : Spatial] components**
4-Position $\mathbf{R} = R^{\mu} = (ct,\mathbf{r})$	[Time (t) : Space ($\mathbf{r}$)]
4-Gradient $\partial_{\mathbf{R}} = \partial = \partial^{\mu} = \partial/\partial R_{\mu} = (\partial_t/c,-\nabla)$	[Time Differential (∂_t) : Spatial Gradient(∇)]

Standard 4-Vector	**Related Gradient 4-Vector (from index-raised Gradient One-Form)**
4-Position $\mathbf{R} = R^{\mu} = (ct,\mathbf{r})$	4-PositionGradient $\partial_{\mathbf{R}} = \partial_R{}^{\mu} = \partial/\partial R_{\mu} = (\partial_{R^t}/c,-\nabla_{\mathbf{R}}) = \partial = \partial^{\mu}$ = 4-Gradient
4-Velocity $\mathbf{U} = U^{\mu} = \gamma(c,\mathbf{u})$	4-VelocityGradient $\partial_{\mathbf{U}} = \partial_U{}^{\mu} = \partial/\partial U_{\mu} = (\partial_{U^t}/c,-\nabla_{\mathbf{U}})$
4-Momentum $\mathbf{P} = P^{\mu} = (E/c,\mathbf{p})$	4-MomentumGradient $\partial_{\mathbf{P}} = \partial_P{}^{\mu} = \partial/\partial P_{\mu} = (\partial_{P^t}/c,-\nabla_{\mathbf{P}})$
4-WaveVector $\mathbf{K} = K^{\mu} = (\omega/c,\mathbf{k})$	4-WaveGradient $\partial_{\mathbf{K}} = \partial_K{}^{\mu} = \partial/\partial K_{\mu} = (\partial_{K^t}/c,-\nabla_{\mathbf{K}})$

In each case, the (Whichever)Gradient 4-Vector is derived from an SR One-Form or 4-CoVector,
which is a type (0,1)-Tensor
ex. One-Form PositionGradient $\partial_{R^{\nu}} = \partial/\partial R^{\nu} = (\partial_{R^t}/c,\nabla_{\mathbf{R}})$

The (Whichever)Gradient 4-Vector is the index-raised version of the SR One-Form (Whichever)Gradient
ex. 4-PositionGradient $\partial_R{}^{\mu} = \partial/\partial R_{\mu} = (\partial_{R^t}/c,-\nabla_{\mathbf{R}}) = \eta^{\mu\nu}\partial_{R^{\nu}} = \eta^{\mu\nu}\partial/\partial R^{\nu} = \eta^{\mu\nu}(\partial_{R^t}/c,\nabla_{\mathbf{R}})_{\nu} = \eta^{\mu\nu}$(One-Form PositionGradient)$_{\nu}$

This is why the 4-Gradient is commonly seen with a minus sign in the spatial component,
unlike the other regular 4-Vectors, which have all positive components.

4-Tensors can be constructed from the Tensor Outer Product of 4-Vectors

SR → QM

Some Basic 4-Vectors Minkowski SpaceTime Diagram Events & Dimensions

4-Vector SRQM Interpretation of QM

A Tensor Study of Physical 4-Vectors

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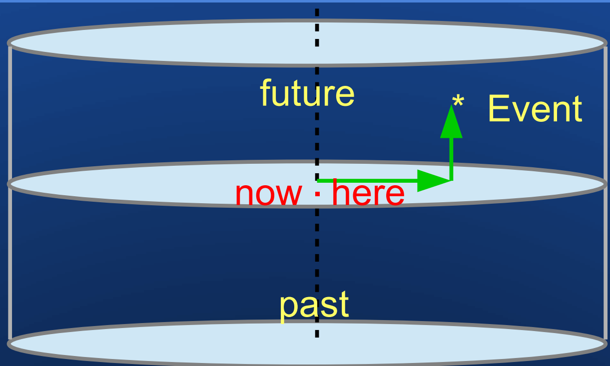


"Stack of Motion Picture Photos"

Classical Mechanics

Δt time-like interval

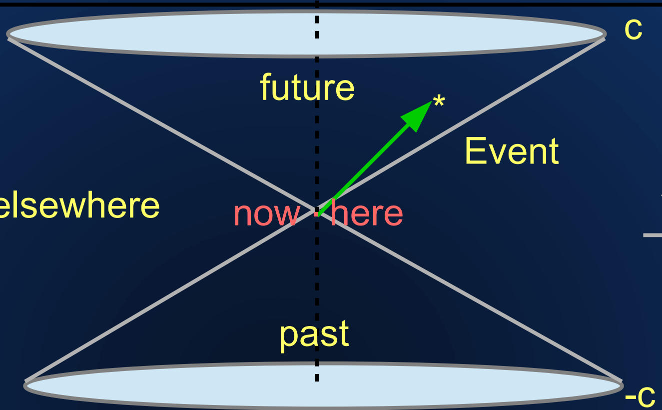
$\mathbf{\Delta r}$ space-like interval

4-Displacement$_{CM}$
$\mathbf{\Delta R}_{CM}=(c\Delta t \; \mathbf{\Delta r})$

1/c → time displacement Δt

3-displacement $\mathbf{\Delta r} = \Delta r^i \to (\Delta x,\Delta y,\Delta z)$

Note the separate dimensional units: (time + 3D space)
Δt is [time], $|\mathbf{\Delta r}|$ is [length]



LightCone

Special Relativity

Δt time-like interval (+)

c light-like interval (0) = null

$\mathbf{\Delta r}$ space-like interval (-)

4-Displacement
$\mathbf{\Delta R}=(c\Delta t,\mathbf{\Delta r})$

4-Position
$\mathbf{R}=(ct,\mathbf{r})$

$\mathbf{\Delta R}\cdot\mathbf{\Delta R} = [(c\Delta t)^2 - \mathbf{\Delta r}\cdot\mathbf{\Delta r}] = 0$

$(c\Delta\tau)^2$ Time-Like (+)
Light-like:Null (0)
$-(\Delta r_o)^2$ Space-like (-)

Note the matching dimensional units: (4D SpaceTime)
$(c\Delta t)$ is [length/time]*[time] = [length], $|\mathbf{\Delta r}|$ is [length], $|\mathbf{\Delta R}|$ is [length]
τ is the Proper Time = "rest-time", time as measured by something not moving spatially
The Minkowski Diagram provides a great visual representation of SpaceTime

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SR → QM

4-Vector SRQM Interpretation of QM

Some Basic 4-Vectors
Minkowski SpaceTime Diagram, WorldLines, LightSpeed to the Future!

A Tensor Study of Physical 4-Vectors

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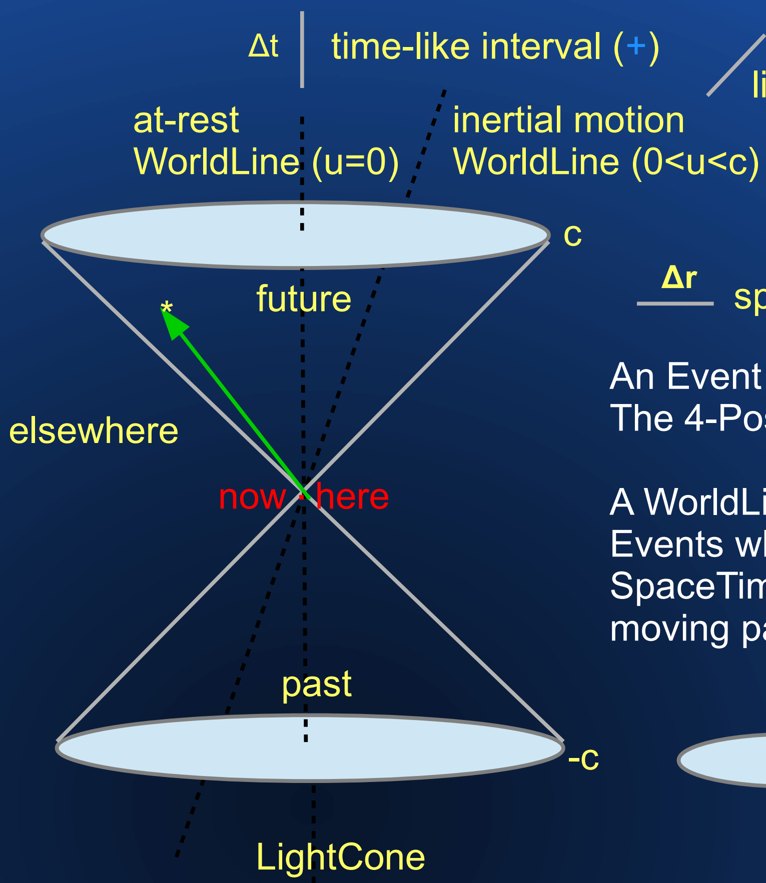
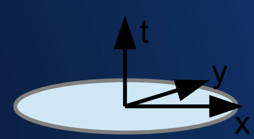


An Event (*) is a point in SpaceTime
The 4-Position points to an Event.

A WorldLine is a series of connected Events which trace out a path in SpaceTime, such as the track of a moving particle.



4-Displacement
$\mathbf{\Delta R}=(c\Delta t,\mathbf{\Delta r})$

4-Position
$\mathbf{R}=(ct,\mathbf{r})$=<Event>

The 4-Position is a particular type of 4-Displacement, for which the vector base is at the origin (0,0,0,0) = 4-Zero.

4-Position is Lorentz Invariant, but not Poincaré Invariant.
A standard 4-Displacement is both.

$$\mathbf{\Delta R}\cdot\mathbf{\Delta R} = [(c\Delta t)^2 - \mathbf{\Delta r}\cdot\mathbf{\Delta r}] = 0 \quad \begin{cases} (c\Delta\tau)^2 & \text{for time-like } (+) \\ 0 & \text{for light-like } (0) \\ -(\Delta r_o)^2 & \text{for space-like } (-) \end{cases}$$

4-Velocity	4-Velocity (rest-frame)	4-Velocity (photonic)
$\mathbf{U}=\gamma(c,\mathbf{u})=d\mathbf{R}/d\tau$	$\mathbf{U}_o=(c,\mathbf{0})$	$\mathbf{U}_c=\gamma_c(c,c\hat{\mathbf{n}})$
$\mathbf{U}\cdot\mathbf{U}=c^2$	$\mathbf{U}_o\cdot\mathbf{U}_o=c^2$	$\mathbf{U}_c\cdot\mathbf{U}_c=c^2$

$$\mathbf{U}\cdot\mathbf{U} = \gamma(c,\mathbf{u})\cdot\gamma(c,\mathbf{u}) = \gamma^2(c^2-\mathbf{u}\cdot\mathbf{u}) = (c^2)$$
$$\gamma = 1/\sqrt{[1-(u/c)^2]} = 1/\sqrt{[1-(\beta)^2]}$$

Massive particles move temporally into future at the speed-of-light (c) in their own rest-frame.

Massless particles (photonic) move nully into the future at the speed-of-light (c), and have no rest-frame.

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SR Invariant Intervals

Minkowski Diagram:Lorentz Transform

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SR:Lorentz Transform

$\partial_\nu[R^{\mu'}] = \partial R^{\mu'}/\partial R^\nu = \Lambda^{\mu'}{}_\nu$

$\Lambda_\mu{}^\nu = (\Lambda^{-1})_\nu{}^\mu : \Lambda^\mu{}_\alpha \Lambda^\alpha{}_\nu = \eta^\mu{}_\nu = \delta^\mu{}_\nu$

$\eta_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta}$

$\mathrm{Det}[\Lambda^\mu{}_\nu]=\pm 1$ $\Lambda_{\mu\nu}\Lambda^{\mu\nu}=4$

SR:Minkowski Metric

$\partial[\mathbf{R}] = \partial^\mu R^\nu = \eta^{\mu\nu} = V^{\mu\nu} + H^{\mu\nu} \rightarrow$

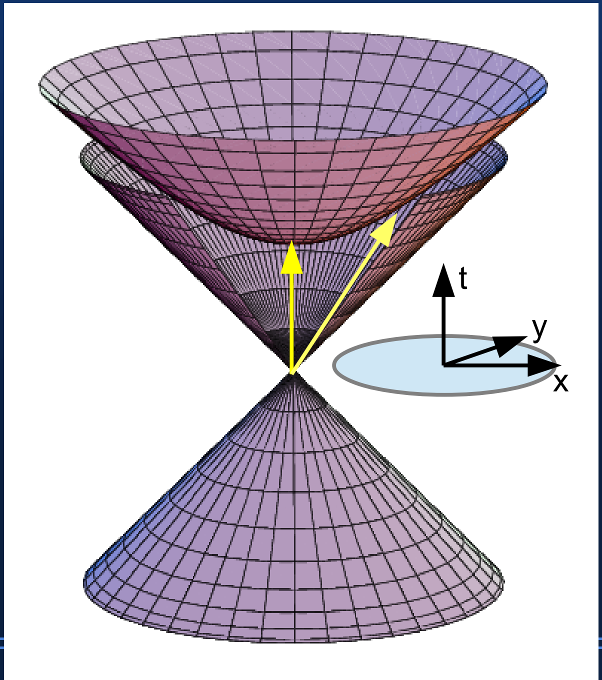
$\mathrm{Diag}[1,-1,-1,-1] = \mathrm{Diag}[1,-I_{(3)}] = \mathrm{Diag}[1,-\delta^{jk}]$

{in Cartesian form} "Particle Physics" Convention

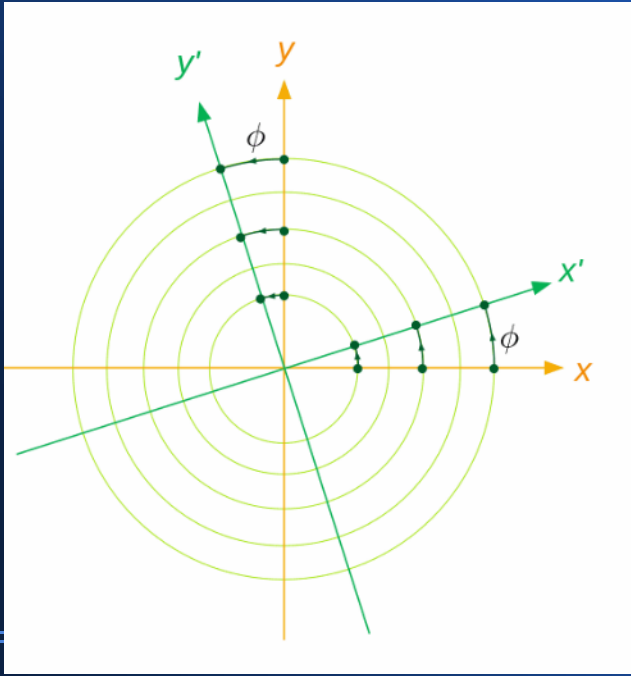
$\{\eta_{\mu\mu}\} = 1/\{\eta^{\mu\mu}\} : \eta_\mu{}^\nu = \delta_\mu{}^\nu$ $\mathrm{Tr}[\eta^{\mu\nu}]=4$

Since the SpaceTime magnitude of **U** is a constant (c), changes in the components of **U** are like rotating the 4-Vector without changing its length. It keeps the same magnitude.
Rotations, purely spatial changes, {eg. along x,y} result in circular displacements.
Boosts, or temporal-spatial changes, {eg. along x,t} result in hyperbolic displacements.
The interval between the origin and a given topograph-line is a Lorentz Invariant Constant.

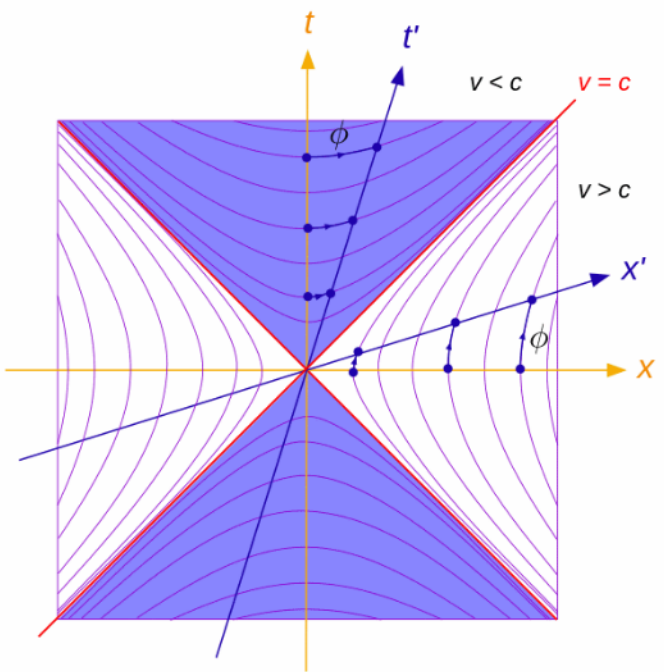
$\mathbf{U}\cdot\mathbf{U} = \gamma(c,\mathbf{u})\cdot\gamma(c,\mathbf{u}) = \gamma^2(c^2-\mathbf{u}\cdot\mathbf{u}) = (c^2)$



Rotation (x,y): Purely Spatial



Boost (x,t): Spatial-Temporal



The Minkowski Diagram provides a great visual representation of SpaceTime

SR Invariant Intervals
Minkowski Diagram

A Tensor Study of Physical 4-Vectors

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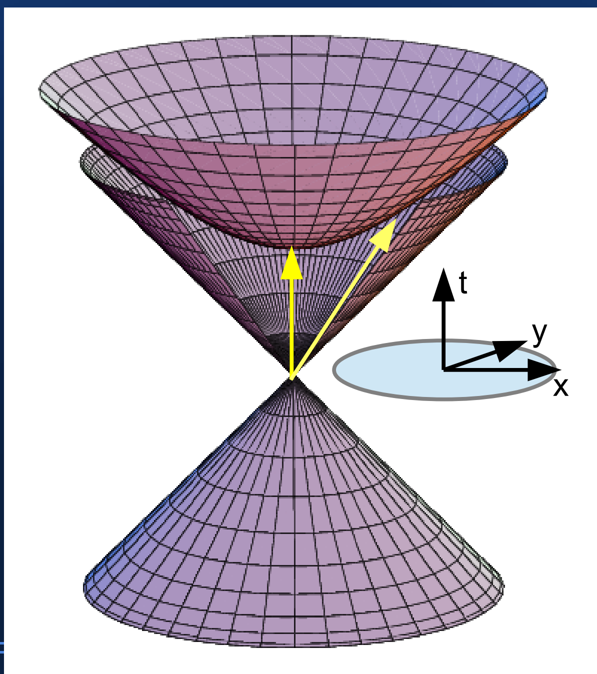
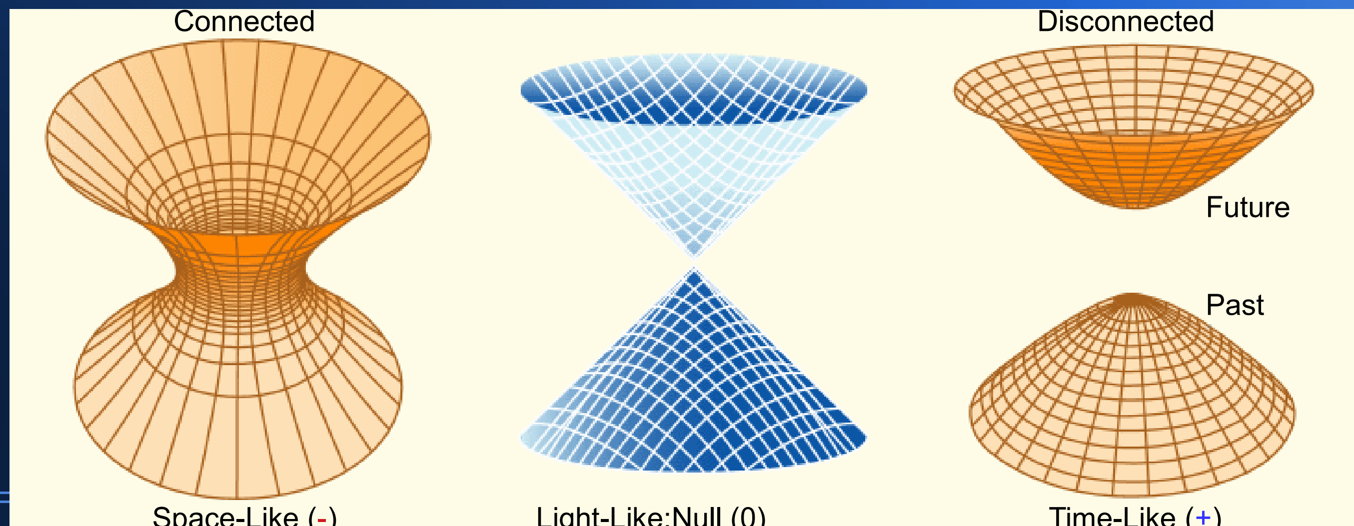


$\mathbf{\Delta R}\cdot\mathbf{\Delta R} = [(c\Delta t)^2 - \mathbf{\Delta r}\cdot\mathbf{\Delta r}] =$

$(c\Delta\tau)^2$	Time-like:Temporal	(+) {causal = temporally-ordered}
(0)	Light-like:Null:Photonic	(0) {causal, maximum signal speed ($\lvert\mathbf{\Delta r}/\Delta t\rvert = c$)}
$-(\Delta r_o)^2$	Space-like:Spatial	(–) {non-causal, spatially-extended}



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SRQM: Some Basic 4-Vectors
4-Position, 4-Velocity, 4-Acceleration
SpaceTime Kinematics

A Tensor Study of Physical 4-Vectors

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ProperTime
$\mathbf{R}\cdot\mathbf{U}/\mathbf{U}\cdot\mathbf{U}=(ct,\mathbf{r})\cdot\gamma(c,\mathbf{u})/c^2=\gamma(c^2t - \mathbf{r}\cdot\mathbf{u})/c^2=(c^2t_o)/c^2$
$= t_o = \tau$

Δ

4-Gradient
$\partial=(\partial_t/c,-\nabla)\rightarrow(\partial_t/c,-\partial_x,-\partial_y,-\partial_z)$

ProperTime Derivative
$\mathbf{U}\cdot\partial=\gamma(c,\mathbf{u})\cdot(\partial_t/c,-\nabla)=\gamma(\partial_t + \mathbf{u}\cdot\nabla)=\gamma d/dt$
$= d/d\tau$

Special Relativity
$|\mathbf{v}| = |\mathbf{u}| = \{0 \leftrightarrow c\}$
$\gamma = 1/\sqrt{[1-(v/c)^2]}$

4-Position $\mathbf{R}=(ct,\mathbf{r})$ → $\mathbf{U}\cdot\partial$ [..], $d/d\tau$ [..], $\gamma d/dt$ [..] → 4-Velocity $\mathbf{U}=\gamma(c,\mathbf{u})$ → $\mathbf{U}\cdot\partial$ [..], $d/d\tau$ [..], $\gamma d/dt$ [..] → 4-Acceleration $\mathbf{A}=\gamma(c\gamma',\gamma'\mathbf{u}+\gamma\mathbf{a})$

4-Vectors:
$\mathbf{R}$ = <Event>
$\mathbf{U} = d\mathbf{R}/d\tau$
$\mathbf{A} = d\mathbf{U}/d\tau$

↓ Newtonian/Classical Limit ↓

Classical Mechanics
$|\mathbf{v}| = |\mathbf{u}| << c$
$\gamma\rightarrow 1+O[(v/c)^2]$
$\gamma'\rightarrow 0$

4-Position$_{CM}$ $\mathbf{R}_{CM}=(ct, \mathbf{r})$ → d/dt [..] → 4-Velocity$_{CM}$ $\mathbf{U}_{CM}=(c, \mathbf{u})$ → d/dt [..] → 4-Acceleration$_{CM}$ $\mathbf{A}_{CM}=(0, \mathbf{a})$

Since time:space don't mix in CM, Typically use time t & 3-position **r** separately

Since temporal velocity (c) always constant in CM Typically use just 3-velocity **u**

Since temporal acceleration (0) always constant in CM, Typically use just 3-acceleration **a**

time t

3-position $\mathbf{r} = r^i\rightarrow(x,y,z)$ → d/dt [..] → 3-velocity $\mathbf{u}\rightarrow(u^x,u^y,u^z)$ → d/dt [..] → 3-acceleration $\mathbf{a}\rightarrow(a^x,a^y,a^z)$

scalar:
time

3-vectors:
$\mathbf{r}$ = <location>
$\mathbf{u} = d\mathbf{r}/dt$
$\mathbf{a} = d\mathbf{u}/dt$

The relativistic Gamma factor $\gamma = 1/\sqrt{[1-(v/c)^2]}$
The 1st order Newtonian Limit gives $\gamma \sim 1 + O[(v/c)^2]$
The 2nd order Newtonian Limit gives $\gamma \sim 1 + (v/c)^2/2 + O[(v/c)^4]$

For historical reasons, velocity can be represented by either (v) or (u)

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$\mathbf{V}\cdot\mathbf{V} = V^{\mu}\eta_{\mu\nu}V^{\nu} = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0_o)^2$
= Lorentz Scalar

SRQM: Some Basic 4-Vectors
4-Position, 4-Velocity, 4-Acceleration, 4-Momentum, 4-Force
SpaceTime Dynamics

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ProperTime
$\mathbf{R}\cdot\mathbf{U}/\mathbf{U}\cdot\mathbf{U}=(ct,\mathbf{r})\cdot\gamma(c,\mathbf{u})/c^2=\gamma(c^2t - \mathbf{r}\cdot\mathbf{u})/c^2=(c^2t_o)/c^2 = t_o = \tau$

4-Gradient
$\partial=(\partial_t/c,-\nabla)\rightarrow(\partial_t/c,-\partial_x,-\partial_y,-\partial_z)$

ProperTime Derivative
$\mathbf{U}\cdot\partial=\gamma(c,\mathbf{u})\cdot(\partial_t/c,-\nabla)=\gamma(\partial_t + \mathbf{u}\cdot\nabla)=\gamma d/dt = d/d\tau$

Special Relativity
$|\mathbf{v}| = |\mathbf{u}| = \{0 \leftrightarrow c\}$
$\gamma = 1/\sqrt{[1-(v/c)^2]}$

4-Position
$\mathbf{R}=(ct,\mathbf{r})$

$\mathbf{U}\cdot\partial$ [..]
$d/d\tau$ [..]
$\gamma d/dt$ [..]

4-Velocity
$\mathbf{U}=\gamma(c,\mathbf{u})$

$\mathbf{U}\cdot\partial$ [..]
$d/d\tau$ [..]
$\gamma d/dt$ [..]

4-Acceleration
$\mathbf{A}=\gamma(c\gamma',\gamma'\mathbf{u}+\gamma\mathbf{a})$

$E_o/c^2 = m_o$

4-Momentum
$\mathbf{P}=(E/c,\mathbf{p})=(mc,\mathbf{p})$

$\mathbf{U}\cdot\partial$ [..]
$d/d\tau$ [..]
$\gamma d/dt$ [..]

4-Force
$\mathbf{F}=\gamma(\dot{E}/c,\mathbf{f})$

4-Vectors:
$\mathbf{R}$ = <Event>
$\mathbf{U} = d\mathbf{R}/d\tau$
$\mathbf{A} = d\mathbf{U}/d\tau$

$\mathbf{P} = m_o\mathbf{U}$
$\mathbf{F} = d\mathbf{P}/d\tau$

This group of 4-Vectors are the main ones that are connected by the ProperTime Derivative.
$\mathbf{U}\cdot\partial = d/d\tau = \gamma d/dt = \gamma(c\partial_t/c+\mathbf{u}\cdot\nabla) = \gamma(\partial_t + \mathbf{u}\cdot\nabla)$

The classical part of it, the convective derivative, $(\partial_t + \mathbf{u}\cdot\nabla)$, is known by many different names:
The convective derivative is a derivative taken with respect to a moving coordinate system. It is also called the advective derivative, derivative following the motion, hydrodynamic derivative, Lagrangian derivative, material derivative, particle derivative, substantial derivative, substantive derivative, Stokes derivative, or total derivative

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SRQM: Some Basic 4-Vectors
4-Velocity, 4-Momentum, $E=mc^2$

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$\mathbf{P}\cdot\mathbf{U} = \gamma(E - \mathbf{p}\cdot\mathbf{u}) = E_o = m_oc^2$

Special Relativity
$|\mathbf{v}| = |\mathbf{u}| = \{0 \leftrightarrow c\}$

4-Velocity
$\mathbf{U}=\gamma(c,\mathbf{u})$

$E_o/c^2 = m_o$

RestEnergy
RestMass

4-Momentum
$\mathbf{P}=(E/c,\mathbf{p})=(mc,\mathbf{p})$

$\mathbf{U}\cdot\mathbf{U} = (c)^2$

$\mathbf{P}\cdot\mathbf{P} = (E_o/c)^2 = (m_oc)^2$

$\mathbf{U} = \gamma(c,\mathbf{u})$
$\mathbf{P} = (E/c,\mathbf{p}) = m_o\mathbf{U} = \gamma m_o(c,\mathbf{u}) = m(c,\mathbf{u})$

Temporal part: {energy} $E = \gamma m_oc^2 = mc^2$
$E = m_oc^2 + (\gamma-1)m_oc^2$
$E = E_o + (\gamma-1)E_o$
(rest) + (kinetic)

Spatial part: **{momentum}** $\mathbf{p} = \gamma m_o\mathbf{u} = m\mathbf{u}$

↓ Newtonian/Classical Limit ↓

Classical Mechanics
$|\mathbf{v}| = |\mathbf{u}| << c$

4-Velocity$_{CM}$
$\mathbf{U}_{CM}=(1+(v/c)^2)(c \;\blacktriangle\; \mathbf{u})$

m_o

Classical Mass

4-Momentum$_{CM}$
$\mathbf{P}_{CM}=(E/c \;\blacktriangle\; \mathbf{p})=(mc \;\blacktriangle\; \mathbf{p})$

c

energy
E

3-momentum
$\mathbf{p}\rightarrow(p^x,p^y,p^z)$

$\mathbf{u} \rightarrow (u_x,u_y,u_z)$
$\mathbf{P} = (E/c,\mathbf{p}) \sim (1+(v/c)^2/2)m_o(c,\mathbf{u})$

Temporal part: {energy} $E \sim (1+(v/c)^2/2)m_oc^2 = m_oc^2 + m_ov^2/2$
$E_o + |\mathbf{p}|^2/2m_o$
(rest) + (kinetic)

Spatial part: **{momentum}** $\mathbf{p} \sim (1)m_o\mathbf{u} = m_o\mathbf{u} \rightarrow m\mathbf{u}$

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SRQM: Some Basic 4-Vectors
4-Velocity, 4-Acceleration,
SpaceTime Orthogonality

A Tensor Study of Physical 4-Vectors

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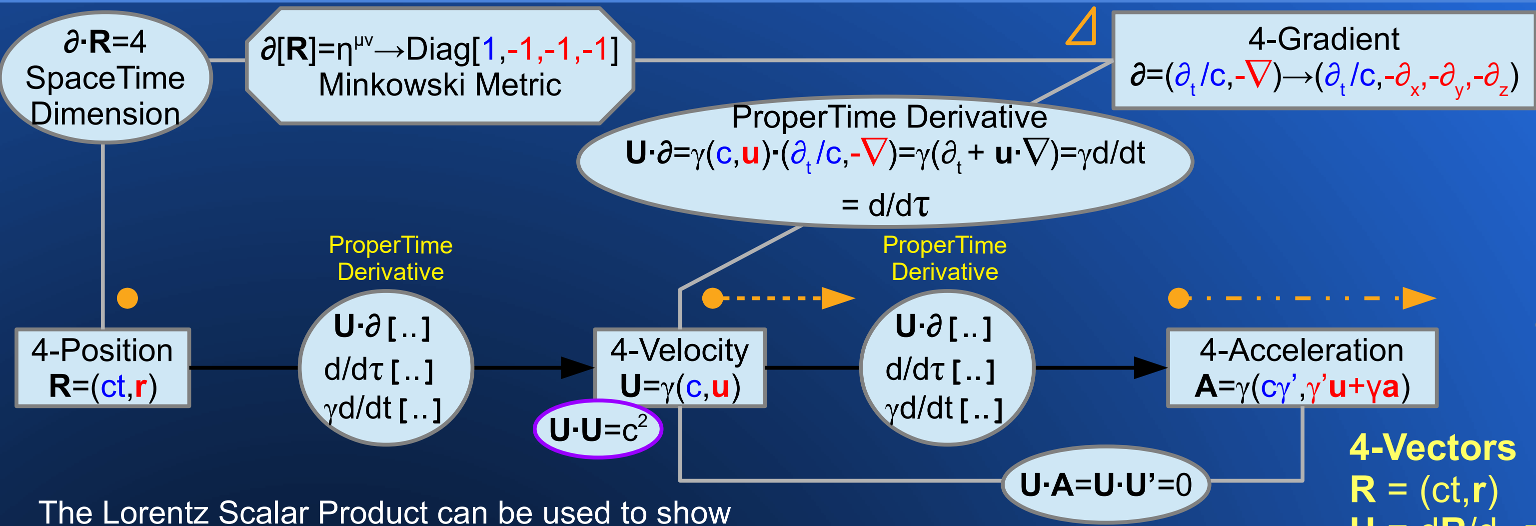
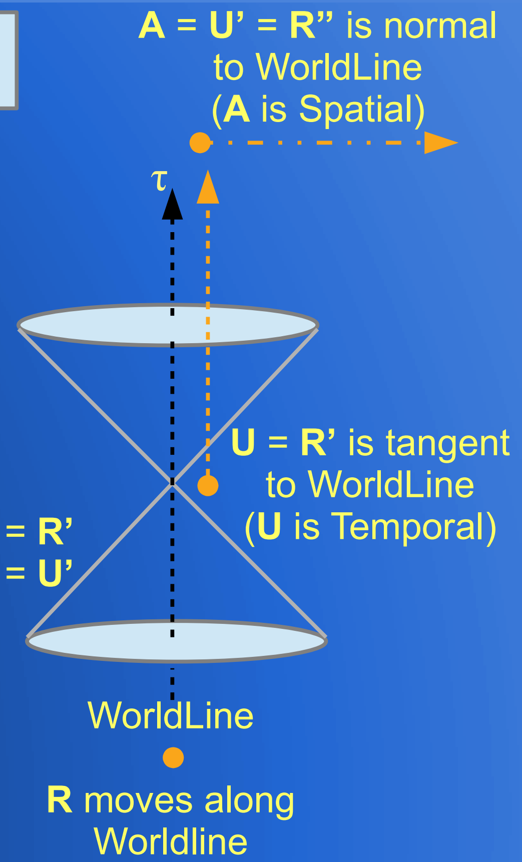
John B. Wilson



4-Velocity **U** (a Temporal 4-Vector) is orthogonal to its own 4-Acceleration **A** = **U'** (a Spatial 4-Vector)

4-Vectors
R = (ct,**r**)
U = d**R**/dτ = **R'**
A = d**U**/dτ = **U'**



The Lorentz Scalar Product can be used to show SpaceTime orthogonality when the result is zero.

$\mathbf{U}\cdot\mathbf{U} = c^2$

$d/d\tau[\mathbf{U}\cdot\mathbf{U}] = d/d\tau[c^2] = 0$

$d/d\tau[\mathbf{U}\cdot\mathbf{U}] = d/d\tau[\mathbf{U}]\cdot\mathbf{U} + \mathbf{U}\cdot d/d\tau[\mathbf{U}] = \mathbf{A}\cdot\mathbf{U} + \mathbf{U}\cdot\mathbf{A} = 2(\mathbf{U}\cdot\mathbf{A}) = 0$

U·A = **U·U'** = 0: The 4-Velocity is SpaceTime orthogonal to it's 4-Acceleration.

4-Velocity is the direction along a WorldLine.
4-Acceleration is the thing which causes a WorldLine to bend/curve.

SR → QM

SRQM: Some Basic 4-Vectors
4-Displacement, 4-Velocity,
Relativity of Simultaneity

4-Vector SRQM Interpretation of QM

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$\partial \cdot \mathbf{X}=4$ SpaceTime Dimension

$\partial_\nu[X^{\mu'}]=\partial X^{\mu'}/\partial X^\nu=\Lambda^{\mu'}{}_\nu$ Lorentz Transform

$\partial[\mathbf{X}]=\eta^{\mu\nu}\rightarrow \mathrm{Diag}[1,-1,-1,-1]$ Minkowski Metric

4-Gradient $\partial=(\partial_t/c,-\nabla)$

4-Displacement $\Delta\mathbf{X}=(c\Delta t,\Delta\mathbf{x})$ $d\mathbf{X}=(cdt,d\mathbf{x})$

4-Position $\mathbf{X}=(ct,\mathbf{x})$

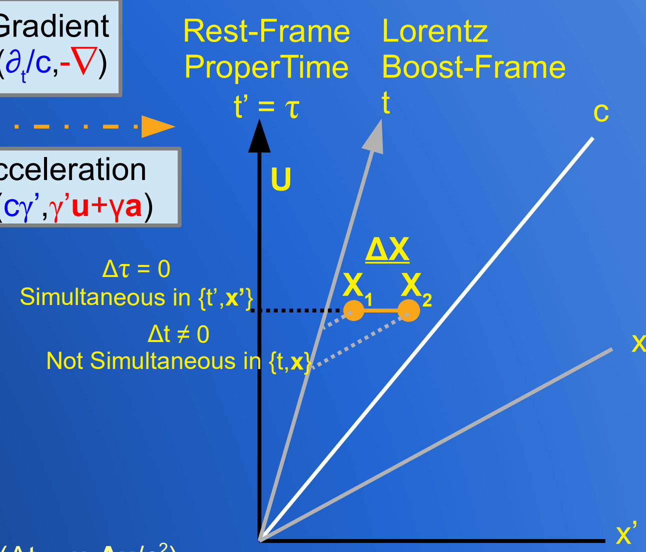
$\mathbf{U}\cdot\partial[..]$ $\gamma d/dt[..]$ $d/d\tau[..]$ ProperTime Derivative

4-Velocity $\mathbf{U}=\gamma(c,\mathbf{u})$

$\mathbf{U}\cdot\partial[..]$ $\gamma d/dt[..]$ $d/d\tau[..]$ ProperTime Derivative

4-Acceleration $\mathbf{A}=\gamma(c\gamma',\gamma'\mathbf{u}+\gamma\mathbf{a})$

$\mathbf{U}\cdot\Delta\mathbf{X} = \gamma(c,\mathbf{u})\cdot(c\Delta t,\Delta\mathbf{x}) = \gamma(c^2\Delta t - \mathbf{u}\cdot\Delta\mathbf{x}) = c^2\Delta t_o = c^2\Delta\tau$



If Lorentz Scalar ($\mathbf{U}\cdot\Delta\mathbf{X} = 0 = c^2\Delta\tau$), then the ProperTime displacement ($\Delta\tau$) is zero, and the event separation ($\Delta\mathbf{X} = \mathbf{X}_2 - \mathbf{X}_1$) is orthogonal to the worldline $\mathbf{U}$.
$\mathbf{X}_1$ and $\mathbf{X}_2$ are therefore simultaneous for the observer on this worldline $\mathbf{U}$.

Examining the equation we get $\gamma(c^2\Delta t - \mathbf{u}\cdot\Delta\mathbf{x}) = 0$. The coordinate time difference between the events is ($\Delta t = \mathbf{u}\cdot\Delta\mathbf{x}/c^2$)
The condition for simultaneity in an alternate frame (moving at 3-velocity **u** wrt. the worldline **U**) is $\Delta t = 0$, which implies ($\mathbf{u}\cdot\Delta\mathbf{x}$) = 0.

This can be met by:
($|\mathbf{u}| = 0$), the alternate observer is not moving wrt. the events, i.e. is on worldline **U** or on a worldline parallel to **U**.
($|\Delta\mathbf{x}| = 0$), the events are at the same spatial location (co-local).
($\mathbf{u}\cdot\Delta\mathbf{x} = 0$), the alternate observer's motion is perpendicular (orthogonal) to the spatial separation $\Delta\mathbf{x}$ of the events in that frame.

If none of these conditions is met, then the events will not be simultaneous in the alternate reference frame.
This is the mathematics behind the concept of Relativity of Simultaneity.

SR 4-Tensor (2,0)-Tensor $T^{\mu\nu}$ (1,1)-Tensor $T^\mu{}_\nu$ or $T_\mu{}^\nu$ (0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector (1,0)-Tensor $V^\mu = \mathbf{V} = (v^0,\mathbf{v})$ **SR 4-CoVector** (0,1)-Tensor $V_\mu = (v_0,-\mathbf{v})$

SR 4-Scalar (0,0)-Tensor S Lorentz Scalar

$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu{}_\mu = T$
$\mathbf{V}\cdot\mathbf{V} = V^\mu\eta_{\mu\nu}V^\nu = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$ = Lorentz Scalar

SR → QM

4-Vector SRQM Interpretation of QM

SR Diagram: SR Motion * Lorentz Scalar = Interesting Physical 4-Vector

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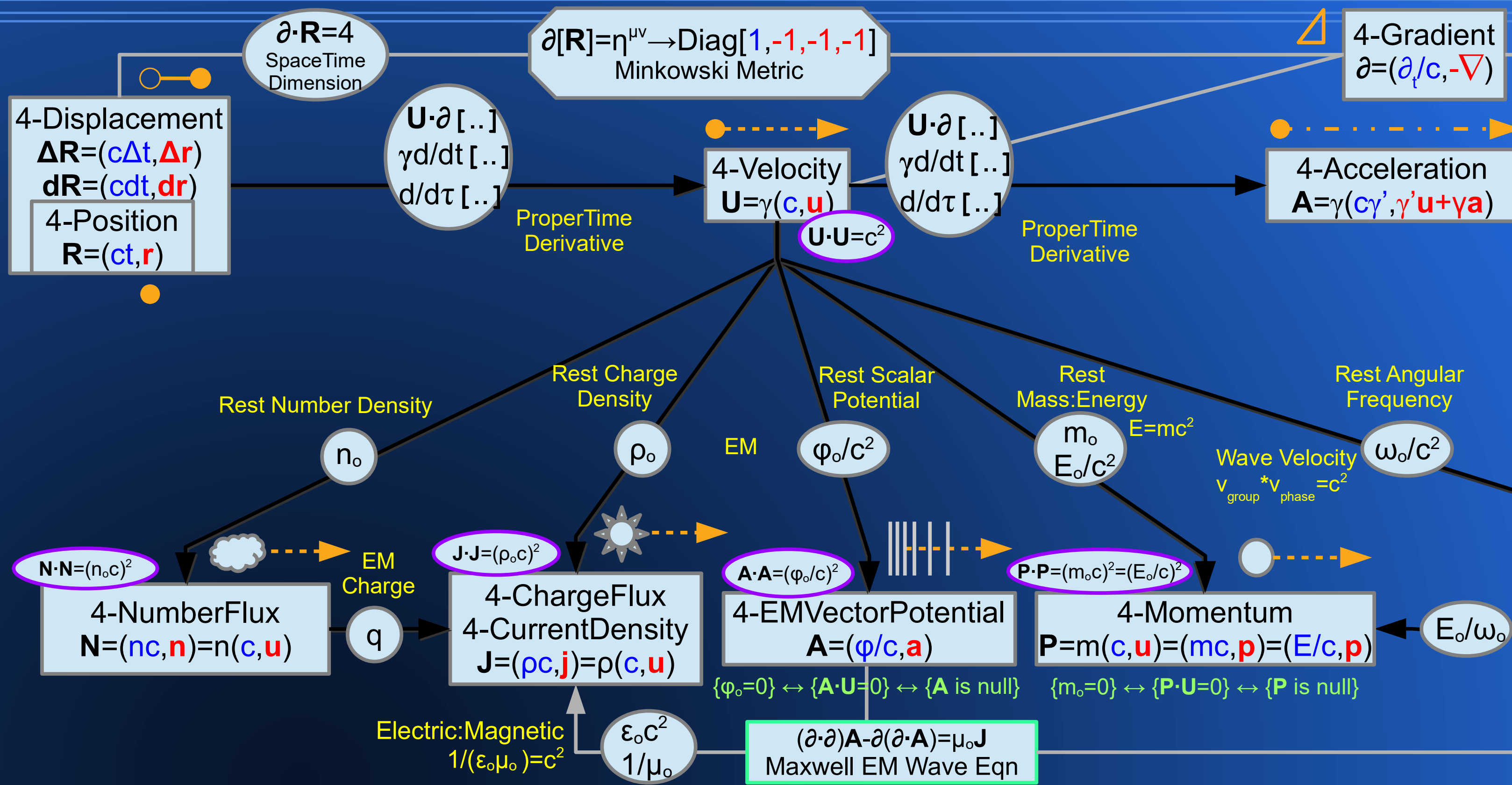

Interesting note:

Most 4-Vectors have 4 independent components. (1 temporal, 3 spatial)

The 4-Velocity has only the 3 spatial however, due to its invariant magnitude $\mathbf{U}\cdot\mathbf{U}=c^2$.

This fact allows one to multiply it by a Lorentz Scalar to make a new 4-Vector with 4 independent components, as shown in the diagram.

Proof of non-varying (c).

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(2,0)-Tensor $T^{\mu\nu}$
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SRQM Diagram:
SRQM Motion * Lorentz Scalar
= Interesting Physical 4-Vector

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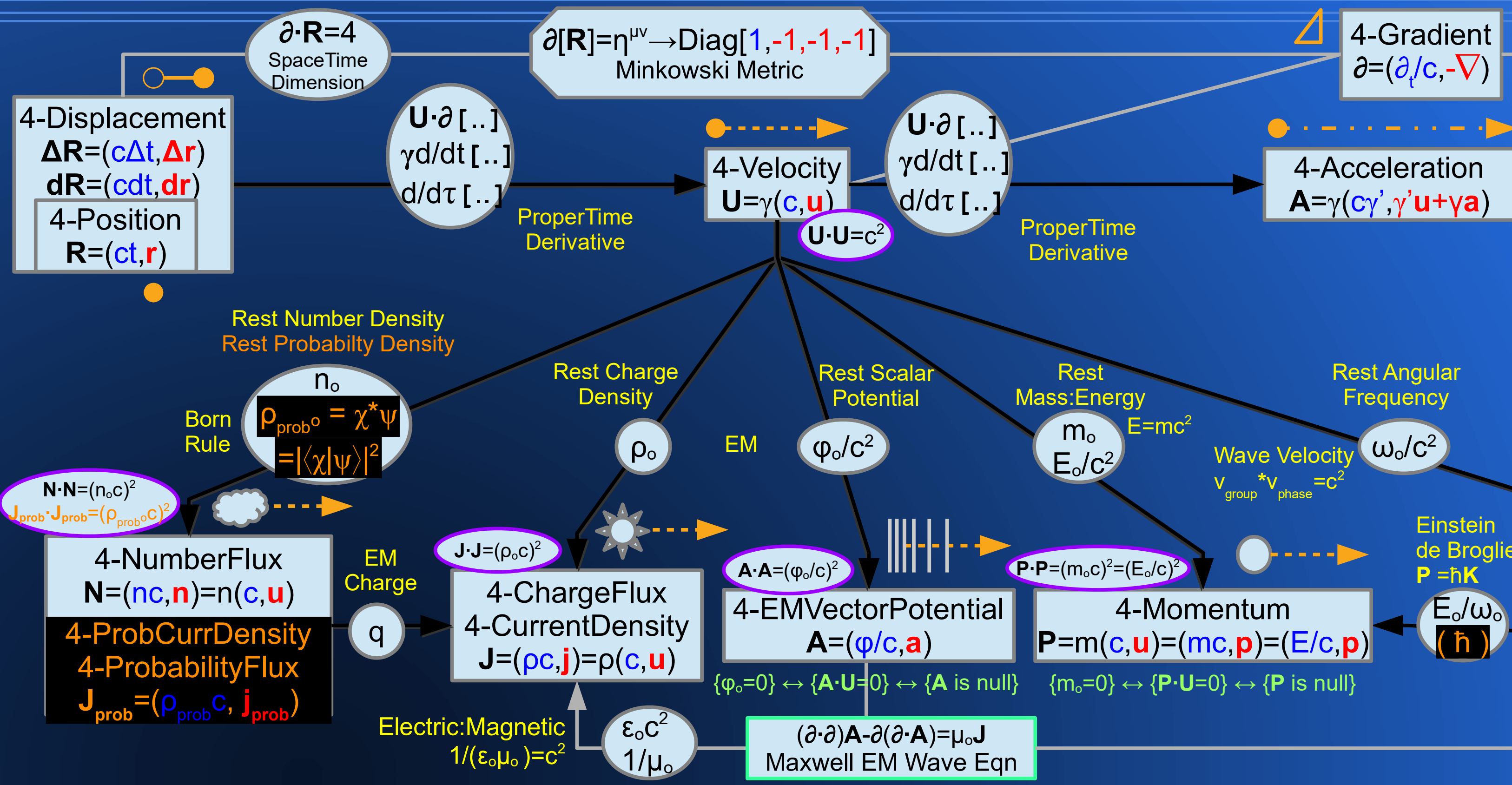

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Existing SR Rules
Quantum Principles

Trace[$T^{\mu\nu}$] = $\eta_{\mu\nu}T^{\mu\nu}$ = $T^{\mu}{}_{\mu}$ = T
V·V = $V^{\mu}\eta_{\mu\nu}V^{\nu}$ = [$(v^0)^2$ - **v·v**] = $(v^0{}_o)^2$
= Lorentz Scalar

SR → QM

4-Vector SRQM Interpretation of QM

SRQM Diagram:
ProperTime Derivative Very Fundamental Results

A Tensor Study of Physical 4-Vectors

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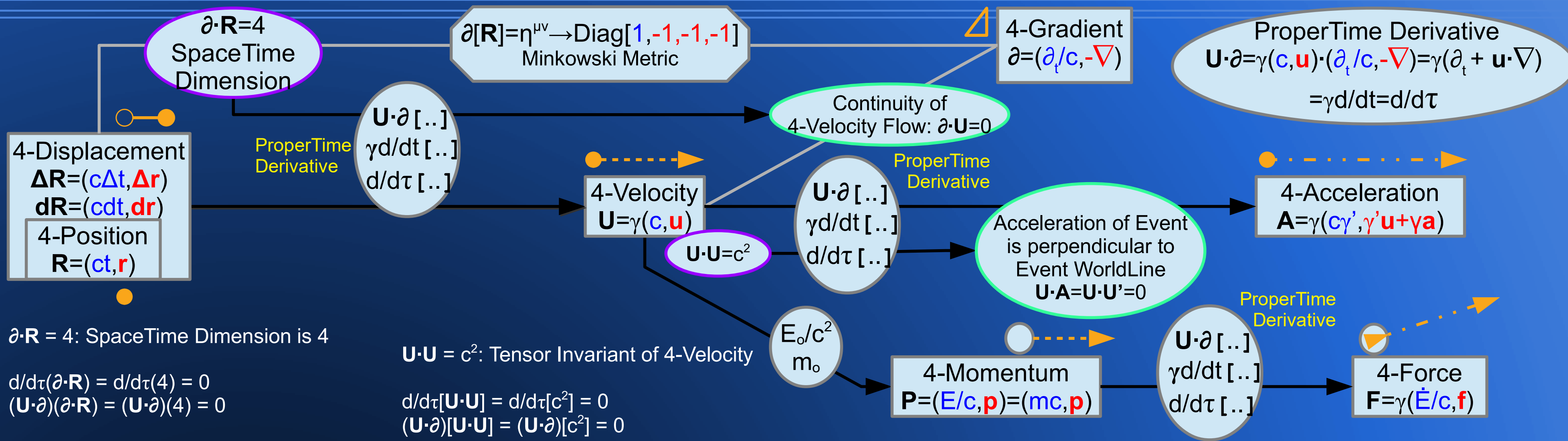

$\partial\cdot\mathbf{R} = 4$: SpaceTime Dimension is 4

$d/d\tau(\partial\cdot\mathbf{R}) = d/d\tau(4) = 0$
$(\mathbf{U}\cdot\partial)(\partial\cdot\mathbf{R}) = (\mathbf{U}\cdot\partial)(4) = 0$

$d/d\tau(\partial\cdot\mathbf{R}) = d/d\tau(\partial)\cdot\mathbf{R} + \partial\cdot d/d\tau(\mathbf{R}) = 0$
$d/d\tau(\partial\cdot\mathbf{R}) = d/d\tau[\partial]\cdot\mathbf{R} + \partial\cdot\mathbf{U} = 0$
$\partial\cdot\mathbf{U} = -d/d\tau[\partial]\cdot\mathbf{R}$
$\partial\cdot\mathbf{U} = -(\mathbf{U}\cdot\partial)[\partial]\cdot\mathbf{R}$
$\partial\cdot\mathbf{U} = -(U_\nu\partial^\nu)[\partial_\mu]R^\mu$
$\partial\cdot\mathbf{U} = -U_\nu\partial^\nu\partial_\mu R^\mu$
$\partial\cdot\mathbf{U} = -U_\nu\partial_\mu\partial^\nu R^\mu$
$\partial\cdot\mathbf{U} = -U_\nu\partial_\mu\eta^{\nu\mu}$
$\partial\cdot\mathbf{U} = -U_\nu(0^\nu)$
$\partial\cdot\mathbf{U} = 0$: Conservation of the 4-Velocity Flow (4-Velocity Flow-Field)

$\mathbf{U}\cdot\mathbf{U} = c^2$: Tensor Invariant of 4-Velocity

$d/d\tau[\mathbf{U}\cdot\mathbf{U}] = d/d\tau[c^2] = 0$
$(\mathbf{U}\cdot\partial)[\mathbf{U}\cdot\mathbf{U}] = (\mathbf{U}\cdot\partial)[c^2] = 0$

$d/d\tau[\mathbf{U}\cdot\mathbf{U}] = d/d\tau[\mathbf{U}]\cdot\mathbf{U} + \mathbf{U}\cdot d/d\tau[\mathbf{U}] = \mathbf{A}\cdot\mathbf{U} + \mathbf{U}\cdot\mathbf{A} = 2(\mathbf{U}\cdot\mathbf{A}) = 0$
$\mathbf{U}\cdot\mathbf{A} = \mathbf{U}\cdot\mathbf{U}' = 0$: The 4-Velocity is SpaceTime orthogonal to it's 4-Acceleration.

4-Velocity is the direction of an Event along a WorldLine.
4-Acceleration of an Event is the thing which causes a WorldLine to bend.

4-Vectors:
$\mathbf{R}$ = <Event>
$\mathbf{U} = d\mathbf{R}/d\tau$
$\mathbf{A} = d\mathbf{U}/d\tau$

$\mathbf{P} = m_o\mathbf{U}$
$\mathbf{F} = d\mathbf{P}/d\tau$

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= Lorentz Scalar

SR → QM

SRQM Diagram:

Local Continuity of 4-Velocity leads to all the Conservation Laws

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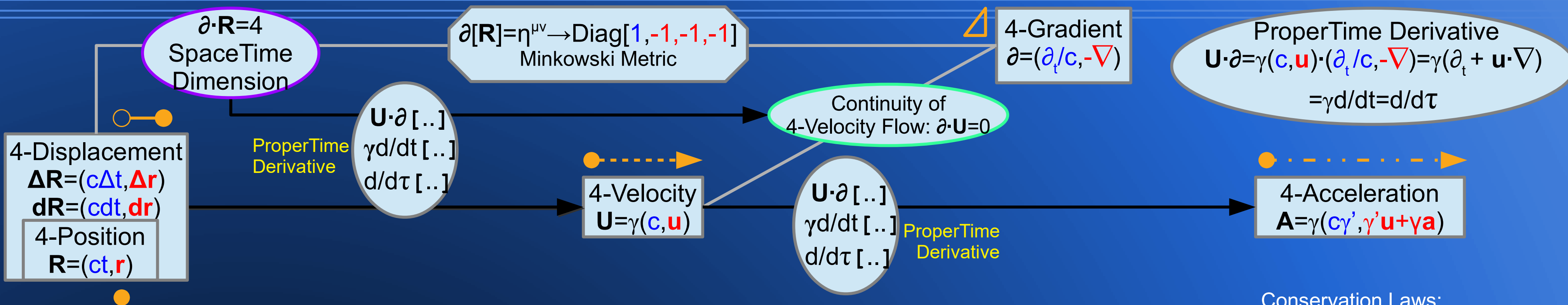

$\partial\cdot\mathbf{R} = 4$
$d/d\tau(\partial\cdot\mathbf{R}) = d/d\tau(4) = 0$

$d/d\tau(\partial\cdot\mathbf{R}) = d/d\tau(\partial)\cdot\mathbf{R} + \partial\cdot d/d\tau(\mathbf{R}) = 0$
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$\partial\cdot\mathbf{U} = -d/d\tau[\partial]\cdot\mathbf{R}$
$\partial\cdot\mathbf{U} = -(\mathbf{U}\cdot\partial)[\partial]\cdot\mathbf{R}$
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$\partial\cdot\mathbf{U} = -U_\nu\partial^\nu\partial_\mu R^\mu$
$\partial\cdot\mathbf{U} = -U_\nu\partial_\mu\partial^\nu R^\mu$: I believe this is legit, partials commute
$\partial\cdot\mathbf{U} = -U_\nu\partial_\mu\eta^{\nu\mu}$
$\partial\cdot\mathbf{U} = -U_\nu(0^\nu)$
$\partial\cdot\mathbf{U} = 0$
Conservation of the 4-Velocity Flow
(4-Velocity Flow-Field)

$\partial\cdot\mathbf{U} = 0$
$\partial\cdot$(Lorentz Scalar)$\mathbf{U}$ = 0(Lorentz Scalar)
$\partial\cdot$(Lorentz Scalar)$\mathbf{U}$ = 0
$\partial\cdot$(**Interesting 4-Vector**) = 0

Example:
$\partial\cdot(\rho_o)\mathbf{U} = 0$
$\partial\cdot\mathbf{J} = 0$
$(\partial_t/c\ \rho c + \nabla\cdot\mathbf{j}) = 0$
$(\partial_t\rho + \nabla\cdot\mathbf{j}) = 0$
= Conservation of Charge
= A Continuity Equation

Conservation Laws:

All of the Physical Conservation Laws are in the form of a 4-Divergence, which is a Lorentz Invariant Scalar equation.

These are local continuity equations which basically say that the temporal change in a quantity is balanced by the flow of that quantity into or out of a local spatial region.

Conservation of Charge:
$\partial\cdot\mathbf{J} = (\partial_t\rho + \nabla\cdot\mathbf{j}) = 0$

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SRQM Diagram: SRQM Motion * Lorentz Scalar
Conservation Laws, Continuity Eqns

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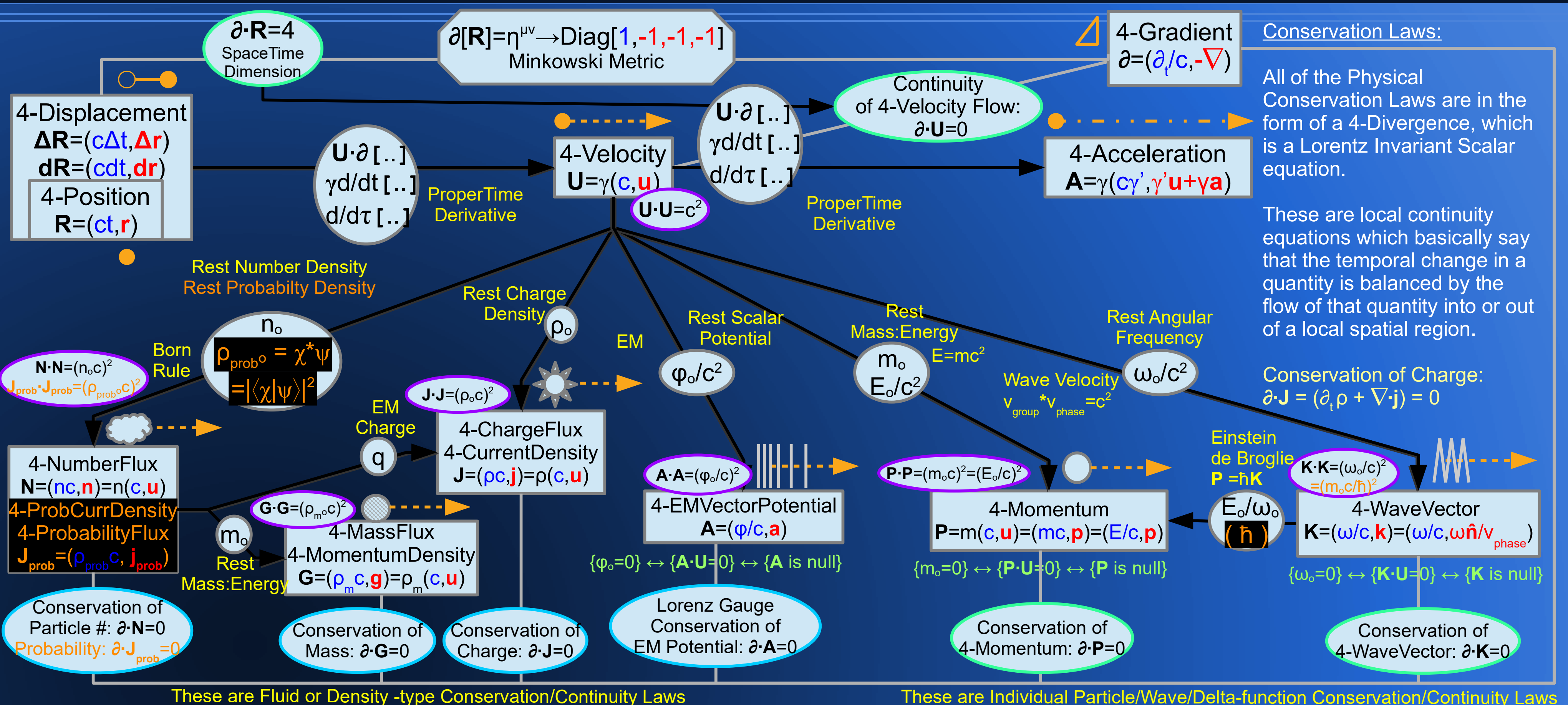

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SRQM: Some Basic 4-Vectors
4-Velocity, 4-Gradient, Time Dilation

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at-rest
worldline $\mathbf{U_o}$
(u=0)
fully temporal

const inertial motion
worldline $\mathbf{U}$
(0<u<c)
trades some time for space

4-Gradient
$\partial=(\partial_t/c,-\nabla)$

ProperTime
$\mathbf{U}\cdot\partial=d/d\tau=\gamma d/dt$
Derivative

4-Velocity
$\mathbf{U}=\gamma(c,\mathbf{u})$

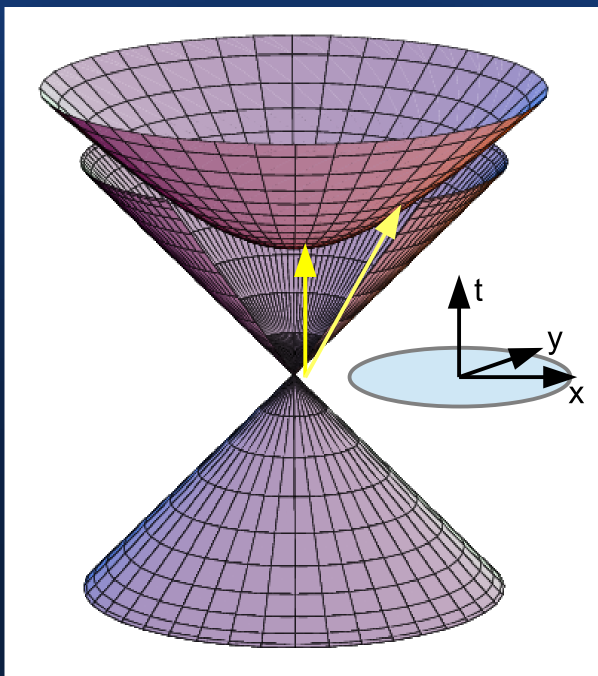
$\mathbf{U}\cdot\mathbf{U} = \gamma(c,\mathbf{u})\cdot\gamma(c,\mathbf{u}) = \gamma^2(c^2-\mathbf{u}\cdot\mathbf{u}) = (c^2)$

$\gamma = 1/\sqrt{[1-(u/c)^2]} = 1/\sqrt{[1-\beta^2]}$

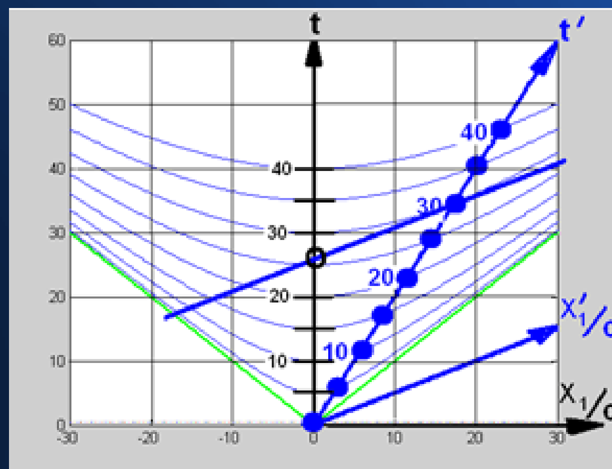
ProperTime
$d\tau=(1/\gamma)dt$
Differential

4-Velocity$_{\text{(at-rest)}}$
$\mathbf{U_o}=(c,\mathbf{0})$

Everything moves into future (+t) at the speed-of-light (c) in its own spatial rest-frame



The Minkowski Diagram provides a great visual representation of SpaceTime



Since the SpaceTime magnitude of **U** is a constant, changes in the components of **U** are like “rotating” the 4-Vector without changing its length. However, as **U** gains some spatial velocity, it loses some “relative” temporal velocity. Objects that move in some reference frame “age” more slowly relative to those at rest in the same reference frame.

Time Dilation!

$\Delta t = \gamma\Delta\tau = \gamma\Delta t_o$

$dt = \gamma d\tau$

$d/d\tau = \gamma d/dt$

Each observer will see the other as aging more slowly; similarly to two people moving oppositely along a train track, seeing the other as appearing smaller in the distance.

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SRQM: Some Basic 4-Vectors
SR 4-WaveVector K

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4-WaveVector, aka. Wave 4-Vector, solution of d'Alembertian Wave Eqn.
$\mathbf{K} = (\omega/c,\mathbf{k}) = (\omega/c,\omega\hat{\mathbf{n}}/v_{phase}) = (\omega/c,\omega\mathbf{u}/c^2) = (\omega/c^2)(c,\mathbf{u}) = (\omega/c)(1,\boldsymbol{\beta}) = (1/c\text{Ŧ},\hat{\mathbf{n}}/\text{ƛ}) = -\partial[\Phi_{phase,plane}]$

There are multiple ways of writing out the components of the 4-WaveVector, with each one giving an interesting take on what the 4-WaveVector means.

$\psi_n(\mathbf{X}) = A_n\, e^{\wedge}\text{-}i(\mathbf{K}_n\cdot\mathbf{X})$: Explicit form of an SR plane wave
$\psi(\mathbf{X}) = \Sigma_n[\ \psi_n(\mathbf{X})\ ]$: Complete wave is a superposition of multiple plane waves.
$\partial[\ \psi(\mathbf{X})\ ] = \partial[\ Ae^{\wedge}\text{-}i(\mathbf{K}\cdot\mathbf{X})\ ] = -i\mathbf{K}\ [\ Ae^{\wedge}\text{-}i(\mathbf{K}\cdot\mathbf{X})\ ] = -i\mathbf{K}[\ \psi(\mathbf{X})\ ]$
$\partial = -i\mathbf{K}$ as the condition for a complex-valued plane wave.

An SR wave Ψ is actually composed of two tensors:
(1) 4-Vector propagation part = K^α, (the engine)
(2) Variable amplitude part = A (the load), depends on what is waving...

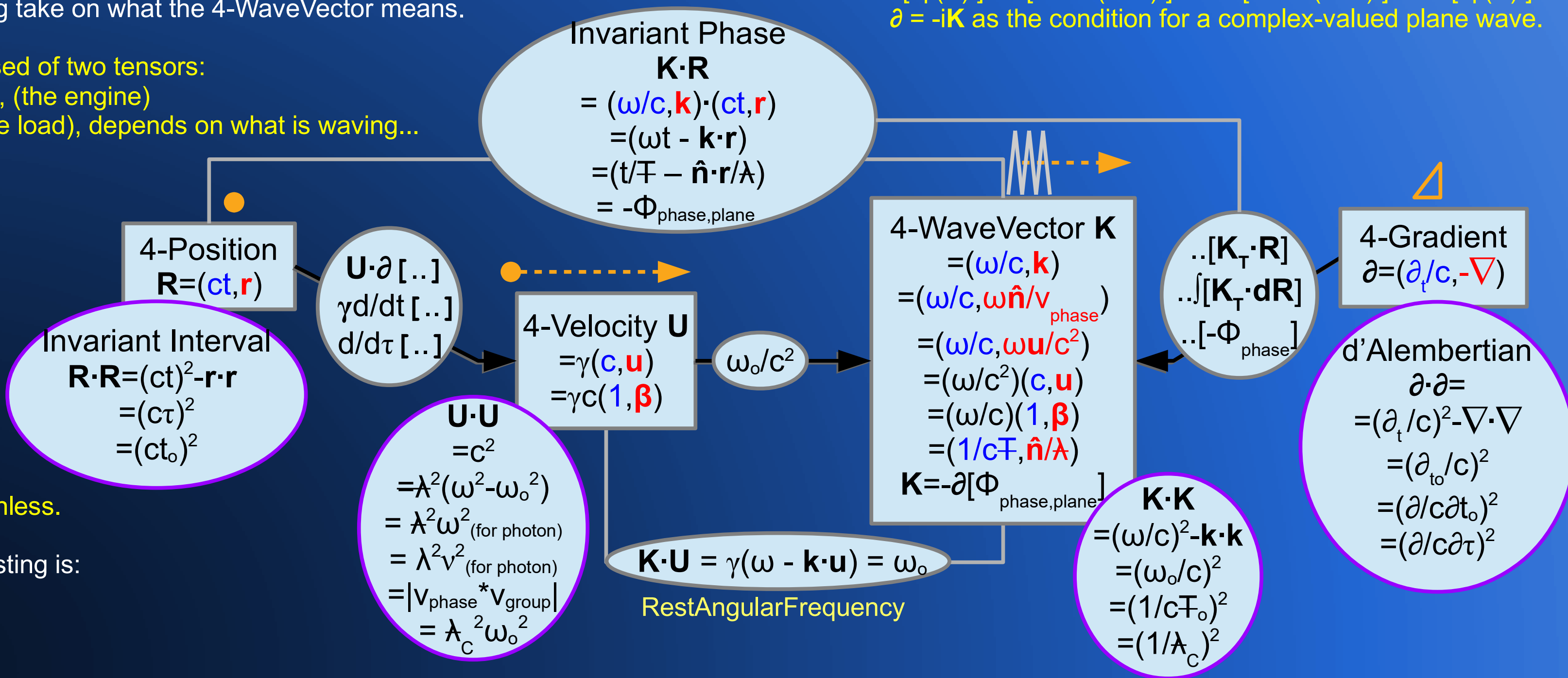
4-Scalar A: $\Psi = A\, e^{\wedge}(-iK^\alpha X_\alpha)$
ex. KG Quantum Wave

4-Vector A^μ: $\Psi^\mu = A^\mu\, e^{\wedge}(-iK^\alpha X_\alpha)$
ex. Maxwell Photon Wave

4-Tensor $A^{\mu\nu}$: $\Psi^{\mu\nu} = A^{\mu\nu}\, e^{\wedge}(-iK^\alpha X_\alpha)$
ex. Gravitational Wave Approx.

The Ψ tensor-type will match the A tensor-type, as the propagation part $e^{\wedge}(-iK^\alpha X_\alpha)$ is overall dimensionless.

One comparison I find very interesting is:
$\mathbf{R}\cdot\mathbf{R} = (ct_o)^2 = (c\tau)^2$
$\mathbf{K}\cdot\mathbf{K} = (1/c\text{Ŧ}_o)^2$
$\partial\cdot\partial = (\partial/c\partial t_o)^2 = (\partial/c\partial\tau)^2$



I believe the last one is correct: $(\partial\cdot\partial)[\mathbf{R}] = \mathbf{0} = (\partial/c\partial\tau)^2[\mathbf{R}] = \mathbf{A}_o/c^2 = \mathbf{0}$: The 4-Acceleration seen in the ProperTime Frame = RestFrame = **0**
Normally $(d/d\tau)^2[\mathbf{R}] = \mathbf{A}$, which could be non-zero. But that is for the total derivative, not the partial derivative.

SRQM: Some Basic 4-Vectors 4-Velocity, 4-WaveVector Wave Properties, Relativistic Doppler Effect

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$\mathbf{K}\cdot\mathbf{U} = \gamma(\omega - \mathbf{k}\cdot\mathbf{u}) = \omega_o$

4-Velocity $\mathbf{U}=\gamma(c,\mathbf{u})$ $\xrightarrow{\omega_o/c^2}$ 4-WaveVector $\mathbf{K}=(\omega/c,\mathbf{k})=(\omega/c,\omega\hat{\mathbf{n}}/v_{phase})$

RestAngularFrequency

$\mathbf{U}\cdot\mathbf{U} = (c)^2$

$\mathbf{K}\cdot\mathbf{K} = (\omega_o/c)^2$

$\mathbf{K} = (\omega/c,\mathbf{k}) = (\omega/c,\omega\hat{\mathbf{n}}/v_{phase}) = (\omega_o/c^2)\mathbf{U}$
$= (\omega_o/c^2)\gamma(c,\mathbf{u}) = (\omega/c^2)(c,\mathbf{u}) = (\omega/c,(\omega/c^2)\mathbf{u})$

$(\omega/c,\omega\hat{\mathbf{n}}/v_{phase}) = (\omega/c,(\omega/c^2)\mathbf{u})$
Taking just the spatial components of the 4-WaveVector:
$\omega\mathbf{n}/v_{phase} = (\omega/c^2)\mathbf{u}$
$\hat{\mathbf{n}}/v_{phase} = (\mathbf{u}/c^2)$

$u * v_{phase} = c^2$
$v_{group} * v_{phase} = c^2$, with $u = v_{group}$

Wave Group velocity (v_{group}) is mathematically the same as Particle velocity (u).
Wave Phase velocity (v_{phase}) is the speed of an individual plane-wave.

Relativistic SR Doppler Effect

($\hat{\mathbf{n}}$) here is the unit-directional 3-vector of the photon

Choose an observer frame for which:
$\mathbf{K} = (\omega/c,\mathbf{k})$, with $\mathbf{k},\hat{\mathbf{n}}$ pointing toward observer
$\mathbf{U}_{obs} = (c,\mathbf{0})$ $\quad$ $\mathbf{K}\cdot\mathbf{U}_{obs} = (\omega/c,\mathbf{k})\cdot(c,\mathbf{0}) = \omega = \omega_{obs^o}$
$\mathbf{U}_{emit} = \gamma(c,\mathbf{u})$ $\quad$ $\mathbf{K}\cdot\mathbf{U}_{emit} = (\omega/c,\mathbf{k})\cdot\gamma(c,\mathbf{u}) = \gamma(\omega - \mathbf{k}\cdot\mathbf{u}) = \omega_{emit^o}$

$\mathbf{K}\cdot\mathbf{U}_{obs} / \mathbf{K}\cdot\mathbf{U}_{emit} = \omega_{obs^o}/\omega_{emit^o} = \omega/[\gamma(\omega - \mathbf{k}\cdot\mathbf{u})]$
For photons, $\mathbf{K}$ is null → $\mathbf{K}\cdot\mathbf{K} = 0$ → $\mathbf{k} = (\omega/c)\hat{\mathbf{n}}$
$\omega_{obs^o}/\omega_{emit^o} = \omega/[\gamma(\omega - (\omega/c)\hat{\mathbf{n}}\cdot\mathbf{u})] = 1/[\gamma(1 - \hat{\mathbf{n}}\cdot\boldsymbol{\beta})] = 1/[\gamma(1 - |\boldsymbol{\beta}|\cos[\theta_{obs}])]$
$\omega_{obs}/\omega_{emit} = \gamma\omega_{obs^o}/(\gamma\omega_{emit^o}) = \omega_{obs^o}/\omega_{emit^o}$

$\omega_{obs} = \omega_{emit}/[\gamma(1 - \hat{\mathbf{n}}\cdot\boldsymbol{\beta})] = \omega_{emit}*\sqrt{[1+|\beta|]}*\sqrt{[1-|\beta|]}/(1 - \hat{\mathbf{n}}\cdot\boldsymbol{\beta})$
with $\gamma = 1/\sqrt{[1-\beta^2]} = 1/(\sqrt{[1+|\beta|]}*\sqrt{[1-|\beta|]})$

For motion of emitter $\boldsymbol{\beta}$: (in observer frame of reference)
Away from obs, $(\hat{\mathbf{n}}\cdot\boldsymbol{\beta}) = -\beta$, $\omega_{obs} = \omega_{emit}*\sqrt{[1-|\beta|]}/\sqrt{(1 + |\beta|)}$ = Red Shift
Toward obs, $(\hat{\mathbf{n}}\cdot\boldsymbol{\beta}) = +\beta$, $\omega_{obs} = \omega_{emit}*\sqrt{[1+|\beta|]}/\sqrt{(1 - |\beta|)}$ = Blue Shift
Transverse, $(\hat{\mathbf{n}}\cdot\boldsymbol{\beta}) = 0$, $\omega_{obs} = \omega_{emit}/\gamma$ = Transverse Doppler Shift

The Phase Velocity of a Photon $\{v_{phase} = c\}$ equals the Particle Velocity of a Photon $\{u = c\}$
The Phase Velocity of a Massive Particle $\{v_{phase} > c\}$ is greater than the Velocity of a Massive Particle $\{u < c\}$

SRQM: Some Basic 4-Vectors 4-Velocity, 4-WaveVector Wave Properties, Relativistic Aberration

$\mathbf{K}\cdot\mathbf{U} = \gamma(\omega - \mathbf{k}\cdot\mathbf{u}) = \omega_o$

4-Velocity $\mathbf{U}=\gamma(c,\mathbf{u})$ $\xrightarrow{\omega_o/c^2}$ 4-WaveVector $\mathbf{K}=(\omega/c,\mathbf{k})=(\omega/c,\omega\hat{\mathbf{n}}/v_{phase})$

RestAngularFrequency

$\mathbf{U}\cdot\mathbf{U} = (c)^2$ $\mathbf{K}\cdot\mathbf{K} = (\omega_o/c)^2$

$\mathbf{K} = (\omega/c,\mathbf{k}) = (\omega/c,\omega\hat{\mathbf{n}}/v_{phase}) = (\omega_o/c^2)\mathbf{U}$
$= (\omega_o/c^2)\gamma(c,\mathbf{u}) = (\omega/c^2)(c,\mathbf{u}) = (\omega/c,(\omega/c^2)\mathbf{u})$

$(\omega/c,\omega\hat{\mathbf{n}}/v_{phase}) = (\omega/c,(\omega/c^2)\mathbf{u})$

Taking just the spatial components of the 4-WaveVector:

$\omega\mathbf{n}/v_{phase} = (\omega/c^2)\mathbf{u}$

$\hat{\mathbf{n}}/v_{phase} = (\mathbf{u}/c^2)$

$u * v_{phase} = c^2$

$v_{group} * v_{phase} = c^2$, with $u = v_{group}$

Wave Group velocity (v_{group}) is mathematically the same as Particle velocity (u).
Wave Phase velocity (v_{phase}) is the speed of an individual plane-wave.

Relativistic SR Doppler Effect

($\hat{\mathbf{n}}$) here is the unit-directional 3-vector of the photon

$\omega_{obs} = \omega_{emit}/[\gamma(1 - \hat{\mathbf{n}}\cdot\boldsymbol{\beta})] = \omega_{emit}/[\gamma(1 - |\boldsymbol{\beta}|\cos[\theta_{obs}])]$

Change reference frames with {obs→emit} &{ $\boldsymbol{\beta} \rightarrow -\boldsymbol{\beta}$ }

$\omega_{emit} = \omega_{obs}/[\gamma(1 + \hat{\mathbf{n}}\cdot\boldsymbol{\beta})] = \omega_{obs}/[\gamma(1 + |\boldsymbol{\beta}|\cos[\theta_{emit}])]$

$(\omega_{obs})*(\omega_{emit}) =(\omega_{emit}/[\gamma(1 - |\boldsymbol{\beta}|\cos[\theta_{obs}])])*(\omega_{obs}/[\gamma(1 + |\boldsymbol{\beta}|\cos[\theta_{emit}])])$

$1 = (1/[\gamma(1 - |\boldsymbol{\beta}|\cos[\theta_{obs}])])*(1/[\gamma(1 + |\boldsymbol{\beta}|\cos[\theta_{emit}])])$
$1 = (\gamma(1 - |\boldsymbol{\beta}|\cos[\theta_{obs}]))*(\gamma(1 + |\boldsymbol{\beta}|\cos[\theta_{emit}]))$
$1 = \gamma^2(1 - |\boldsymbol{\beta}|\cos[\theta_{obs}])*(1 + |\boldsymbol{\beta}|\cos[\theta_{emit}])$

Solve for $|\boldsymbol{\beta}|\cos[\theta_{obs}]$ and use $\{(\gamma^2-1) = \beta^2\gamma^2\}$

Relativistic SR Aberration Effect

$\cos[\theta_{obs}] = (\cos[\theta_{emit}] + |\boldsymbol{\beta}|) / (1 + |\boldsymbol{\beta}|\cos[\theta_{emit}])$

The Phase Velocity of a Photon $\{v_{phase} = c\}$ equals the Particle Velocity of a Photon $\{u = c\}$
The Phase Velocity of a Massive Particle $\{v_{phase} > c\}$ is greater than the Velocity of a Massive Particle $\{u < c\}$

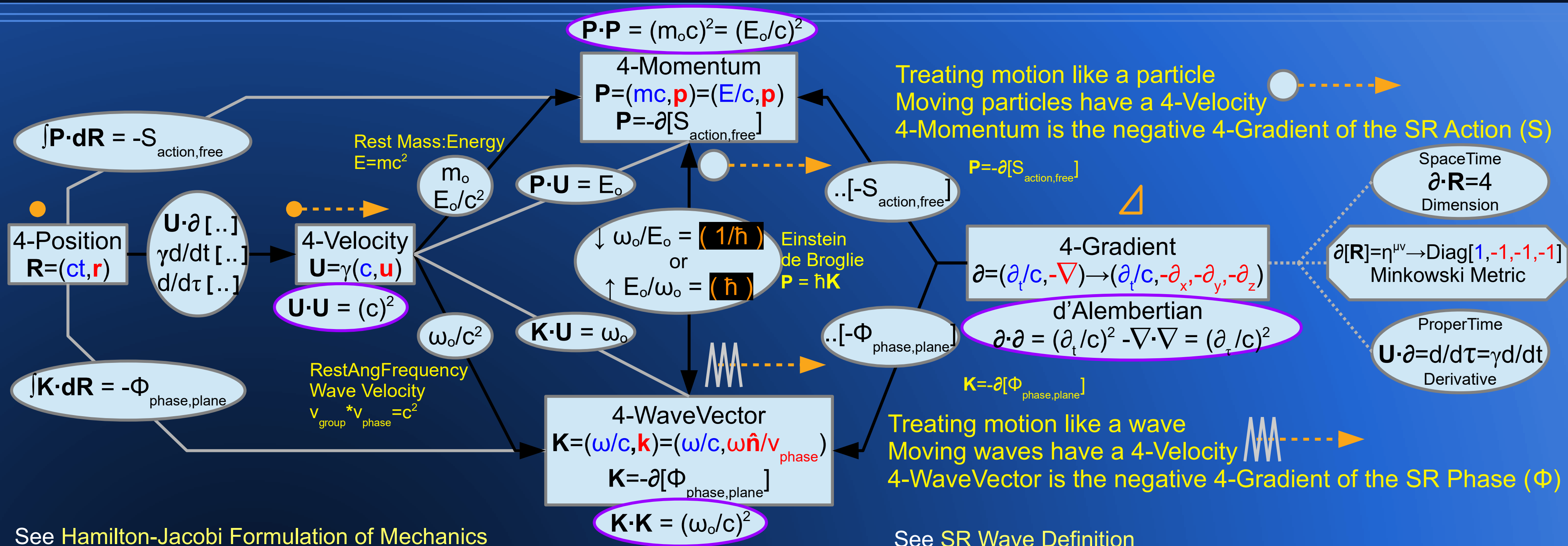
SR → QM

SRQM: Some Basic 4-Vectors 4-Momentum, 4-WaveVector, 4-Position, 4-Velocity, 4-Gradient, Wave-Particle

4-Vector SRQM Interpretation of QM

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson



See Hamilton-Jacobi Formulation of Mechanics
for info on the Lorentz Scalar Invariant SR Action.
{ $\mathbf{P} = (E/c,\mathbf{p}) = -\partial[S] = (-\partial/c\partial t[S],\nabla[S])$ }
{temporal component} $E = -\partial/\partial t[S] = -\partial_t[S]$
{spatial component} $\mathbf{p} = \nabla[S]$
Note This is the Action (S_{action}) for a free particle.
Generally Action is for the 4-TotalMomentum $\mathbf{P_T}$ of a system.

See SR Wave Definition
for info on the Lorentz Scalar Invariant SR WavePhase.
{ $\mathbf{K} = (\omega/c,\mathbf{k}) = -\partial[\Phi] = (-\partial/c\partial t[\Phi],\nabla[\Phi])$ }
{temporal component} $\omega = -\partial/\partial t[\Phi] = -\partial_t[\Phi]$
{spatial component} $\mathbf{k} = \nabla[\Phi]$
Note This is the Phase (Φ) for a single plane-wave.
Generally WavePhase is for the 4-TotalWaveVector $\mathbf{K_T}$ of a system.

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^{\mu}{}_{\nu}$ or $T_{\mu}{}^{\nu}$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0,\mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0,-\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Existing SR Rules
Quantum Principles

$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^{\mu}{}_{\mu} = T$
$\mathbf{V}\cdot\mathbf{V} = V^\mu\eta_{\mu\nu}V^\nu = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$
= Lorentz Scalar

Some Cool Minkowski Metric Tensor Tricks
4-Gradient, 4-Position, 4-Velocity
SpaceTime is 4D

A Tensor Study of Physical 4-Vectors

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ProperTime Derivative
$\mathbf{U}\cdot\partial = \gamma(c,\mathbf{u})\cdot(\partial_t/c,-\nabla) = \gamma(\partial_t + \mathbf{u}\cdot\nabla)$
$= d/d\tau = \gamma d/dt$

4-Gradient
$\partial = (\partial_t/c,-\nabla) \to (\partial_t/c,-\partial_x,-\partial_y,-\partial_z)$

$\partial[\mathbf{R}] = \eta^{\mu\nu}$
$\to$ Diag[1,-1,-1,-1]
Minkowski Metric

$\partial\cdot\mathbf{R}=4$
SpaceTime Dimension

$\mathbf{U}\cdot\partial$ [..]
$d/d\tau$ [..]
$\gamma d/dt$ [..]

4-Velocity
$\mathbf{U}=\gamma(c,\mathbf{u})$

4-Position
$\mathbf{R}=(ct,\mathbf{r})$

=

$Tr[\eta^{\mu\nu}]=\eta_\mu{}^\mu=4$
SpaceTime Dimension

Index Raise

$\partial[\mathbf{R}] = \eta^{\mu\nu}$
$\to$ Diag[1,-1,-1,-1]
Index-Raised Minkowski Metric

$\eta_\mu{}^\nu$
$\to$ Diag[1,1,1,1]
Index-Mixed Minkowski Metric

=

$\delta_\mu{}^\nu$
=Diag[1,1,1,1]
Kronecker Delta

$\{\eta^{\mu\mu}\} = 1/\{\eta_{\mu\mu}\}$

Index Lower

$\eta_{\mu\nu}$
$\to$ Diag[1,-1,-1,-1]
Index-Lowered Minkowski Metric

$\eta_{\alpha\beta}(\eta^{\gamma\beta}) = \eta_\alpha{}^\gamma$ = Diag[1,-1,-1,-1]*Diag[1,-1,-1,-1] = Diag[1,1,1,1]
thus
Single Index-Lowering the Minkowski Metric ($\eta^{\gamma\beta}$) gives the Kronecker Delta

$(\partial\cdot\mathbf{R}) = (\partial^\alpha\cdot R^\beta) = (\partial^\alpha\eta_{\alpha\beta}R^\beta) = \eta_{\alpha\beta}(\partial^\alpha R^\beta) = \eta_{\alpha\beta}(\eta^{\alpha\beta}) = \eta_\alpha{}^\alpha = \delta_\alpha{}^\alpha = 4$
Trace[Minkowski Metric] = $Tr[\eta^{\alpha\beta}] = \eta_{\alpha\beta}[\eta^{\alpha\beta}] = \eta_\alpha{}^\alpha = \delta_\alpha{}^\alpha = 4$
thus
The Divergence of 4-Position ($\partial\cdot\mathbf{R}$) = “Magnitude” of the Minkowski Metric $Tr[\eta^{\alpha\beta}]$ = the Dimension of SpaceTime (4)

$(\mathbf{U}\cdot\partial)[\mathbf{R}] = (U^\alpha\cdot\partial^\beta)[R^\gamma] = (U^\alpha\eta_{\alpha\beta}\partial^\beta)[R^\gamma] = (U_\beta\partial^\beta)[R^\gamma] = (U_\beta)\partial^\beta[R^\gamma] = (U_\beta)\eta^{\beta\gamma} = U^\gamma = \mathbf{U} = (d/d\tau)[\mathbf{R}]$
thus
Lorentz Scalar Product ($\mathbf{U}\cdot\partial$) = Derivative wrt. ProperTime ($d/d\tau$) = Relativistic Factor * Derivative wrt. CoordinateTime $\gamma(d/dt)$:

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(1,1)-Tensor $T^\mu{}_\nu$ or $T_\mu{}^\nu$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0,\mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0,-\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

$Trace[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu{}_\mu = T$
$\mathbf{V}\cdot\mathbf{V} = V^\mu\eta_{\mu\nu}V^\nu = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$
= Lorentz Scalar

SRQM+EM Diagram: 4-Vectors

A Tensor Study of Physical 4-Vectors

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4-Displacement
$\mathbf{\Delta R}=(c\Delta t,\mathbf{\Delta r})$
$\mathbf{dR}=(cdt,\mathbf{dr})$

4-Position
$\mathbf{R}=(ct,\mathbf{r})$

4-Acceleration
$\mathbf{A}=\gamma(c\gamma',\gamma'\mathbf{u}+\gamma\mathbf{a})$

4-Polarization:Spin
$\mathbf{E}=(\varepsilon^0,\boldsymbol{\varepsilon})=(\boldsymbol{\varepsilon}\cdot\boldsymbol{\beta},\boldsymbol{\varepsilon})$
$\mathbf{S}=(s^0,\mathbf{s})=(\mathbf{s}\cdot\boldsymbol{\beta},\mathbf{s})$
Complex

4-Gradient
$\partial=(\partial_t/c,-\nabla)$

4-WaveVector
$\mathbf{K}=(\omega/c,\mathbf{k})$

4-TotalWaveVector
$\mathbf{K}_T=(\omega_T/c,\mathbf{k}_T)$

4-UnitTemporal
$\mathbf{T}=\gamma(1,\boldsymbol{\beta})$

4-Velocity
$\mathbf{U}=\gamma(c,\mathbf{u})$

4-UnitSpatial
$\mathbf{S}=\gamma_{\beta n}(\hat{\mathbf{n}}\cdot\boldsymbol{\beta},\hat{\mathbf{n}})_\perp$

4-Momentum
$\mathbf{P}=(mc,\mathbf{p})=(E/c,\mathbf{p})$

4-Force
$\mathbf{F}=\gamma(E'/c,\mathbf{f})$

4-TotalMomentum
$\mathbf{P}_T=(E_T/c,\mathbf{p}_T)=(H/c,\mathbf{p}_T)$

4-NumberFlux
$\mathbf{N}=(nc,\mathbf{n})=n(c,\mathbf{u})$

4-ProbCurrDensity
4-ProbabilityFlux
$\mathbf{J}_{prob}=(\rho_{prob}c,\mathbf{j}_{prob})$
$=(i\hbar/2m_o)(\psi^*\partial[\psi]-\partial[\psi^*]\psi)+(q/m_o)(\psi^*\psi)\mathbf{A}$
Complex

4-ChargeFlux
4-CurrentDensity
$\mathbf{J}=(\rho c,\mathbf{j})=\rho(c,\mathbf{u})$

4-EMVectorPotential
$\mathbf{A}=(\varphi/c,\mathbf{a})$

4-MassFlux
4-MomentumDensity
$\mathbf{G}=(\rho_m c,\mathbf{g})=(\rho_e/c,\mathbf{g})$

4-ForceDensity
$\mathbf{F}_{den}=\gamma(E_{den}'/c,\mathbf{f}_{den})$

4-MomentumField
$\mathbf{P}_f=(E_f/c,\mathbf{p}_f)$
$=\mathbf{P}+\mathbf{Q}=\mathbf{P}+q\mathbf{A}$

4-EMPotentialMomentum
$\mathbf{Q}=(U/c,\mathbf{q})=q\mathbf{A}$

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^{\mu}{}_{\nu}$ or $T_{\mu}{}^{\nu}$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor V^μ = $\mathbf{V}=(v^0,\mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu=(v_0,-\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Existing SR Rules
Quantum Principles

$\mathrm{Trace}[T^{\mu\nu}]=\eta_{\mu\nu}T^{\mu\nu}=T^{\mu}{}_{\mu}=T$
$\mathbf{V}\cdot\mathbf{V}=V^\mu\eta_{\mu\nu}V^\nu=[(v^0)^2-\mathbf{v}\cdot\mathbf{v}]=(v^0{}_o)^2$
= Lorentz Scalar

SRQM+EM Diagram: 4-Vectors, 4-Tensors

A Tensor Study of Physical 4-Vectors

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John B. Wilson

$\partial[\mathbf{R}]=\eta^{\mu\nu}\rightarrow \mathrm{Diag}[1,-1,-1,-1]$
Minkowski Metric

4-Displacement
$\Delta\mathbf{R}=(c\Delta t,\Delta\mathbf{r})$
$\mathbf{dR}=(cdt,\mathbf{dr})$
4-Position
$\mathbf{R}=(ct,\mathbf{r})$

4-Acceleration
$\mathbf{A}=\gamma(c\gamma',\gamma'\mathbf{u}+\gamma\mathbf{a})$

4-Polarization:Spin
$\mathbf{E}=(\varepsilon^0,\boldsymbol{\varepsilon})=(\boldsymbol{\varepsilon}\cdot\boldsymbol{\beta},\boldsymbol{\varepsilon})$
$\mathbf{S}=(s^0,\mathbf{s})=(\mathbf{s}\cdot\boldsymbol{\beta},\mathbf{s})$
Complex

SR Perfect Fluid
$T^{\mu\nu}=((\rho_{eo}+p_o)/c^2)U^\mu U^\nu-(p_o)\eta^{\mu\nu}$
$T^{\mu\nu}=(\rho_{eo})V^{\mu\nu}-(p_o)H^{\mu\nu}$
StressEnergy 4-Tensor

Einstein GR
$G^{\mu\nu}=R^{\mu\nu}-g^{\mu\nu}R/2$
4-Tensor

4-Gradient
$\partial=(\partial_t/c,-\nabla)$

4-WaveVector
$\mathbf{K}=(\omega/c,\mathbf{k})$

4-TotalWaveVector
$\mathbf{K_T}=(\omega_T/c,\mathbf{k_T})$

$\eta_{\mu\nu}$ ·

EM Faraday
$F^{\alpha\beta}=\partial^\alpha A^\beta-\partial^\beta A^\alpha$
$=[\ 0\ ,-e^j/c]$
$[+e^i/c,-\varepsilon^{ij}{}_k b^k]$
4-Tensor

$\eta_{\mu\nu}$ ·

4-UnitTemporal
$\mathbf{T}=\gamma(1,\boldsymbol{\beta})$

4-Velocity
$\mathbf{U}=\gamma(c,\mathbf{u})$

4-Force
$\mathbf{F}=\gamma(E'/c,\mathbf{f})$

4-TotalMomentum
$\mathbf{P_T}=(E_T/c,\mathbf{p_T})=(H/c,\mathbf{p_T})$

4-Momentum
$\mathbf{P}=(mc,\mathbf{p})=(E/c,\mathbf{p})$

4-UnitSpatial
$\mathbf{S}=\gamma_{\beta n}(\hat{\mathbf{n}}\cdot\boldsymbol{\beta},\hat{\mathbf{n}})_\perp$

4-MassFlux
4-MomentumDensity
$\mathbf{G}=(\rho_m c,\mathbf{g})=(\rho_e/c,\mathbf{g})$

4-ForceDensity
$\mathbf{F_{den}}=\gamma(E_{den}'/c,\mathbf{f_{den}})$

4-MomentumField
$\mathbf{P_f}=(E_f/c,\mathbf{p_f})$
$=\mathbf{P}+\mathbf{Q}=\mathbf{P}+q\mathbf{A}$

4-NumberFlux
$\mathbf{N}=(nc,\mathbf{n})=n(c,\mathbf{u})$
4-ProbCurrDensity
4-ProbabilityFlux
$\mathbf{J_{prob}}=(\rho_{prob}c,\mathbf{j_{prob}})$
$=(i\hbar/2m_o)(\psi^*\partial[\psi]-\partial[\psi^*]\psi)+(q/m_o)(\psi^*\psi)\mathbf{A}$
Complex

4-ChargeFlux
4-CurrentDensity
$\mathbf{J}=(\rho c,\mathbf{j})=\rho(c,\mathbf{u})$

4-EMVectorPotential
$\mathbf{A}=(\varphi/c,\mathbf{a})$

4-EMPotentialMomentum
$\mathbf{Q}=(U/c,\mathbf{q})=q\mathbf{A}$

SR 4-Tensor
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SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Existing SR Rules
Quantum Principles

$\mathrm{Trace}[T^{\mu\nu}]=\eta_{\mu\nu}T^{\mu\nu}=T^\mu{}_\mu=T$
$\mathbf{V}\cdot\mathbf{V}=V^\mu\eta_{\mu\nu}V^\nu=[(v^0)^2-\mathbf{v}\cdot\mathbf{v}]=(v^0{}_o)^2$
$=$ Lorentz Scalar

SRQM+EM Diagram: 4-Vectors, 4-Tensors Lorentz Scalars / Physical Constants

A Tensor Study of Physical 4-Vectors

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$\partial \cdot \mathbf{R}=4$
SpaceTime Dimension

$\partial[\mathbf{R}]=\eta^{\mu\nu} \rightarrow \mathrm{Diag}[1,-1,-1,-1]$
Minkowski Metric

$-\partial \cdot T^{\mu\nu}=\mathbf{F_{den}}$
SR Conservation of StressEnergy if $\mathbf{F_{den}}=0^{\mu}$

$\partial \cdot G^{\mu\nu}=0^{\mu}$
SR Conservation of Einstein Tensor

4-Acceleration
$\mathbf{A}=\gamma(c\gamma',\gamma'\mathbf{u}+\gamma\mathbf{a})$

4-Polarization:Spin
$\mathbf{E}=(\varepsilon^0,\boldsymbol{\varepsilon})=(\boldsymbol{\varepsilon}\cdot\boldsymbol{\beta},\boldsymbol{\varepsilon})$
$\mathbf{S}=(s^0,\mathbf{s})=(\mathbf{s}\cdot\boldsymbol{\beta},\mathbf{s})$
Complex

$-p_o$

$(\rho_{eo}+p_o)/c^2$

SR Perfect Fluid
$T^{\mu\nu}=((\rho_{eo}+p_o)/c^2)U^{\mu}U^{\nu}-(p_o)\eta^{\mu\nu}$
$T^{\mu\nu}=(\rho_{eo})V^{\mu\nu}-(p_o)H^{\mu\nu}$
StressEnergy 4-Tensor

$8\pi G/c^4$

Einstein GR
$G^{\mu\nu}=R^{\mu\nu}-g^{\mu\nu}R/2$
4-Tensor

4-Gradient
$\partial=(\partial_t/c,-\nabla)$

4-Displacement
$\Delta\mathbf{R}=(c\Delta t,\Delta\mathbf{r})$
$d\mathbf{R}=(cdt,d\mathbf{r})$

4-Position
$\mathbf{R}=(ct,\mathbf{r})$

$\mathbf{U}\cdot\partial\,[..]$
$d/d\tau\,[..]=\gamma d/dt\,[..]$

$\mathbf{K}\cdot\mathbf{R}=-\Phi_{phase}$

4-WaveVector
$\mathbf{K}=(\omega/c,\mathbf{k})$

4-TotalWaveVector
$\mathbf{K_T}=(\omega_T/c,\mathbf{k_T})$

$..[\mathbf{K_T}\cdot\mathbf{R}]$
$..[-\Phi_{Tphase}]$
$(\,i\,)$

$\mathbf{T}\cdot\Delta\mathbf{R}/c=\Delta\tau$
ProperTime

$\mathbf{U}\cdot\mathbf{A}=\mathbf{U}\cdot\mathbf{U}'=0$

$\mathbf{U}\cdot\mathbf{E}=0$
$\mathbf{U}\cdot\mathbf{S}=0$

ProperTime
$\mathbf{U}\cdot\partial=d/d\tau=\gamma d/dt$
Derivative

E_{To}/ω_{To}
$(\,\hbar\,)$

$..[\mathbf{P_T}\cdot\mathbf{R}]$
$..[-S_{action}]$

EM Faraday
$F^{\alpha\beta}=\partial^{\alpha}A^{\beta}-\partial^{\beta}A^{\alpha}$
$=[\;0\;,-e^j/c]$
$[+e^i/c,-\varepsilon^{ij}{}_k b^k]$
4-Tensor

$\eta_{\mu\nu}$

ω_o/c^2

4-UnitTemporal
$\mathbf{T}=\gamma(1,\boldsymbol{\beta})$

c

4-Velocity
$\mathbf{U}=\gamma(c,\mathbf{u})$

E_o/ω_o
$(\,\hbar\,)$

$\eta_{\mu\nu}$

q

$\mathbf{P_T}\cdot\mathbf{R}$
$=\int\mathbf{P_T}\cdot d\mathbf{R}$
$=-S_{action}$

$\varepsilon_o c^2$
$1/\mu_o$

$\mathbf{T}\cdot\mathbf{S}=0$

m_o
E_o/c^2

n_o
$\rho_{probo}=\chi^*\psi$
$=|\langle\chi|\psi\rangle|^2$

4-Force
$\mathbf{F}=\gamma(E'/c,\mathbf{f})$

4-TotalMomentum
$\mathbf{P_T}=(E_T/c,\mathbf{p_T})=(H/c,\mathbf{p_T})$

4-UnitSpatial
$\mathbf{S}=\gamma_{\beta n}(\hat{\mathbf{n}}\cdot\boldsymbol{\beta},\hat{\mathbf{n}})_{\perp}$

ρ_o

φ_o/c^2

4-Momentum
$\mathbf{P}=(mc,\mathbf{p})=(E/c,\mathbf{p})$

$\mathbf{U}\cdot\partial\,[..]$
$d/d\tau\,[..]$

n_o

n_o

4-MassFlux
4-MomentumDensity
$\mathbf{G}=(\rho_m c,\mathbf{g})=(\rho_e/c,\mathbf{g})$

$\mathbf{U}\cdot\partial\,[..]$
$d/d\tau\,[..]$

4-ForceDensity
$\mathbf{F_{den}}=\gamma(E_{den}'/c,\mathbf{f_{den}})$

m_o
E_o/c^2

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$\mathbf{J_{prob}}=(\rho_{prob}c,\mathbf{j_{prob}})$
$=(i\hbar/2m_o)(\psi^*\partial[\psi]-\partial[\psi^*]\psi)+(q/m_o)(\psi^*\psi)\mathbf{A}$
Complex

q

4-ChargeFlux
4-CurrentDensity
$\mathbf{J}=(\rho c,\mathbf{j})=\rho(c,\mathbf{u})$

4-EMVectorPotential
$\mathbf{A}=(\varphi/c,\mathbf{a})$

q

4-EMPotentialMomentum
$\mathbf{Q}=(U/c,\mathbf{q})=q\mathbf{A}$

$\varepsilon_o c^2$
$1/\mu_o$

$(\partial\cdot\partial)\mathbf{A}-\partial(\partial\cdot\mathbf{A})=\mu_o\mathbf{J}$
Maxwell EM Wave Eqn

$\partial\cdot\mathbf{A}=0$
Conservation of EM Field = Lorenz Gauge

$\partial\cdot\mathbf{N}=0$: $\partial\cdot\mathbf{J_{prob}}=0$
Conservation of Particle # : Probabilty

$\partial\cdot\mathbf{J}=0$
Conservation of Charge

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(2,0)-Tensor $T^{\mu\nu}$
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SR → QM

4-Vector SRQM Interpretation of QM

SRQM+EM Diagram: 4-Vectors, 4-Tensors
Lorentz Scalars / Physical Constants

A Tensor Study of Physical 4-Vectors

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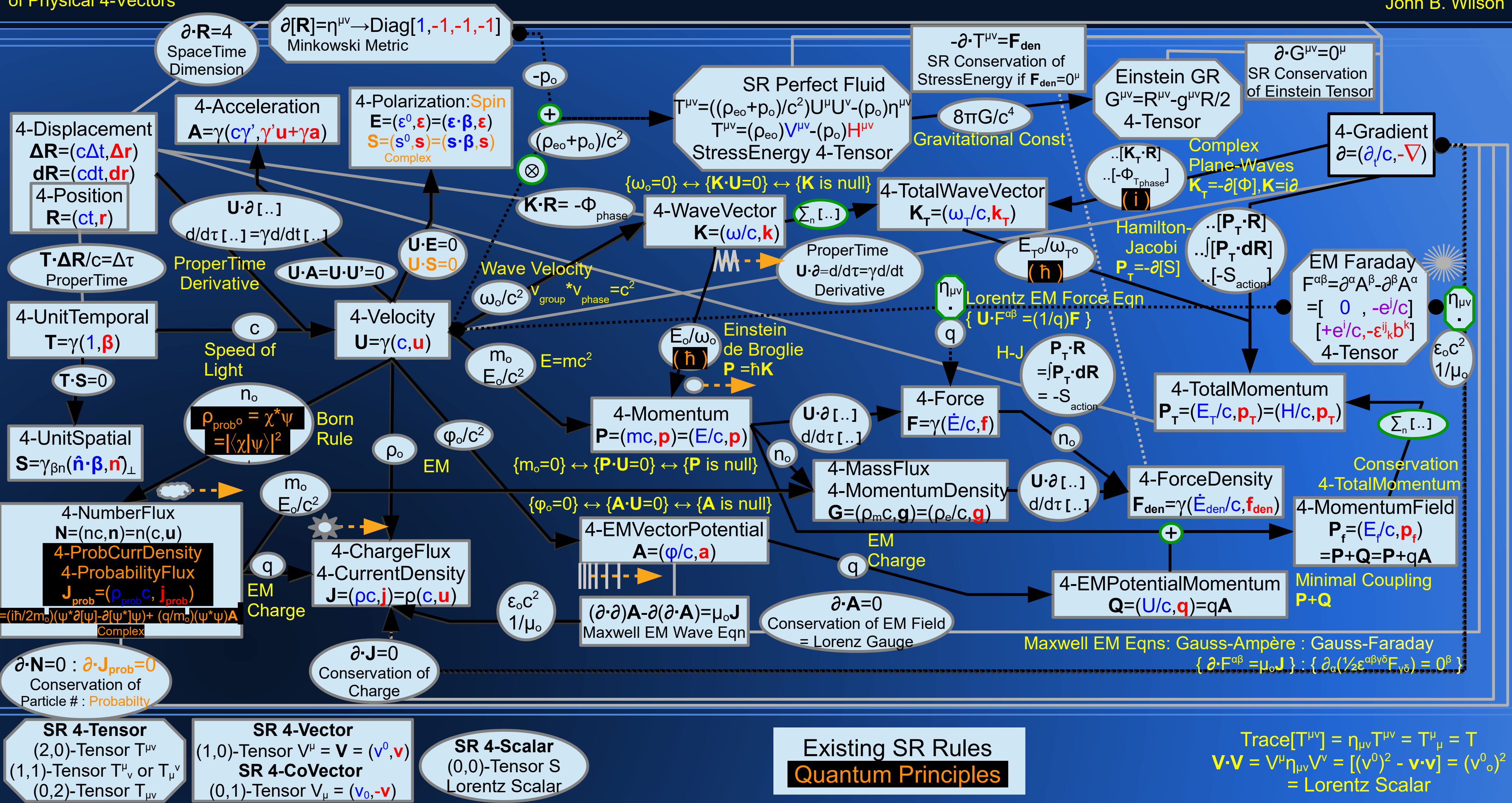

SR → QM

4-Vector SRQM Interpretation of QM

SRQM+EM Diagram: 4-Vectors, 4-Tensors Lorentz Scalars / Physical Constants with Tensor Invariants

A Tensor Study of Physical 4-Vectors

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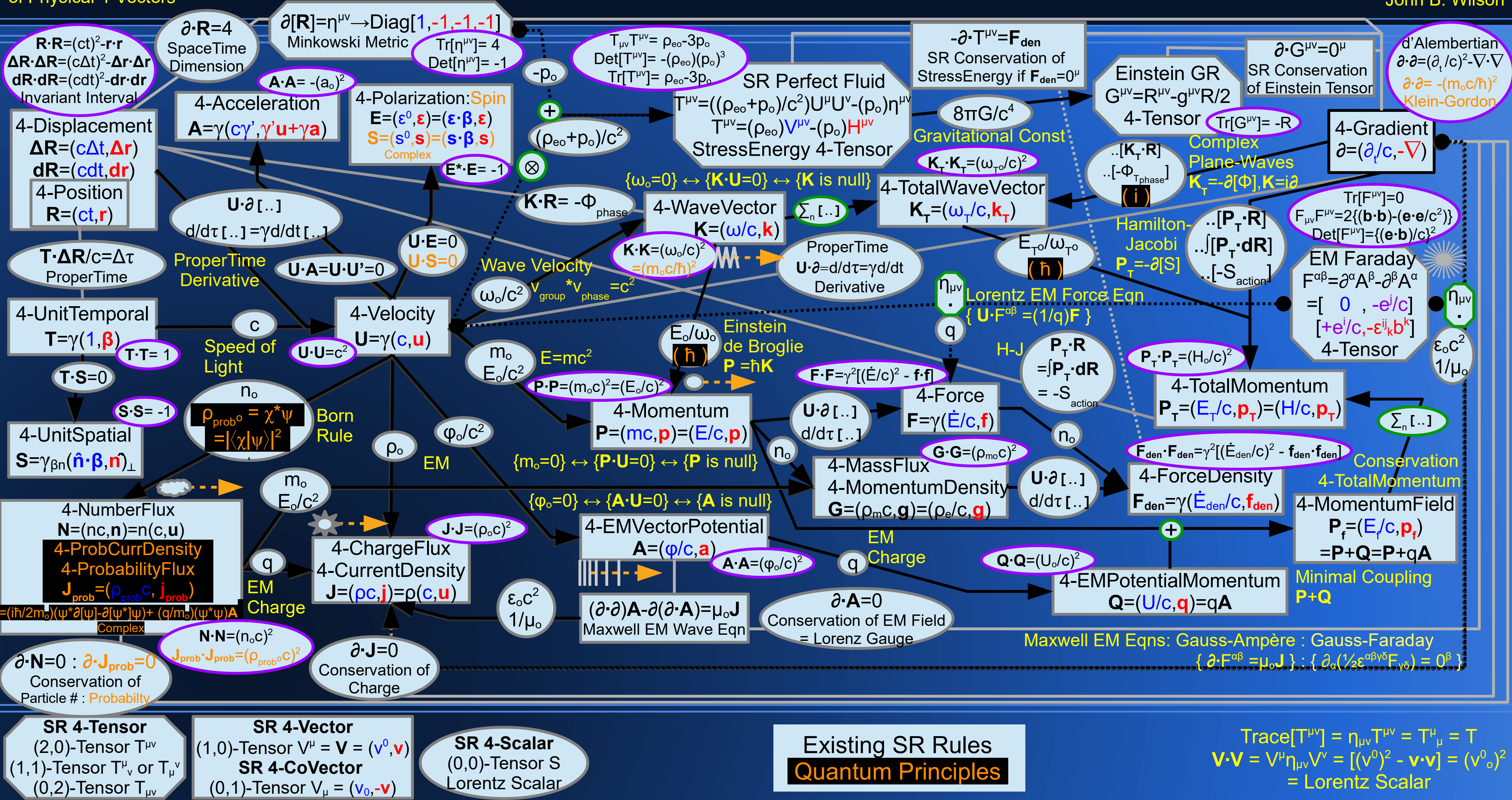

SR → QM

4-Vector SRQM Interpretation of QM

SRQM Diagram:
Physical Constants Emphasized

A Tensor Study of Physical 4-Vectors

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$\partial\cdot\mathbf{R}=4$ SpaceTime Dimension

$\partial[\mathbf{R}]=\eta^{\mu\nu}\rightarrow\mathrm{Diag}[1,-1,-1,-1]$ Minkowski Metric

$-p_o$

$(\rho_{eo}+p_o)/c^2$

$\mathbf{U}\cdot\partial\,[..]$
$d/d\tau\,[..]$
$\gamma d/dt\,[..]$

$\mathbf{U}\cdot\mathbf{E}=0$
$\mathbf{U}\cdot\mathbf{S}=0$

$\mathbf{K}\cdot\mathbf{R}=-\Phi_{phase}$

$\mathbf{T}\cdot\Delta\mathbf{R}/c=\Delta\tau$ ProperTime

$\mathbf{U}\cdot\mathbf{A}=\mathbf{U}\cdot\mathbf{U}'=0$

ω_o/c^2

$\mathbf{T}\cdot\mathbf{T}=1$

c

$\mathbf{T}\cdot\mathbf{S}=0$

$\mathbf{S}\cdot\mathbf{S}=-1$

n_o
$\rho_{prob}{}^o=\chi^*\psi$
$=|\langle\chi|\psi\rangle|^2$

m_o
E_o/c^2

φ_o/c^2

ρ_o

m_o
E_o/c^2

q

$\varepsilon_o c^2$
$1/\mu_o$

$\partial\cdot\mathbf{N}=0$: $\partial\cdot\mathbf{J}_{prob}=0$
Conservation of Particle # : Probabilty

$\partial\cdot\mathbf{J}=0$ Conservation of Charge

$8\pi G/c^4$

$-\partial\cdot T^{\mu\nu}=\mathbf{F}_{den}$ SR Conservation of StressEnergy if $\mathbf{F}_{den}=0^\mu$

$\partial\cdot G^{\mu\nu}=0^\mu$ SR Conservation of Einstein Tensor

$..[\mathbf{K}\cdot\mathbf{R}]$
$..[-\Phi_{phase}]$
$(\,i\,)$

ProperTime
$\mathbf{U}\cdot\partial=d/d\tau=\gamma d/dt$
Derivative

E_{To}/ω_{To}
$(\,\hbar\,)$

$..[\mathbf{P_T}\cdot\mathbf{R}]$
$..[-S_{action}]$

$\eta_{\mu\nu}$ ·

$\eta_{\mu\nu}$ ·

q

$\varepsilon_o c^2$
$1/\mu_o$

E_o/ω_o
$(\,\hbar\,)$

$\int\mathbf{P_T}\cdot d\mathbf{R}=-S_{action}$
$\mathbf{P_T}\cdot\mathbf{R}=-S_{action}$

$\mathbf{U}\cdot\partial\,[..]$
$d/d\tau\,[..]$

n_o

n_o

$\mathbf{U}\cdot\partial\,[..]$
$d/d\tau\,[..]$

q

$\partial\cdot\mathbf{A}=0$ Conservation of EM Field = Lorenz Gauge

Notice that all the main "Universal" or "Fundamental" Physical Constants are here: $G,c,\hbar,\varepsilon_o,\mu_o$.

Some depend on the actual particle type: q,m_o,ω_o
Some depend on regional conditions: $\tau,\rho_{eo},p_o,\rho_o,\varphi_o,\psi^*\psi$
Some depend on interaction: Φ_{phase},S_{action}
Some are mathematical: $0,4,\pi,i,\mathrm{Diag}[1,-1,-1,-1],d/d\tau$
Conservation Laws are also a type of "zero" constant in this regard.

The majority of the constants are Lorentz Scalars, but some are 4-Vector or 4-Tensor, and all are valid for all inertial observers.

Fundamental Physical Constants are SR Lorentz Scalars

The fact that these "tie together" a network of 4-Vectors is a good argument for why their values are constant. Changing even one would change the relationship properties among all of the 4-Vectors.

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^{\mu}{}_{\nu}$ or $T_{\mu}{}^{\nu}$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0,\mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0,-\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Existing SR Rules
Quantum Principles

SRQM Diagram: Projection Tensors
Temporal, Spatial, Null, SpaceTime

A Tensor Study of Physical 4-Vectors

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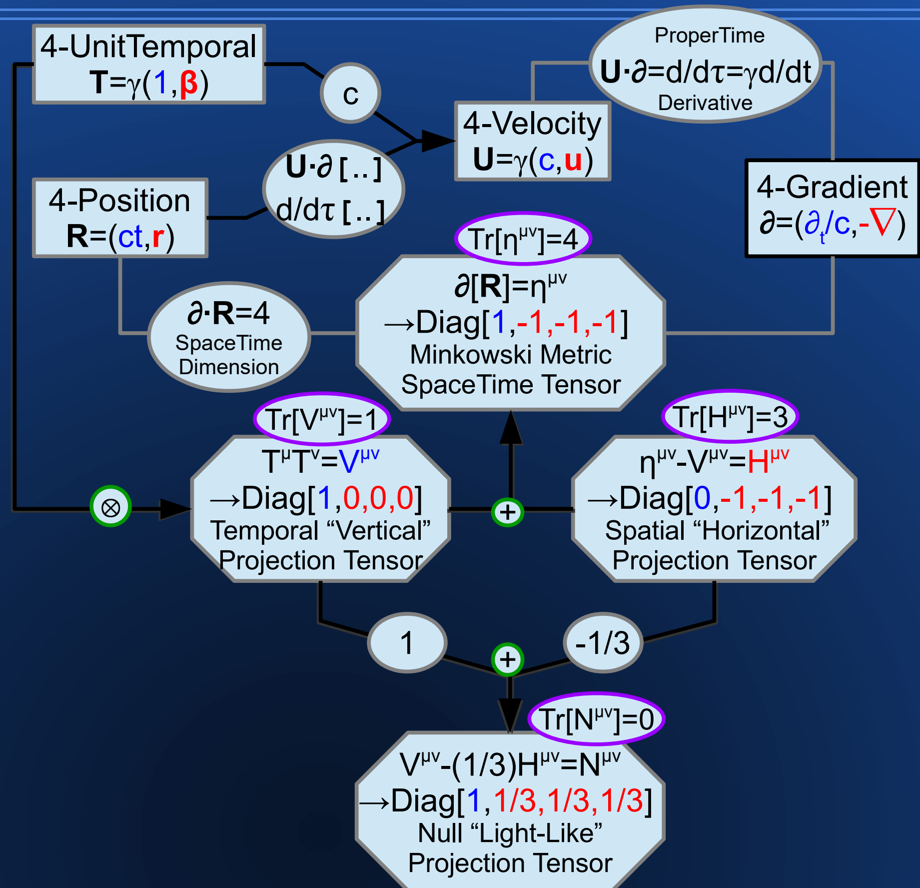


Projection Tensors act as follows:

Generic 4-Vector:

$A^\nu = (a^0,\mathbf{a}) = (a^0,a^1,a^2,a^3)$

Temporal Projection:

$V^\mu{}_\nu = \eta_{\omega\nu}V^{\mu\omega} \rightarrow \mathrm{Diag}[1,0,0,0]$

$V^\mu{}_\nu A^\nu = (a^0,0,0,0) = (a^0,\mathbf{0})$

Spatial Projection:

$H^\mu{}_\nu = \eta_{\omega\nu}H^{\mu\omega} \rightarrow \mathrm{Diag}[0,1,1,1]$

$H^\mu{}_\nu A^\nu = (0,a^1,a^2,a^3) = (0,\mathbf{a})$

SpaceTime Projection:

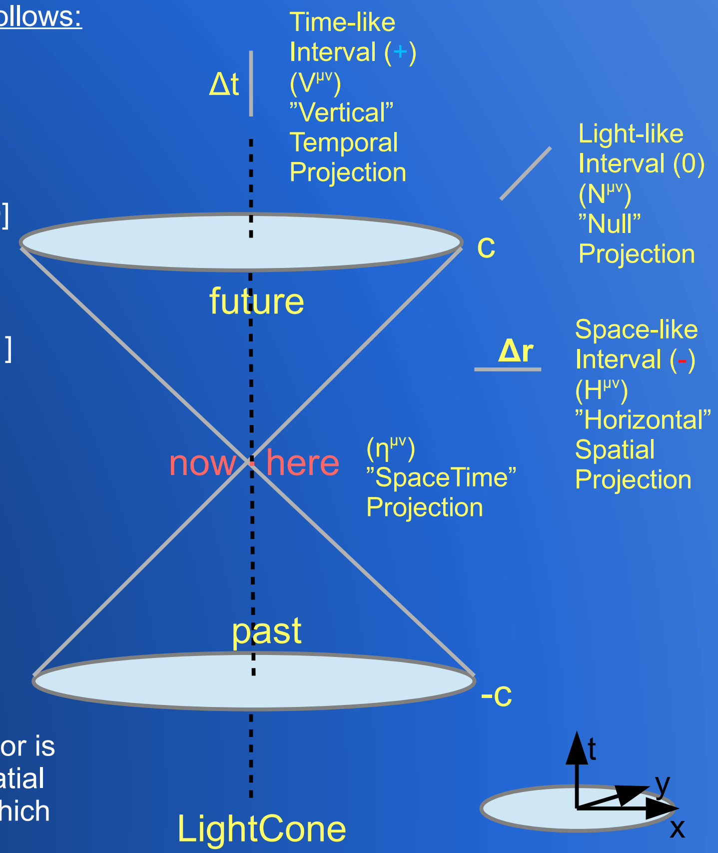
$V^\mu{}_\nu A^\nu + H^\mu{}_\nu A^\nu = \eta^\mu{}_\nu A^\nu$

$= \delta^\mu{}_\nu A^\nu = A^\mu = (a^0,\mathbf{a})$

$V^\mu{}_\nu + H^\mu{}_\nu = \eta^\mu{}_\nu = \delta^\mu{}_\nu$

$V^{\mu\nu} + H^{\mu\nu} = \eta^{\mu\nu}$

The Minkowski Metric Tensor is the Sum of Temporal & Spatial Projection Tensors, all of which are dimensionless.



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Lorentz Scalar

$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu{}_\mu = T$

$\mathbf{V}\cdot\mathbf{V} = V^\mu\eta_{\mu\nu}V^\nu = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0_o)^2$

= Lorentz Scalar

SRQM Diagram: Projection Tensors & Perfect-Fluid Stress-Energy Tensor

A Tensor Study of Physical 4-Vectors

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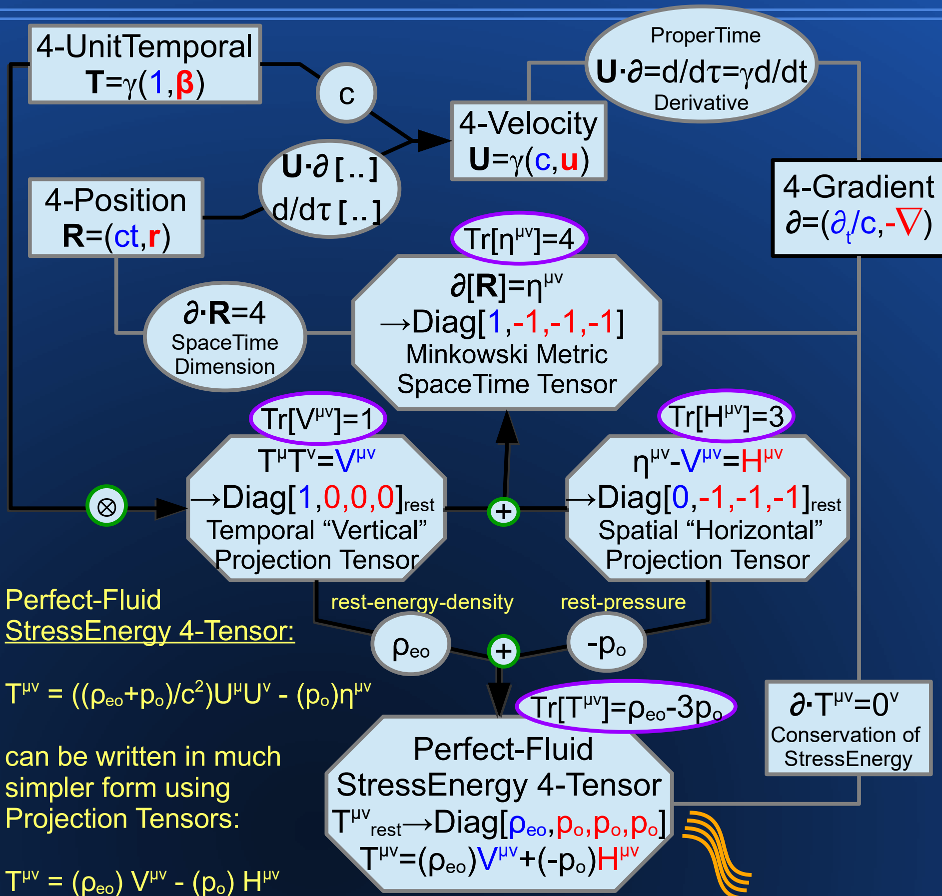


Perfect-Fluid StressEnergy 4-Tensor:

$T^{\mu\nu} = ((\rho_{eo}+p_o)/c^2)U^\mu U^\nu - (p_o)\eta^{\mu\nu}$

can be written in much simpler form using Projection Tensors:

$T^{\mu\nu} = (\rho_{eo})\, V^{\mu\nu} - (p_o)\, H^{\mu\nu}$

Projection Tensors act as follows:

$A^\nu = (a^0,\mathbf{a}) = (a^0,a^1,a^2,a^3)$

$V^\mu{}_\nu = \eta_{\omega\nu}V^{\mu\omega} \rightarrow \mathrm{Diag}[1,0,0,0]$
$V^\mu{}_\nu A^\nu = (a^0,0,0,0) = (a^0,\mathbf{0})$

$H^\mu{}_\nu = \eta_{\omega\nu}H^{\mu\omega} \rightarrow \mathrm{Diag}[0,1,1,1]$
$H^\mu{}_\nu A^\nu = (0,a^1,a^2,a^3) = (0,\mathbf{a})$

$V^\mu{}_\nu A^\nu + H^\mu{}_\nu A^\nu = \eta^\mu{}_\nu A^\nu$
$= \delta^\mu{}_\nu A^\nu = A^\mu = (a^0,\mathbf{a})$

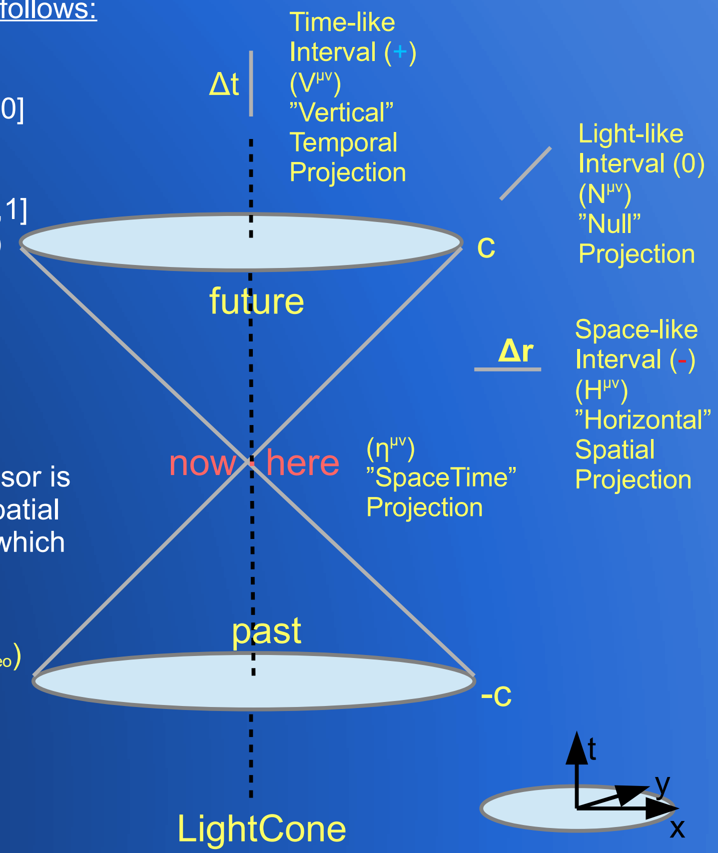
$V^\mu{}_\nu + H^\mu{}_\nu = \eta^\mu{}_\nu$
$V^{\mu\nu} + H^{\mu\nu} = \eta^{\mu\nu}$

The Minkowski Metric Tensor is the Sum of Temporal & Spatial Projection Tensors, all of which are dimensionless.

The rest-energy-density (ρ_{eo}) is the Temporal Projection.

The neg rest-pressure $-(p_o)$ is the Spatial Projection.

$T^{\mu\nu}{}_{rest} \rightarrow \mathrm{Diag}[\rho_{eo},p_o,p_o,p_o]$



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(0,0)-Tensor S
Lorentz Scalar

$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu{}_\mu = T$
$\mathbf{V}\cdot\mathbf{V} = V^\mu\eta_{\mu\nu}V^\nu = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$
= Lorentz Scalar

SR → QM

4-Vector SRQM Interpretation of QM

SRQM+EM Diagram: Projection Tensors & Stress-Energy Tensors: Special Cases

A Tensor Study of Physical 4-Vectors

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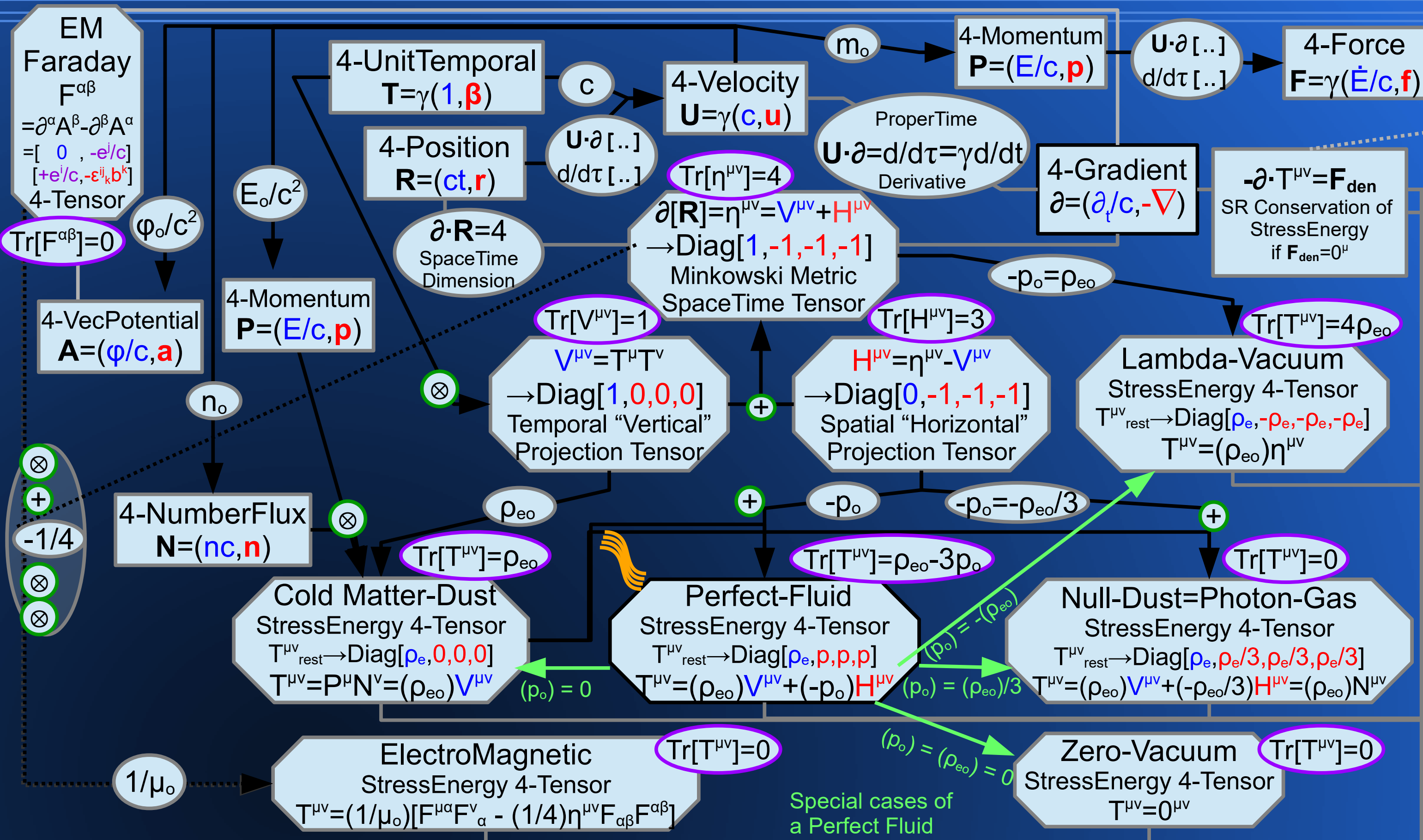

A few interesting special cases:

{for Perfect Fluid (no viscosity)}
$T^{\mu\nu}{}_{PerfectFluid} = (\rho_{eo})V^{\mu\nu} - (p_o)H^{\mu\nu}$
$T^{\mu\nu}{}_{PerfectFluid} = ((\rho_{eo}+p_o)/c^2)U^\mu U^\nu - (p_o)\eta^{\mu\nu}$
$Tr[T^{\mu\nu}] = 1(\rho_{eo}) - 3(p_o)$

If $(p_o) = (\rho_{eo})/3$:
then {NullDust = PhotonGas = Radiation}
$T^{\mu\nu}{}_{PhotonGas} = (\rho_{eo})V^{\mu\nu} - (\rho_{eo}/3)H^{\mu\nu} = (\rho_{eo})N^{\mu\nu}$
$T^{\mu\nu}{}_{PhotonGas} = (p_o)(4V^{\mu\nu} - \eta^{\mu\nu})$
$Tr[T^{\mu\nu}{}_{PhotonGas}] = 0$: Null (Light-Like) Projection

If $(p_o) = 0$:
then {Cold Matter Dust (pressureless) }
$T^{\mu\nu}{}_{MatterDust} = P^\mu N^\nu = (\rho_{mo})U^\mu U^\nu = (\rho_{eo})V^{\mu\nu}$
$Tr[T^{\mu\nu}{}_{MatterDust}] = (\rho_{eo})$: Temporal Projection

If $(p_o) = -(\rho_{eo})$:
then {LambdaVacuum Energy}
$T^{\mu\nu}{}_{VaccEnergy} = (\rho_{eo})V^{\mu\nu} + (\rho_{eo})H^{\mu\nu} = (\rho_{eo})\eta^{\mu\nu}$
$Tr[T^{\mu\nu}{}_{VaccEnergy}] = 4(\rho_{eo})$: SpaceTime Projection

If $(p_o) = (\rho_{eo}) = 0$:
then {ZeroVacuum Energy}
$T^{\mu\nu}{}_{VaccEnergy} = 0^{\mu\nu}$
$Tr[T^{\mu\nu}{}_{VaccEnergy}] = 0$: Zero Projection

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(0,0)-Tensor S
Lorentz Scalar

Tr[ ] = Trace Function = $\eta_{\mu\nu}$
$N^{\mu\nu} = V^{\mu\nu} - (1/3)H^{\mu\nu}$ = Null Projection Tensor
$N^{\mu\nu} \to Diag[1,1/3,1/3,1/3]$ with $Tr[N^{\mu\nu}] = 0$

$Trace[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu{}_\mu = T$
$\mathbf{V}\cdot\mathbf{V} = V^\mu\eta_{\mu\nu}V^\nu = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$
= Lorentz Scalar

SRQM Study:
4D Gauss' Theorem

Gauss' Theorem in SR:

$\int_\Omega d^4\mathbf{X}\ (\partial_\mu V^\mu) = \oint_{\partial\Omega} dS\ (V^\mu N_\mu)$

$\int_\Omega d^4\mathbf{X}\ (\boldsymbol{\partial}\cdot\mathbf{V}) = \oint_{\partial\Omega} dS\ (\mathbf{V}\cdot\mathbf{N})$

where:

V = V^μ is a 4-Vector field defined in Ω

$(\boldsymbol{\partial}\cdot\mathbf{V}) = (\partial_\mu V^\mu)$ is the 4-Divergence of **V**

$(\mathbf{V}\cdot\mathbf{N}) = (V^\mu N_\mu)$ si the component of **V** along the **N**-direction

Ω is a 4D simply-connected region of Minkowski SpaceTime

∂Ω = S is its 3D boundary with its own 3D Volume element dS and outward pointing normal **N**.

N = N^μ is the outward-pointing normal

$d^4\mathbf{X} = (c\ dt)(d^3\mathbf{x}) = (c\ dt)(dx\ dy\ dz)$ is the 4D differential volume element

In vector calculus, the divergence theorem, also known as Gauss's theorem or Ostrogradsky's theorem,
is a result that relates the flow (that is, flux) of a vector field through a surface to the behavior of the vector field inside the surface.
More precisely, the divergence theorem states that the outward flux of a vector field through a closed surface
is equal to the volume integral of the divergence over the region inside the surface.
Intuitively, it states that the sum of all sources minus the sum of all sinks gives the net flow out of a region.
In vector calculus, and more generally in differential geometry,
the generalized Stokes' theorem is a statement about the integration of differential forms on manifolds,
which both simplifies and generalizes several theorems from vector calculus.

SRQM Diagram:
Minimal Coupling = Potential Interaction
Conservation of 4-TotalMomentum

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$\mathbf{P} = (E/c, \mathbf{p})$: 4-Momentum
$\mathbf{Q} = (V/c, \mathbf{q})$: 4-PotentialMomentum
$\mathbf{A} = (\varphi/c, \mathbf{a})$: 4-VectorPotential
$\mathbf{P_f} = (E_f/c, \mathbf{p_f})$: 4-MomentumIncPotentialField
$\mathbf{P_T} = (E_T/c, \mathbf{p_T}) = (H/c, \mathbf{p_T})$: 4-TotalMomentum

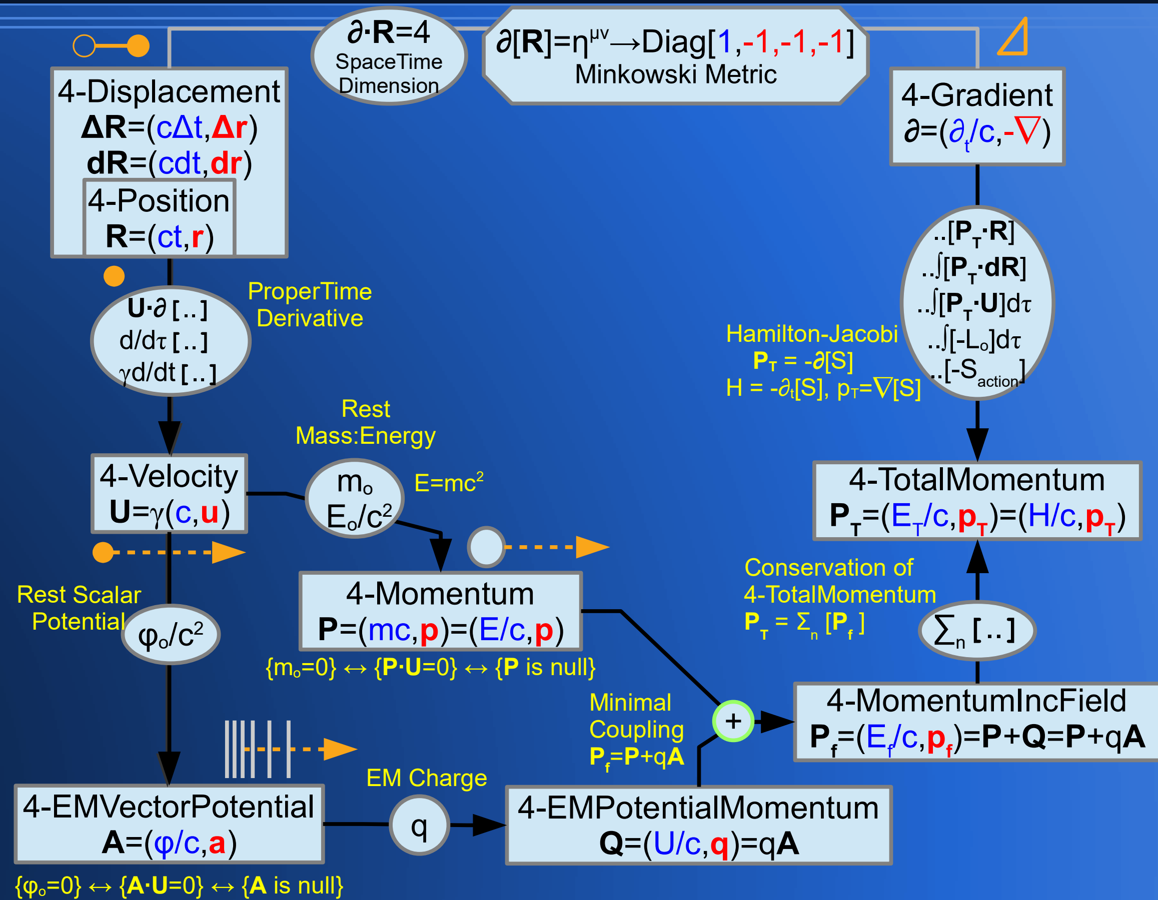
$\mathbf{P} = \mathbf{P_f} - q\mathbf{A} = (E_f/c - q\varphi/c, \mathbf{p_f} - q\mathbf{a})$: Minimal Coupling Relation

$\mathbf{P_f} = \mathbf{P} + \mathbf{Q} = \mathbf{P} + q\mathbf{A}$: Conservation of 4-MomentumIncPotentialField

$\mathbf{P_f} = \mathbf{P} + \mathbf{Q}$
$\mathbf{P_f} = \mathbf{P} + q\mathbf{A}$
$\mathbf{P_f} = (m_o)\mathbf{U} + (q\varphi_o/c^2)\mathbf{U}$
$\mathbf{P_f} = (E_o/c^2)\mathbf{U} + (q\varphi_o/c^2)\mathbf{U}$
$\mathbf{P_f} = ((E_o+q\varphi_o)/c^2)\mathbf{U}$
$\mathbf{P_f} = ((E+q\varphi)/c^2)(c, \mathbf{u})$
$\mathbf{P_f} = ((E+q\varphi)/c, \mathbf{p}+q\mathbf{a})$

4-MomentumIncPotentialField has a contribution from
a Mass "charge" (m_o)
an EM charge (q) interacting with a potential (φ_o)

$\mathbf{P_T} = \Sigma_n [\, \mathbf{P_f} \,]$: Conservation of 4-TotalMomentum
4-TotalMomentum is the Sum over all such 4-Momenta



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Lorentz Scalar

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$\mathbf{V}\cdot\mathbf{V} = V^{\mu}\eta_{\mu\nu}V^{\nu} = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$
= Lorentz Scalar

SRQM Hamiltonian:Lagrangian Connection

$H + L = (\mathbf{p_T}\cdot\mathbf{u}) = \gamma(\mathbf{P_T}\cdot\mathbf{U}) + -(\mathbf{P_T}\cdot\mathbf{U})/\gamma$

4-Momentum $\mathbf{P} = m_o\mathbf{U} = (E_o/c^2)\mathbf{U}$; 4-VectorPotential $\mathbf{A} = (\varphi_o/c^2)\mathbf{U}$
4-TotalMomentum $\mathbf{P_T} = (\mathbf{P} + q\mathbf{A}) = (H/c,\mathbf{p_T})$

$\mathbf{P}\cdot\mathbf{U} = \gamma(E - \mathbf{p}\cdot\mathbf{u}) = E_o = m_oc^2$; $\mathbf{A}\cdot\mathbf{U} = \gamma(\varphi - \mathbf{a}\cdot\mathbf{u}) = \varphi_o$
$\mathbf{P_T}\cdot\mathbf{U} = (\mathbf{P}\cdot\mathbf{U} + q\mathbf{A}\cdot\mathbf{U}) = E_o + q\varphi_o = m_oc^2 + q\varphi_o$

$\gamma = 1/\mathrm{Sqrt}[1-\boldsymbol{\beta}\cdot\boldsymbol{\beta}]$: Relativistic Gamma Identity
$(\gamma - 1/\gamma) = (\gamma\boldsymbol{\beta}\cdot\boldsymbol{\beta})$: Manipulate into this form… still an identity
$(\gamma - 1/\gamma)(\mathbf{P_T}\cdot\mathbf{U}) = (\gamma\boldsymbol{\beta}\cdot\boldsymbol{\beta})(\mathbf{P_T}\cdot\mathbf{U})$: Still covariant with Lorentz Scalar
$\gamma(\mathbf{P_T}\cdot\mathbf{U}) + -(\mathbf{P_T}\cdot\mathbf{U})/\gamma = (\gamma\boldsymbol{\beta}\cdot\boldsymbol{\beta})(\mathbf{P_T}\cdot\mathbf{U})$
$\gamma(\mathbf{P_T}\cdot\mathbf{U}) + -(\mathbf{P_T}\cdot\mathbf{U})/\gamma = (\gamma\boldsymbol{\beta}\cdot\boldsymbol{\beta})(E_o + q\varphi_o)$
$\gamma(\mathbf{P_T}\cdot\mathbf{U}) + -(\mathbf{P_T}\cdot\mathbf{U})/\gamma = (\gamma\mathbf{u}\cdot\mathbf{u})(E_o + q\varphi_o)/c^2$
$\gamma(\mathbf{P_T}\cdot\mathbf{U}) + -(\mathbf{P_T}\cdot\mathbf{U})/\gamma = (\gamma(E_o/c^2 + q\varphi_o/c^2)\mathbf{u}\cdot\mathbf{u})$
$\gamma(\mathbf{P_T}\cdot\mathbf{U}) + -(\mathbf{P_T}\cdot\mathbf{U})/\gamma = ((\gamma E_o\mathbf{u}/c^2 + \gamma q\varphi_o\mathbf{u}/c^2)\cdot\mathbf{u})$
$\gamma(\mathbf{P_T}\cdot\mathbf{U}) + -(\mathbf{P_T}\cdot\mathbf{U})/\gamma = ((E\mathbf{u}/c^2 + q\varphi\mathbf{u}/c^2)\cdot\mathbf{u})$
$\gamma(\mathbf{P_T}\cdot\mathbf{U}) + -(\mathbf{P_T}\cdot\mathbf{U})/\gamma = ((\mathbf{p}+q\mathbf{a})\cdot\mathbf{u})$
$\gamma(\mathbf{P_T}\cdot\mathbf{U}) + -(\mathbf{P_T}\cdot\mathbf{U})/\gamma = (\mathbf{p_T}\cdot\mathbf{u})$
$\{\ H\ \} + \{\ L\ \} = (\mathbf{p_T}\cdot\mathbf{u})$: The Hamiltonian/Lagrangian connection

$H = \gamma(\mathbf{P_T}\cdot\mathbf{U}) = \gamma((\mathbf{P}+q\mathbf{A})\cdot\mathbf{U})$ = The Hamiltonian with minimal coupling
$L = -(\mathbf{P_T}\cdot\mathbf{U})/\gamma = -((\mathbf{P}+q\mathbf{A})\cdot\mathbf{U})/\gamma$ = The Lagrangian with minimal coupling

H:L Connection in Density Format
$H + L = (\mathbf{p_T}\cdot\mathbf{u})$
$nH + nL = n(\mathbf{p_T}\cdot\mathbf{u})$, with number density $n = \gamma n_o$
$\mathcal{H} + \mathcal{L} = (\mathbf{g_T}\cdot\mathbf{u})$, with
momentum density $\{\mathbf{g_T} = n\mathbf{p_T}\}$
Hamiltonian density $\{\mathcal{H} = nH\}$
Lagrangian Density $\{\mathcal{L} = nL = (\gamma n_o)(L_o/\gamma) = n_oL_o\}$
Lagrangian Density is Lorentz Scalar

for an EM field (photonic):
$\mathcal{H} = (1/2)\{\varepsilon_o\mathbf{e}\cdot\mathbf{e} + \mathbf{b}\cdot\mathbf{b}/\mu_o\}$
$\mathcal{L} = (1/2)\{\varepsilon_o\mathbf{e}\cdot\mathbf{e} - \mathbf{b}\cdot\mathbf{b}/\mu_o\} = (-1/4\mu_o)F_{\mu\nu}F^{\mu\nu}$
$\mathcal{H} + \mathcal{L} = \varepsilon_o\mathbf{e}\cdot\mathbf{e} = (\mathbf{g_T}\cdot\mathbf{u})$
$|\mathbf{u}| = c$
$|\mathbf{g_T}| = \varepsilon_o\mathbf{e}\cdot\mathbf{e}/c$
Poynting Vector $|\mathbf{s}| = |\mathbf{g}|c^2 \rightarrow c\varepsilon_o\mathbf{e}\cdot\mathbf{e}$

$H_o + L_o = 0$ Calculating the Rest Values

$H_o = (\mathbf{P_T}\cdot\mathbf{U})$ $\quad H = \gamma H_o$
$L_o = -(\mathbf{P_T}\cdot\mathbf{U})$ $\quad L = L_o/\gamma$

4-Vector notation gives a very nice way to find the Hamiltonian/Lagrangian connection:
$(H) + (L) = (\mathbf{p_T}\cdot\mathbf{u})$, where $H = \gamma(\mathbf{P_T}\cdot\mathbf{U})$ & $L = -(\mathbf{P_T}\cdot\mathbf{U})/\gamma$

SRQM Study:

SR Lagrangian, Lagrangian Density, and Relativistic Action (S)

Relativistic Action (S) is Lorentz Scalar Invariant

$S = \int L dt = \int (L_o/\gamma)(\gamma d\tau) = \int (L_o)(d\tau)$

$S = \int L dt = \int (\mathcal{L}/n) dt = \int \mathcal{L}/(n) dt = \int \mathcal{L}(d^3x) dt = \int (\mathcal{L}/c)(d^3x)(cdt) = \int (\mathcal{L}/c)(d^4x)$

Explicitly-Covariant Relativistic Action (S)

Particle Form	Density Form {= n_o*Particle}
$S = \int L_o d\tau = -\int H_o d\tau$	$S = (1/c)\int (n_o L_o)(d^4x) = -(1/c)\int (n_o H_o)(d^4x)$
$S = -\int (\mathbf{P_T}\cdot\mathbf{U}) d\tau$	$S = (1/c)\int (\mathcal{L})(d^4x)$
$S = -\int (\mathbf{P_T}\cdot\mathbf{dR}/d\tau) d\tau$	
$S = -\int (\mathbf{P_T}\cdot\mathbf{dR})$	$S = \int (\mathcal{L}/c)(d^4x)$
$S = -\int (\mathbf{P_T}\cdot\mathbf{U}) d\tau$	$S = -(1/c)\int n_o(\mathbf{P_T}\cdot\mathbf{U})(d^4x)$
$S = -\int ((\mathbf{P} + q\mathbf{A})\cdot\mathbf{U}) d\tau$`	$S = -(1/c)\int n_o((\mathbf{P} + q\mathbf{A})\cdot\mathbf{U})(d^4x)$
$S = -\int (\mathbf{P}\cdot\mathbf{U} + q\mathbf{A}\cdot\mathbf{U}) d\tau$	$S = -(1/c)\int (n_o\mathbf{P}\cdot\mathbf{U} + n_o q\mathbf{A}\cdot\mathbf{U})(d^4x)$
$S = -\int (E_o + q\mathbf{U}\cdot\mathbf{A}) d\tau$	$S = -(1/c)\int (n_o E_o + n_o q\mathbf{U}\cdot\mathbf{A})(d^4x)$
$S = -\int (E_o + q\varphi_o) d\tau$	$S = -(1/c)\int (\rho_{E^o} + \mathbf{J}\cdot\mathbf{A})(d^4x)$
$S = -\int (E_o + V) d\tau$	
$S = -\int (m_o c^2 + V) d\tau$	$S = (1/c)\int (\mathcal{L})(d^4x)$
	$S = (1/c)\int ((1/2)\{\varepsilon_o\mathbf{e}\cdot\mathbf{e} - \mathbf{b}\cdot\mathbf{b}/\mu_o\})(d^4x)$
with $V = q\varphi_o$	$S = (1/c)\int ((-1/4\mu_o)F_{\mu\nu}F^{\mu\nu})(d^4x)$
	for an EM field = no rest frame

Lagrangian {$L = (\mathbf{p_T}\cdot\mathbf{u}) - H$} is *not* Lorentz Scalar Invariant

Rest Lagrangian {$L_o = \gamma L = -(\mathbf{P_T}\cdot\mathbf{U})$} is Lorentz Scalar Invariant

Lagrangian Density {$\mathcal{L} = nL = (\gamma n_o)(L_o/\gamma) = n_o L_o$} is Lorentz Scalar Invariant

$n = \gamma n_o = \#/d^3x = \#/(dx)(dy)(dz)$ = number density

$dt = \gamma d\tau$

$cd\tau = n_o(cdt)(dx)(dy)(dz) = n_o(d^4x)$

$d\tau = (n_o/c)(d^4x)$

H:L Connection in Density Format for Photonic System (no rest-frame)

$H + L = (\mathbf{p_T}\cdot\mathbf{u})$

$nH + nL = n(\mathbf{p_T}\cdot\mathbf{u})$, with number density $n = \gamma n_o$

$\mathcal{H} + \mathcal{L} = (\mathbf{g_T}\cdot\mathbf{u})$, with

momentum density {$\mathbf{g_T} = n\mathbf{p_T}$}

Hamiltonian density {$\mathcal{H} = nH$}

Lagrangian Density {$\mathcal{L} = nL = (\gamma n_o)(L_o/\gamma) = n_o L_o$}

Lagrangian Density is Lorentz Scalar

for an EM field (photonic):

$\mathcal{H} = (1/2)\{\varepsilon_o\mathbf{e}\cdot\mathbf{e} + \mathbf{b}\cdot\mathbf{b}/\mu_o\} = n_o E_o = \rho_{E^o}$ = EM Field Energy Density

$\mathcal{L} = (1/2)\{\varepsilon_o\mathbf{e}\cdot\mathbf{e} - \mathbf{b}\cdot\mathbf{b}/\mu_o\} = (-1/4\mu_o)F_{\mu\nu}F^{\mu\nu} = (-1/4\mu_o)$*Faraday EM Tensor Inner Product

$\mathcal{H} + \mathcal{L} = \varepsilon_o\mathbf{e}\cdot\mathbf{e} = (\mathbf{g_T}\cdot\mathbf{u})$

$|\mathbf{u}| = c$

$|\mathbf{g_T}| = \varepsilon_o\mathbf{e}\cdot\mathbf{e}/c$

Poynting Vector $|\mathbf{s}| = |\mathbf{g}|c^2 \rightarrow c\varepsilon_o\mathbf{e}\cdot\mathbf{e}$

$\varepsilon_o\mu_o = 1/c^2$:Electric:Magnetic Constant Eqn

The Relativistic Action Equation is seen in many different formats

SRQM Study:
SR Hamilton-Jacobi Equation and Relativistic Action (S)

A Tensor Study of Physical 4-Vectors
SciRealm.org
John B. Wilson

Lagrangian $\{L = (\mathbf{p_T}\cdot\mathbf{u}) - H\}$ is *not* a Lorentz Scalar
Rest Lagrangian $\{L_o = \gamma L = -(\mathbf{P_T}\cdot\mathbf{U})\}$ is a Lorentz Scalar

Relativistic Action (S) is Lorentz Scalar
$S = \int L dt$
$S = \int (L_o/\gamma)(\gamma d\tau)$
$S = \int (L_o)(d\tau)$

Explicitly Covariant
Relativistic Action (S)
$S = \int L_o d\tau = -\int H_o d\tau$
$S = -\int(\mathbf{P_T}\cdot\mathbf{U})d\tau$
$S = -\int(\mathbf{P_T}\cdot\mathbf{dR}/d\tau)d\tau$
$S = -\int(\mathbf{P_T}\cdot\mathbf{dR})$
$S = -\int(\mathbf{P_T}\cdot\mathbf{U})d\tau$
$S = -\int((\mathbf{P} + q\mathbf{A})\cdot\mathbf{U})d\tau$
$S = -\int(\mathbf{P}\cdot\mathbf{U} + q\mathbf{A}\cdot\mathbf{U})d\tau$
$S = -\int(E_o + q\varphi_o)d\tau$
$S = -\int(E_o + V)d\tau$ with $V = q\varphi_o$
$S = -\int(m_o c^2 + V)d\tau$
$S = -\int(H_o)d\tau$

4-Scalars
Relativistic Action Eqn
Integral Format

$S_{action} = -\int[\mathbf{P_T}\cdot\mathbf{dR}]$
$= -\int[\mathbf{P_T}\cdot\mathbf{U}]d\tau$
$= -\int[(H/c,\mathbf{p_T})\cdot\gamma(c,\mathbf{u})]d\tau$
$= -\int[\gamma(H-\mathbf{p_T}\cdot\mathbf{u}]d\tau$

Inverse

4-Vectors
Relativistic Hamilton-Jacobi Eqn
Differential Format

4-TotalMomentum
$\mathbf{P_T} = (E_T/c,\mathbf{p_T}) = (H/c,\mathbf{p_T})$
$\mathbf{P_T} = -\partial[S_{action}]$
$(H/c,\mathbf{p_T}) = (-\partial_t/c[S_{action}],\nabla[S_{action}])$

Hamilton-Jacobi Equation
$\partial[-S] = -\partial[S] = \mathbf{P_T}$

$S = -\int(E_o + q\varphi_o)d\tau$
$S = -(E_o + q\varphi_o)\int d\tau$
$S = -(E_o + q\varphi_o)(\tau + \text{const})$

$-S = (E_o + q\varphi_o)(\tau + \text{const})$
$\partial[-S] = (E_o + q\varphi_o)\partial[(\tau + \text{const})]$
$\partial[-S] = (E_o + q\varphi_o)\partial[\tau]$
$\partial[-S] = (E_o + q\varphi_o)\partial[\mathbf{R}\cdot\mathbf{U}/c^2]$
$\partial[-S] = ((E_o + q\varphi_o)/c^2)\partial[\mathbf{R}\cdot\mathbf{U}]$
$\partial[-S] = (E_o/c^2 + q\varphi_o/c^2)\mathbf{U}$
$\partial[-S] = (m_o + q\varphi_o/c^2)\mathbf{U}$
$\partial[-S] = m_o\mathbf{U} + q(\varphi_o/c^2)\mathbf{U}$
$\partial[-S] = \mathbf{P} + q\mathbf{A}$
$\partial[-S] = \mathbf{P_T}$
Verified!

$\mathbf{R}\cdot\mathbf{U} = c^2\tau : \tau = \mathbf{R}\cdot\mathbf{U}/c^2$

The Hamilton-Jacobi Equation is incredibly simple in 4-Vector form

SR → QM

4-Vector SRQM Interpretation of QM

SRQM Diagram:
Relativistic Hamilton-Jacobi Equation
($P_T = -\partial[S]$) Differential Format : 4-Vectors

A Tensor Study of Physical 4-Vectors

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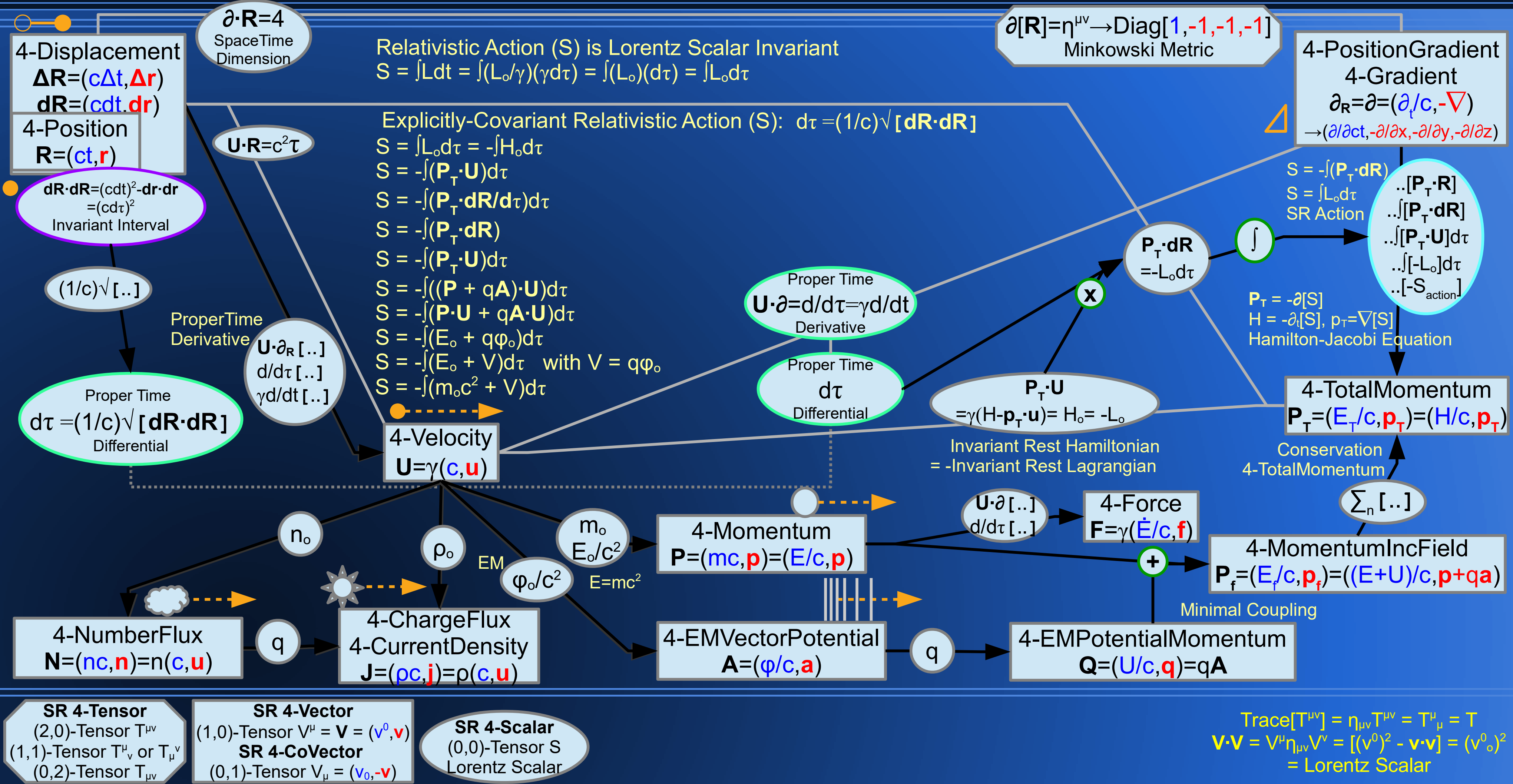

SR → QM

4-Vector SRQM Interpretation of QM

SRQM Diagram:

Relativistic Action Equation

$(S = -\int(P_T \cdot dR))$ Integral Format : 4-Scalars

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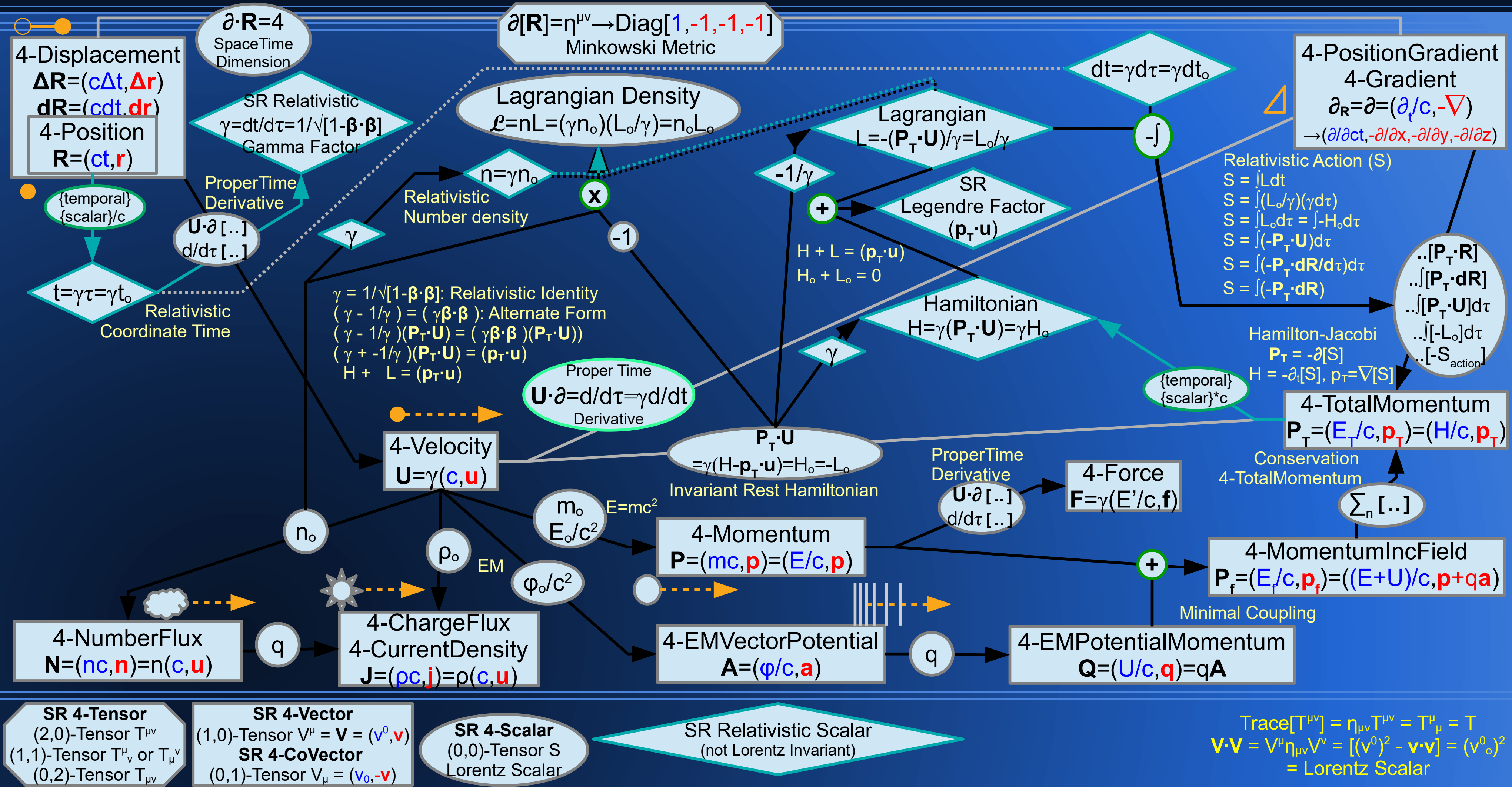

SR → QM

4-Vector SRQM Interpretation of QM

SRQM Diagram: Relativistic Factors
Hamiltonian & Lagrangian
Relativistic Euler-Lagrange Equation

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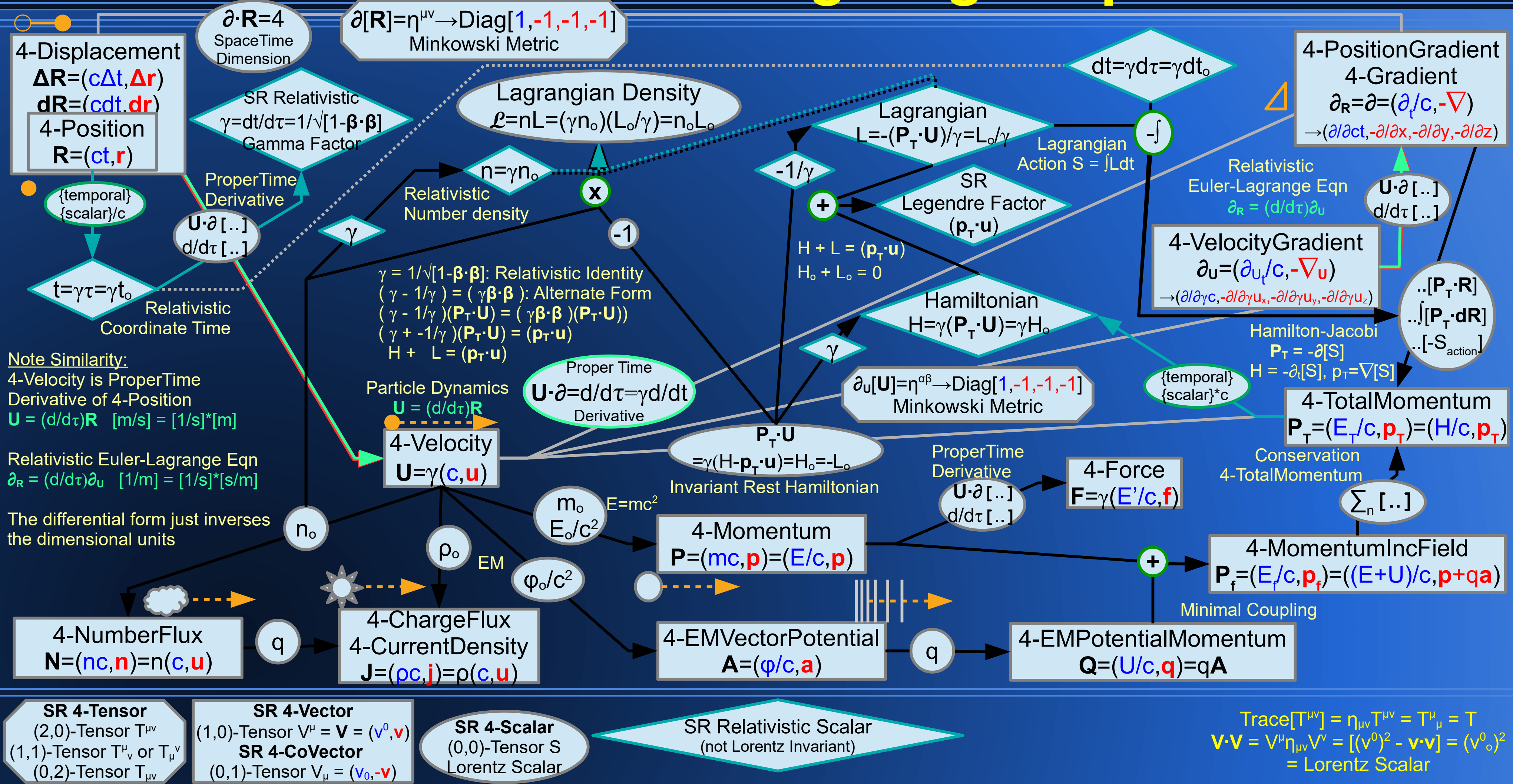

SR → QM

4-Vector SRQM Interpretation of QM

SRQM Diagram:
Relativistic Euler-Lagrange Equation
The Easy Derivation $(\mathbf{U}=(d/d\tau)\mathbf{R}) \rightarrow (\partial_{\mathbf{R}}=(d/d\tau)\partial_{\mathbf{U}})$

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Note Similarity:

4-Velocity is ProperTime Derivative of 4-Position
$\mathbf{U} = (d/d\tau)\mathbf{R}$ [m/s] = [1/s]*[m]

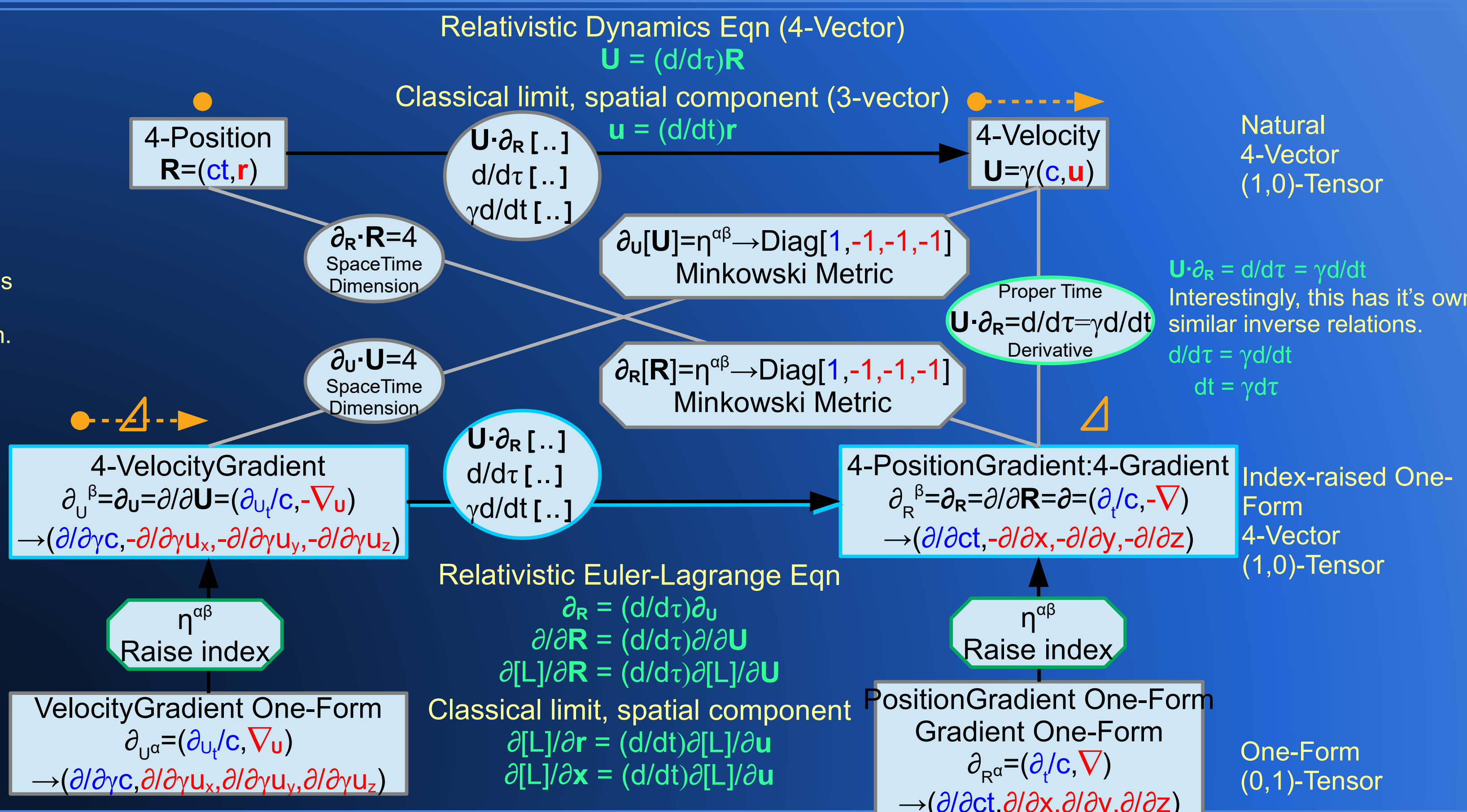
Relativistic Euler-Lagrange Eqn
$\partial_{\mathbf{R}} = (d/d\tau)\partial_{\mathbf{U}}$ [1/m] = [1/s]*[s/m]

The differential form just inverses the dimensional units, so the placement of the **R** and **U** switch.

That is it: so simple!
Much, much easier than how I was taught in Grad School.

To complete the process and create the Equations of Motion, one just applies the base form to a Lagrangian.

This can be:
a classical Lagrangian
a relativistic Lagrangian
a Lorentz scalar Lagrangian
a quantum Lagrangian



SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^{\mu}{}_{\nu}$ or $T_{\mu}{}^{\nu}$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^{\mu} = \mathbf{V} = (v^0,\mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_{\mu} = (v_0,-\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^{\mu}{}_{\mu} = T$
$\mathbf{V}\cdot\mathbf{V} = V^{\mu}\eta_{\mu\nu}V^{\nu} = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$
= Lorentz Scalar

SR → QM

4-Vector SRQM Interpretation of QM

SRQM Diagram:

Relativistic Euler-Lagrange Equation Alternate Forms: Particle vs. Density

A Tensor Study of Physical 4-Vectors

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4-Velocity **U** is ProperTime Derivative of 4-Position **R**. The Euler-Lagrange Eqn can be generated by taking the differential form of the same equation.

Relativistic 4-Vector Kinematical Eqn

$\mathbf{U} = (d/d\tau)\mathbf{R}$

$\mathbf{U}\cdot\mathbf{K} = (d/d\tau)\mathbf{R}\cdot\mathbf{K}$

Relativistic Euler-Lagrange Eqns

{uses gradient-type 4-Vectors}

$\partial_{\mathbf{R}} = (d/d\tau)\partial_{\mathbf{U}}$: {particle format}

$\partial_{\mathbf{R}\cdot\mathbf{K}} = (d/d\tau)\ \partial_{\mathbf{U}\cdot\mathbf{K}}$

$\partial_{(-\Phi)} = (d/d\tau)\ \partial_{\mathbf{U}\cdot\mathbf{K}}$

$\partial_{(-\Phi)} = (\mathbf{U}\cdot\partial_{\mathbf{R}})\ \partial_{\mathbf{U}\cdot\mathbf{K}}$

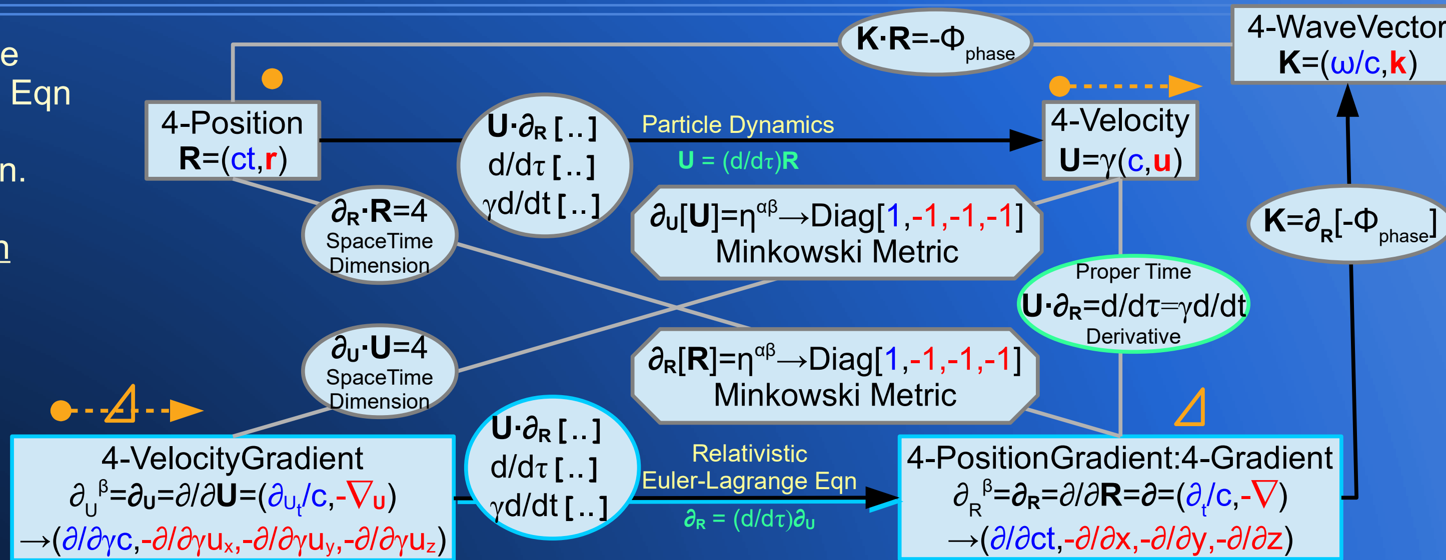
$\partial/\partial(-\Phi) = (\mathbf{U}\cdot\partial_{\mathbf{R}})\ \partial/\partial[\mathbf{U}\cdot\mathbf{K}]$

$\partial/\partial(-\Phi) = (\partial_{\mathbf{R}})\ \partial/\partial[\mathbf{K}]$

$\partial/\partial(-\Phi) = (\partial_{\mathbf{R}})\ \partial/\partial[\partial_{\mathbf{R}}(-\Phi)]$

$\partial/\partial(\Phi) = (\partial_{\mathbf{R}})\ \partial/\partial[\partial_{\mathbf{R}}(\Phi)]$

$\partial_{[\Phi]} = (\partial_{\mathbf{R}})\ \partial_{[\partial_{\mathbf{R}}(\Phi)]}$: {density format}



$\mathcal{L} = (1/2)\{\ \partial_{\mathbf{R}}[\Phi]\cdot\partial_{\mathbf{R}}[\Phi] - (m_o c/\hbar)^2\ \Phi^2\ \}$: KG Lagrangian Density

$\partial_{[\Phi]}\ \mathcal{L} = (\partial_{\mathbf{R}})\ \partial_{[\partial_{\mathbf{R}}(\Phi)]}\ \mathcal{L}$: Euler-Lagrange Eqn {density format}

$-(m_o c/\hbar)^2\ \Phi = (\partial_{\mathbf{R}})\cdot\partial_{\mathbf{R}}[\Phi]$

$(\partial_{\mathbf{R}}\cdot\partial_{\mathbf{R}})[\Phi] = -\ (m_o c/\hbar)^2\ \Phi$

$(\partial\cdot\partial) = -\ (m_o c/\hbar)^2$: KG Eqn of Motion

Klein-Gordon Relativistic Quantum Wave Eqn

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^{\mu}{}_{\nu}$ or $T_{\mu}{}^{\nu}$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^{\mu} = \mathbf{V} = (v^0,\mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_{\mu} = (v_0,-\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^{\mu}{}_{\mu} = T$

$\mathbf{V}\cdot\mathbf{V} = V^{\mu}\eta_{\mu\nu}V^{\nu} = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$

= Lorentz Scalar

SRQM Diagram:
Relativistic Euler-Lagrange Equation Equation of Motion (EoM) for EM particle

A Tensor Study of Physical 4-Vectors

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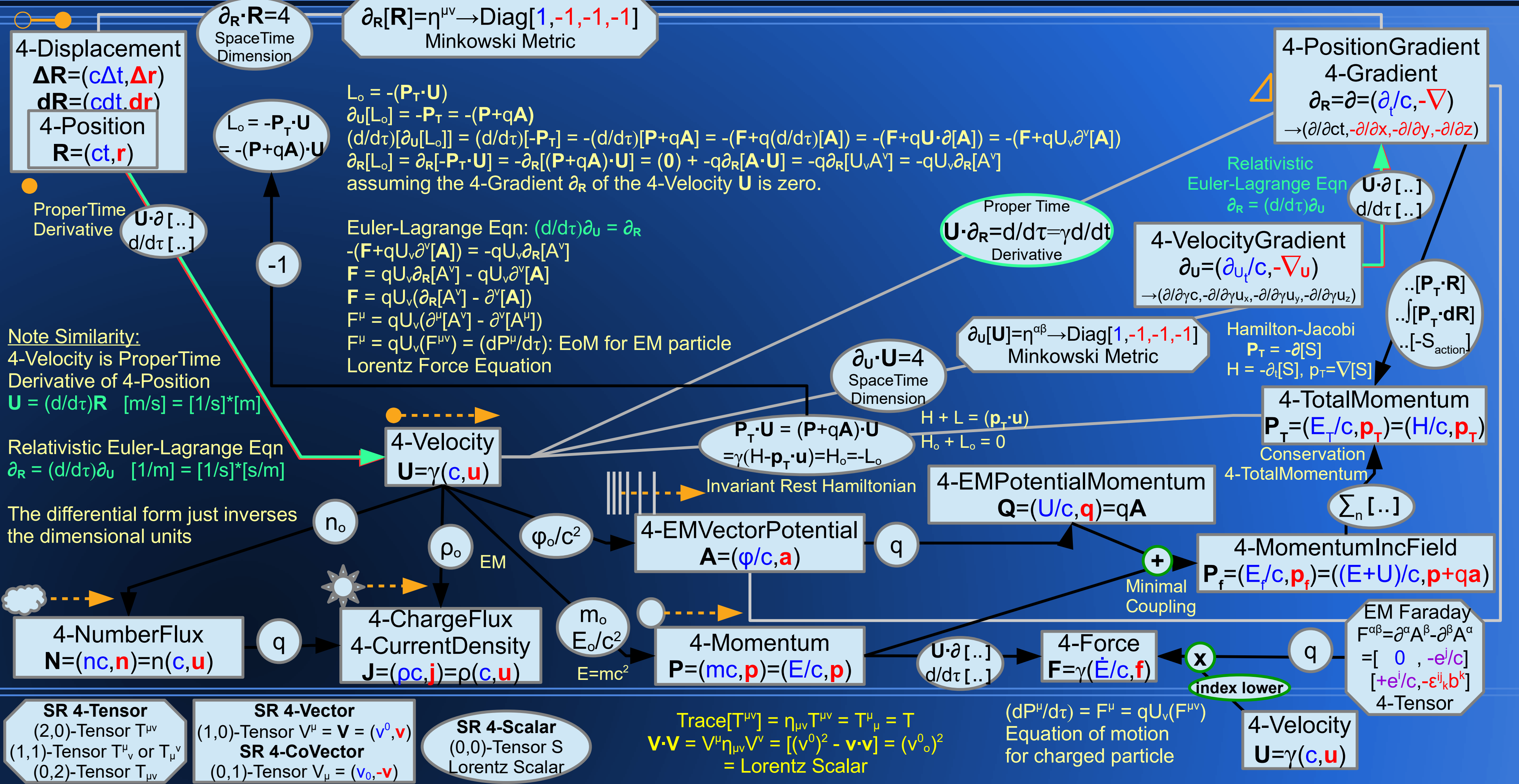

SRQM Diagram:
Relativistic Euler-Lagrange Equation
Equation of Motion (EoM) for EM particle

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$\gamma = 1/\mathrm{Sqrt}[1-\boldsymbol{\beta}\cdot\boldsymbol{\beta}]$: Relativistic Gamma Identity
$(\gamma - 1/\gamma) = (\gamma\boldsymbol{\beta}\cdot\boldsymbol{\beta})$: Manipulate into this form… still an identity
$\gamma(\mathbf{P_T}\cdot\mathbf{U}) + -(\mathbf{P_T}\cdot\mathbf{U})/\gamma = (\gamma\boldsymbol{\beta}\cdot\boldsymbol{\beta})(\mathbf{P_T}\cdot\mathbf{U})$
$\gamma(\mathbf{P_T}\cdot\mathbf{U}) + -(\mathbf{P_T}\cdot\mathbf{U})/\gamma = (\mathbf{p_T}\cdot\mathbf{u})$
$\{ \quad H \quad \} + \{ \quad L \quad \} = (\mathbf{p_T}\cdot\mathbf{u})$: The Hamiltonian/Lagrangian connection

$H = \gamma H_o = \gamma(\mathbf{P_T}\cdot\mathbf{U}) = \gamma((\mathbf{P}+q\mathbf{A})\cdot\mathbf{U})$ = The Hamiltonian with minimal coupling
$L = L_o/\gamma = -(\mathbf{P_T}\cdot\mathbf{U})/\gamma = -((\mathbf{P}+q\mathbf{A})\cdot\mathbf{U})/\gamma$ = The Lagrangian with minimal coupling

$H_o = (\mathbf{P_T}\cdot\mathbf{U}) = -L_o = (\mathbf{U}\cdot\mathbf{P_T})$: Rest Hamiltonian = Total RestEnergy
$L_o = -(\mathbf{P_T}\cdot\mathbf{U}) = -H_o$

$(d/d\tau)\boldsymbol{\partial}_\mathbf{U}[L_o] = \boldsymbol{\partial}_\mathbf{R}[L_o]$

4-Velocity is ProperTime Derivative of 4-Position
$\mathbf{U} = (d/d\tau)\mathbf{R}$ $[m/s] = [1/s]*[m]$

Relativistic Euler-Lagrange Eqn
$\boldsymbol{\partial}_\mathbf{R} = (d/d\tau)\boldsymbol{\partial}_\mathbf{U}$ $[1/m] = [1/s]*[s/m]$

$\partial/\partial\mathbf{R} = (d/d\tau)\partial/\partial\mathbf{U}$
$\partial[L]/\partial\mathbf{R} = (d/d\tau)\partial[L]/\partial\mathbf{U}$

Classical limit, spatial component
$\partial[L]/\partial\mathbf{r} = (d/dt)\partial[L]/\partial\mathbf{u}$
$\partial[L]/\partial\mathbf{x} = (d/dt)\partial[L]/\partial\mathbf{u}$

$\mathbf{F_{EM}} = \gamma q\{ (\mathbf{u}\cdot\mathbf{e})/c, (\mathbf{e}) + (\mathbf{u}\times\mathbf{b}) \}$
$\mathbf{e} = (-\nabla\varphi - \partial_t\mathbf{a})$ and $\mathbf{b} = [\nabla\times \mathbf{a}]$

If $\mathbf{a}$~0, then $\mathbf{f} = -q\nabla\varphi = -\nabla U$, the force is neg grad of a potential

4-TotalMomentum
$\mathbf{P_T}=(E_T/c,\mathbf{p_T})=(H/c,\mathbf{p_T})$

Rest Lagrangian L_o
$= -(\mathbf{P_T}\cdot\mathbf{U})$
$= -(\mathbf{P}+q\mathbf{A})\cdot\mathbf{U}$
$= -\mathbf{P}\cdot\mathbf{U}-q\mathbf{A}\cdot\mathbf{U}$

4-Velocity
$\mathbf{U}=\gamma(c,\mathbf{u})$

ProperTime
$\mathbf{U}\cdot\boldsymbol{\partial}=d/d\tau=\gamma d/dt$
Derivative

X

$(d/d\tau)\boldsymbol{\partial}_\mathbf{U}[L_o]$
$= (d/d\tau)[-\mathbf{P_T}]$
$= -(d/d\tau)[\mathbf{P}+q\mathbf{A}]$
$= -(\mathbf{F}+q(d/d\tau)[\mathbf{A}])$
$= -(\mathbf{F}+q\mathbf{U}\cdot\boldsymbol{\partial}[\mathbf{A}])$
$= -(F^\alpha+qU_\beta\partial^\beta[A^\alpha])$

Relativistic Rest Lagrangian Euler-Lagrange Equations of Motion

$(d/d\tau)\boldsymbol{\partial}_\mathbf{U}[L_o] = \boldsymbol{\partial}_\mathbf{R}[L_o]$

$\boldsymbol{\partial}_\mathbf{R}[L_o]$
$= \boldsymbol{\partial}_\mathbf{R}[-\mathbf{P_T}\cdot\mathbf{U}]$
$= -\boldsymbol{\partial}_\mathbf{R}[(\mathbf{P}+q\mathbf{A})\cdot\mathbf{U}]$
$= (\mathbf{0}) + -q\boldsymbol{\partial}_\mathbf{R}[\mathbf{A}\cdot\mathbf{U}]$
$= -q\boldsymbol{\partial}_\mathbf{R}[U_\beta A^\beta]$
$= -qU_\beta\partial^\alpha[A^\beta]$

$-(F^\alpha+qU_\beta\partial^\beta[A^\alpha]) = -qU_\beta\partial^\alpha[A^\beta]$
$(F^\alpha+qU_\beta\partial^\beta[A^\alpha]) = qU_\beta\partial^\alpha[A^\beta]$
$F^\alpha = qU_\beta\partial^\alpha[A^\beta] - qU_\beta\partial^\beta[A^\alpha]$
$F^\alpha = qU_\beta(\partial^\alpha[A^\beta] - \partial^\beta[A^\alpha])$
$F^\alpha = qU_\beta(F^{\alpha\beta})$
Lorentz Force Equation

SRQM Diagram:
Relativistic Hamilton's Equations
Equation of Motion (EoM) for EM particle

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John B. Wilson

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$\gamma(\mathbf{P_T}\cdot\mathbf{U}) + -(\mathbf{P_T}\cdot\mathbf{U})/\gamma = (\gamma\boldsymbol{\beta}\cdot\boldsymbol{\beta})(\mathbf{P_T}\cdot\mathbf{U})$
$\gamma(\mathbf{P_T}\cdot\mathbf{U}) + -(\mathbf{P_T}\cdot\mathbf{U})/\gamma = (\mathbf{p_T}\cdot\mathbf{u})$
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$L = L_o/\gamma = -(\mathbf{P_T}\cdot\mathbf{U})/\gamma = -((\mathbf{P}+q\mathbf{A})\cdot\mathbf{U})/\gamma$ = The Lagrangian with minimal coupling

$H_o = (\mathbf{P_T}\cdot\mathbf{U}) = -L_o = (\mathbf{U}\cdot\mathbf{P_T})$: Rest Hamiltonian = Total RestEnergy
$L_o = -(\mathbf{P_T}\cdot\mathbf{U}) = -H_o$

$\partial_{\mathbf{P_T}}[H_o] = \partial_{\mathbf{P_T}}[\mathbf{U}\cdot\mathbf{P_T}] = \partial_{\mathbf{P_T}}[\mathbf{U}]\cdot\mathbf{P_T} + \mathbf{U}\cdot\partial_{\mathbf{P_T}}[\mathbf{P_T}] = 0 + \mathbf{U}\cdot\partial_{\mathbf{P_T}}[\mathbf{P_T}] = \mathbf{U} = d/d\tau[\mathbf{X}]$
Thus: $(d/d\tau)[\mathbf{X}] = (\partial/\partial\mathbf{P_T})[H_o]$
$\partial_{\mathbf{X}}[H_o] = \partial_{\mathbf{X}}[\mathbf{U}\cdot\mathbf{P_T}] = \partial_{\mathbf{X}}[\mathbf{U}]\cdot\mathbf{P_T} + \mathbf{U}\cdot\partial_{\mathbf{X}}[\mathbf{P_T}] = 0 + \mathbf{U}\cdot\partial_{\mathbf{X}}[\mathbf{P_T}] = d/d\tau[\mathbf{P_T}]$
Thus: $(d/d\tau)[\mathbf{P_T}] = (\partial/\partial\mathbf{X})[H_o]$

Relativistic Hamilton's Equations (4-Vector):
$(d/d\tau)[\mathbf{X}] = (\partial/\partial\mathbf{P_T})[H_o]$
$(d/d\tau)[\mathbf{P_T}] = (\partial/\partial\mathbf{X})[H_o]$

$(d/d\tau)[\mathbf{X}] = \gamma(d/dt)[\mathbf{X}] = (\partial/\partial\mathbf{P_T})[H_o] = (\partial/\partial\mathbf{P_T})[(\mathbf{P_T}\cdot\mathbf{U})] = \mathbf{U}$
$(d/d\tau)[\mathbf{P_T}] = \gamma(d/dt)[\mathbf{P_T}] = (\partial/\partial\mathbf{X})[H_o] = (\partial/\partial\mathbf{X})[(\mathbf{P_T}\cdot\mathbf{U})] = (\partial/\partial\mathbf{X})[\gamma(H-\mathbf{p_T}\cdot\mathbf{u})]$

Taking just the spatial components:
$\gamma(d/dt)[\mathbf{x}] = (-\partial/\partial\mathbf{p_T})[H_o] = (-\partial/\partial\mathbf{p_T})[H/\gamma]$ {hard}
$\gamma(d/dt)[\mathbf{p_T}] = (-\partial/\partial\mathbf{x})[H_o] = (-\partial/\partial\mathbf{x})[H/\gamma]$ {easy because $(\partial/\partial\mathbf{x})[\gamma]=0$}

$\gamma^2(d/dt)[\mathbf{p_T}] = (-\partial/\partial\mathbf{x})[H]$
Take the Classical limit $\{\gamma\to 1\}$

Classical Hamilton's Equations (3-vector):
$(d/dt)[\mathbf{x}] = (+\partial/\partial\mathbf{p_T})[H]$
$(d/dt)[\mathbf{p_T}] = (-\partial/\partial\mathbf{x})[H]$

Sign-flip difference is interaction of $(-\partial/\partial\mathbf{p_T})$ with $[1/\gamma]$

4-TotalMomentum
$\mathbf{P_T}=(E_T/c,\mathbf{p_T})=(H/c,\mathbf{p_T})$

Rest Hamiltonian H_o
$= (\mathbf{P_T}\cdot\mathbf{U})$
$= (\mathbf{P}+q\mathbf{A})\cdot\mathbf{U}$
$= \mathbf{P}\cdot\mathbf{U}+q\mathbf{A}\cdot\mathbf{U}$

4-Velocity
$\mathbf{U}=\gamma(c,\mathbf{u})$

4-Position
$\mathbf{X}=(ct,\mathbf{x})$

$(d/d\tau)[\mathbf{X}]$
$= \mathbf{U}=\gamma(c,\mathbf{u})$
= 4-Velocity
$= \mathbf{P}/m_o$
$= (\mathbf{P_T}-q\mathbf{A})/m_o$

$(\partial/\partial\mathbf{P_T})[H_o]$
$= (\partial/\partial\mathbf{P_T})[\mathbf{P_T}\cdot\mathbf{U}]$
$= \mathbf{U}=\gamma(c,\mathbf{u})$
= 4-Velocity
$= \mathbf{P}/m_o$
$= (\mathbf{P_T}-q\mathbf{A})/m_o$

Relativistic Rest Hamiltonian
Hamilton's
Equations of Motion

$(d/d\tau)[\mathbf{X}] = (\partial/\partial\mathbf{P_T})[H_o]$

$(d/d\tau)[\mathbf{P_T}] = (\partial/\partial\mathbf{X})[H_o]$

$(d/d\tau)[\mathbf{P_T}]$
$= (d/d\tau)[\mathbf{P}+q\mathbf{A}]$
$= [\mathbf{F} + q(d/d\tau)\mathbf{A}]$
$= [\mathbf{F} + q(\mathbf{U}\cdot\partial)\mathbf{A}]$
$= [F^\alpha + q(U_\beta\partial^\beta)A^\alpha]$

$(\partial/\partial\mathbf{X})[H_o]$
$= (\partial/\partial\mathbf{X})[\mathbf{P}\cdot\mathbf{U}+q\mathbf{A}\cdot\mathbf{U}]$
$= [0 + q(\partial\mathbf{A}/\partial\mathbf{X})\cdot\mathbf{U}]$
$= [q\partial[\mathbf{A}]\cdot\mathbf{U}]$
$= q\partial[\mathbf{A}]\cdot\mathbf{U}$
$= q\partial^\alpha[A^\beta]U_\beta$
$= q(\partial[\mathbf{A}]\cdot(\mathbf{P_T}-q\mathbf{A})/m_o$

$[F^\alpha + q(U_\beta\partial^\beta)A^\alpha] = q(\partial^\alpha[A^\beta]U_\beta$
$F^\alpha = q(\partial^\alpha[A^\beta]U_\beta - q(U_\beta\partial^\beta)A^\alpha$
$F^\alpha = q(\partial^\alpha[A^\beta] - \partial^\beta A^\alpha)U_\beta$
$F^\alpha = q(F^{\alpha\beta})U_\beta$
Lorentz Force Equation

SRQM Diagram:
EM Lorentz Force Eqn
→ Force = - Grad[Potential]

A Tensor Study of Physical 4-Vectors

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Lorentz EM Force Equation:

$F^\alpha = q(F^{\alpha\beta})U_\beta$

$F^\alpha = q(\partial^\alpha A^\beta - \partial^\beta A^\alpha)U_\beta$

Examine just the spatial components of 4-Force **F**:

$F^i = q(\partial^i A^\beta - \partial^\beta A^i)U_\beta$

$F^i = q(\partial^i A^0 - \partial^0 A^i)U_0 + q(\partial^i A^j - \partial^j A^i)U_j$

$\gamma\mathbf{f} = q(-\nabla[\varphi/c] - (\partial^t/c)\mathbf{a})(\gamma c) + q(-\nabla[\mathbf{a}\cdot\mathbf{u}] - -\mathbf{u}\cdot\nabla[\mathbf{a}])\gamma$

$\mathbf{f} = q(-\nabla[\varphi/c] - (\partial^t/c)\mathbf{a})(c) + q(\mathbf{u}\cdot\nabla[\mathbf{a}] - \nabla[\mathbf{a}\cdot\mathbf{u}])$

$\mathbf{f} = q(-\nabla[\varphi] - \partial^t\mathbf{a} + \mathbf{u}\cdot\nabla[\mathbf{a}] - \nabla[\mathbf{a}\cdot\mathbf{u}])$

$\mathbf{f} = q(-\nabla[\varphi] - \partial^t\mathbf{a} + \mathbf{u} \times \mathbf{b})$

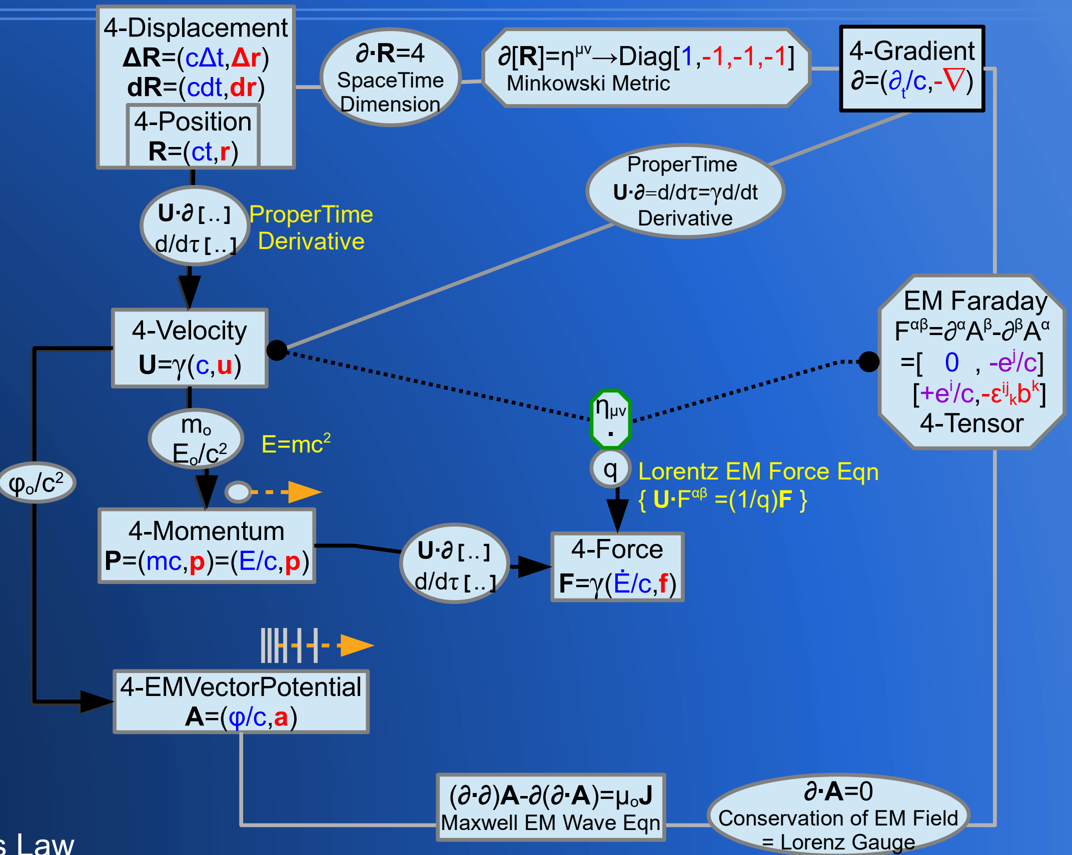
Take the limit of $\{|\nabla[\varphi]| >> |\partial^t\mathbf{a}| + |\mathbf{u} \times \mathbf{b}|\}$

$\mathbf{f} \sim q(-\nabla[\varphi]) = -\nabla[q\varphi] = -\nabla[U]$

The Classical Force = -Grad[Potential]
when $\{|\nabla[\varphi]| >> |\partial^t\mathbf{a}| + |\mathbf{u} \times \mathbf{b}|\}$ or when $\{\mathbf{a} = \mathbf{0}\}$

The majority of non-gravity, non-nuclear potentials dealt with in CM are those mediated by the EM potential.

ex. Spring Potential $\{ U = kx^2/2 \}$, then $\{ \mathbf{f} = -\nabla[kx^2/2] = -k\mathbf{x} \}$ Hooke's Law



SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
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(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0,\mathbf{v})$
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SR 4-Scalar
(0,0)-Tensor S
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= Lorentz Scalar

SR → QM

4-Vector SRQM Interpretation of QM

SRQM: The Speed-of-Light (c)

c^2 Invariant Relations (part 1)

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson

The Speed-of-Light (c) is THE connection between Time and Space: $d\mathbf{R} = (cdt, d\mathbf{r})$

This physical constant appears in several seemingly unrelated places. You don't notice these cool relations when you set c→1.
Also notice that the set of all these relations definitely rules out a variable speed-of-light.
(c) is an Invariant Lorentz Scalar constant.

$\mathbf{U}\cdot\mathbf{U} = \gamma^2(c^2-\mathbf{u}\cdot\mathbf{u}) = c^2$ — Speed of all things into the Future

$(E_o/m_o)=(\gamma E_o/\gamma m_o)=(E/m) = c^2$ — Mass is concentrated Energy, $E = mc^2$

$|u * v_{phase}|=|v_{group} * v_{phase}| = c^2$ — Particle-Wave "Duality" Correlation

$\lambda\!\!\!^-{}^2(\omega^2-\omega_o{}^2) = \lambda^2(f^2-f_o{}^2) = c^2$ — Wavelength-Frequency Relation: $\lambda f = c$ for photons

$(1/\varepsilon_o\mu_o) = c^2$ — Electric (ε_o) and Magnetic (μ_o) EM Field Constants

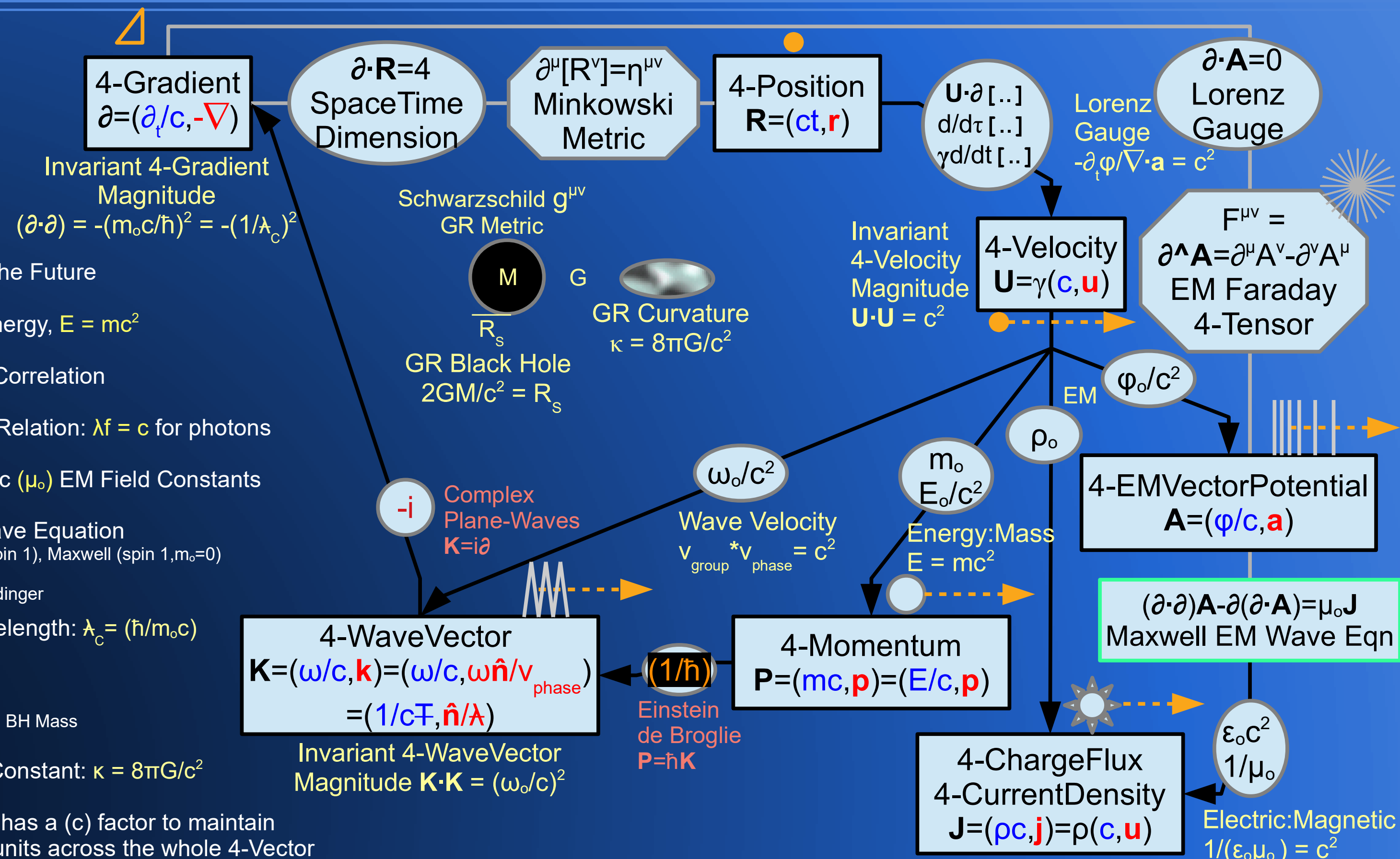
$-(\hbar/m_o)^2(\partial\cdot\partial) = c^2$ — Relativistic Quantum Wave Equation
Klein-Gordon (spin 0), Proca (spin 1), Maxwell (spin 1, m_o=0)
Factors to Dirac (spin ½)
Classical-limit (|**v**|<<c) to Schrödinger

$(\hbar/\lambda\!\!\!^-_C m_o)^2 = c^2$ — Reduced Compton Wavelength: $\lambda\!\!\!^-_C = (\hbar/m_oc)$

$2GM/R_S = c^2$ — GR Black Hole Equation
R_S = Schwarzschild Radius
G = GR GravitationalConst, M = BH Mass

$8\pi G/\kappa = c^2$ — GR Einstein Curvature Constant: $\kappa = 8\pi G/c^2$

($c^{\pm1}$ * scalar, **3-vector**) = **4-Vector** — Every Physical 4-Vector has a (c) factor to maintain equivalent dimensional units across the whole 4-Vector



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SR → QM

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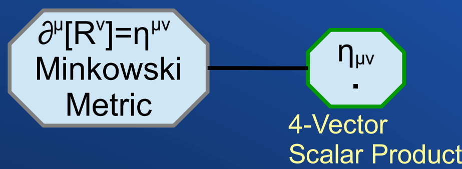

$\mathbf{U}\cdot\mathbf{U} = \gamma^2(c^2-\mathbf{u}\cdot\mathbf{u}) = c^2$	Speed of all things into the Future
$(E_o/m_o)=(\gamma E_o/\gamma m_o)=(E/m) = c^2$	Mass is concentrated Energy, $E = mc^2$
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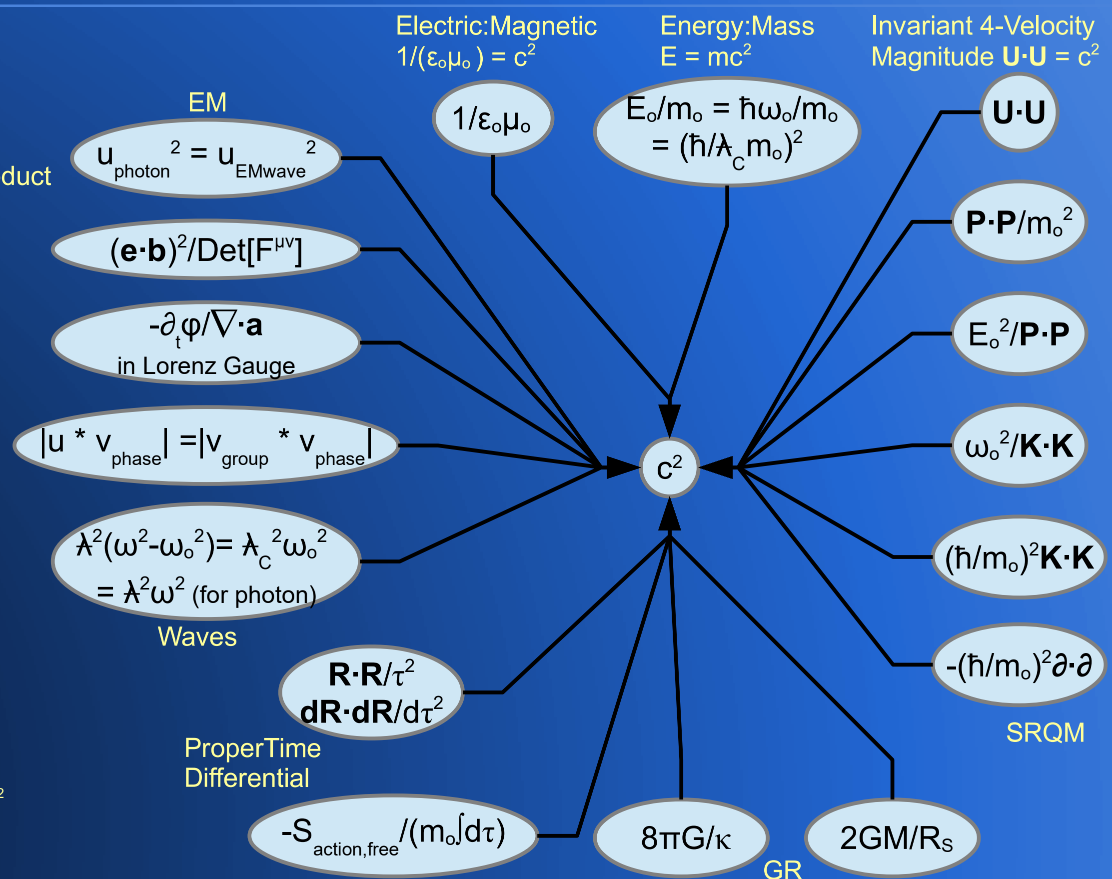
$8\pi G/\kappa = c^2$	GR Einstein Curvature Constant$_{\text{(mass density form)}}$: $\kappa = 8\pi G/c^2$
($c^{\pm1}$ * scalar, **3-vector**) = **4-Vector**	Every Physical 4-Vector has a (c) factor to maintain equivalent dimensional units across the whole 4-Vector



SR 4-Tensor	**SR 4-Vector**	**SR 4-Scalar**
(2,0)-Tensor $T^{\mu\nu}$ (1,1)-Tensor $T^\mu{}_\nu$ or $T_\mu{}^\nu$ (0,2)-Tensor $T_{\mu\nu}$	(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0,\mathbf{v})$ **SR 4-CoVector** (0,1)-Tensor $V_\mu = (v_0,-\mathbf{v})$	(0,0)-Tensor S Lorentz Scalar

$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu{}_\mu = T$
$\mathbf{V}\cdot\mathbf{V} = V^\mu\eta_{\mu\nu}V^\nu = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$ = Lorentz Scalar

SRQM 4-Vector Study: 4-ThermalVector Relativistic Thermodynamics

A Tensor Study of Physical 4-Vectors

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The 4-ThermalVector is used in Relativistic Thermodynamics.

My prime motivation for the form of this 4-Vector is that the probability distributions calculated by statistical mechanics ought to be covariant functions since they are based on counting arguments.

F(state) ~ e^-(E/k_BT) = e^-(βE), with this $\beta = 1/k_BT$, (not v/c)

A covariant way to get this is the Lorentz Scalar Product of the 4-Momentum **P** with the 4-ThermalVector **Θ**.
F(state) ~ e^-(**P·Θ**) = e^-(E_o/k_BT_o)

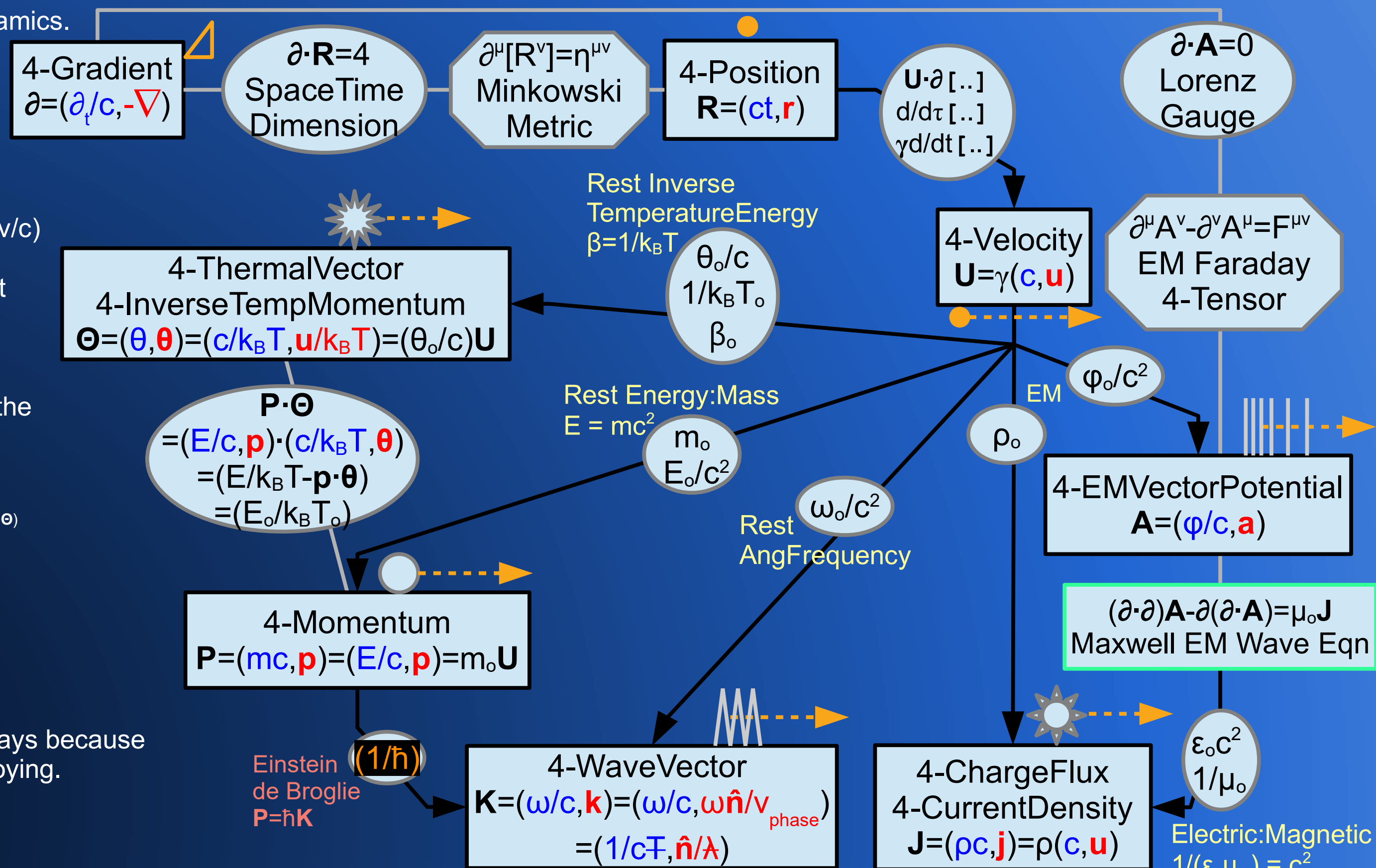
This also gets Boltzmann's constant (k_B) out there with the other Lorentz Scalars like (c) and (ħ)

<u>see (Relativistic) Maxwell-Jüttner distribution</u>

$f\,[\mathbf{P}] = N_o/(2c(m_oc)^d K_{[(d+1)/2]}[m_oc\Theta_o])*(m_oc\Theta_o/2\pi)^{(d-1)/2} * e^{-(\mathbf{P\cdot\Theta})}$

$f\,[\mathbf{P}] = N_o/(2c(m_oc)^3 K_{[2]}[m_oc\Theta_o])*(m_oc\Theta_o/2\pi) * e^{-(\mathbf{P\cdot\Theta})}$
$f\,[\mathbf{P}] = (\Theta_o)N_o/(4\pi c(m_oc)^2 K_{[2]}[m_oc\Theta_o]) * e^{-(\mathbf{P\cdot\Theta})}$
$f\,[\mathbf{P}] = cN_o/(4\pi k_BT_o(m_oc)^2 K_{[2]}[m_oc\Theta_o]\,)* e^{-(\mathbf{P\cdot\Theta})}$
$f\,[\mathbf{P}] = N_o/(4\pi k_BT_om_o^2c\, K_{[2]}[m_oc^2/k_BT_o]\,)* e^{-(\mathbf{P\cdot\Theta})}$

It is possible to find this distribution written in multiple ways because many authors don't show constants, which is quite annoying. Show the damn constants people!
(k_B),(c),(ħ) deserve at least that much respect.



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SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Be careful not to confuse (unfortunate symbol clash):
Thermal $\beta = 1/k_BT$
Relatvisitic β = v/c
These are totally separate uses of (β)

Trace[$T^{\mu\nu}$] = $\eta_{\mu\nu}T^{\mu\nu} = T^{\mu}{}_{\mu}$ = T
V·V = $V^{\mu}\eta_{\mu\nu}V^{\nu}$ = [$(v^0)^2$ - **v·v**] = $(v^0{}_o)^2$
= Lorentz Scalar

SR → QM

SRQM 4-Vector Study: 4-ThermalVector Unruh-Hawking Radiation

4-Vector SRQM Interpretation of QM

A Tensor Study of Physical 4-Vectors

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John B. Wilson

The 4-ThermalVector is used in Relativistic Thermodynamics.
It can be used in a partial derivation of Unruh-Hawking Radiation (up to a numerical constant).

Let a “Unruh-DeWitt thermal detector” be in the Momentarily-Comoving-Rest-Frame (MCRF) of a constant spatial acceleration (**a**), in which $|\mathbf{u}|\to 0$, $\gamma\to 1$, $\gamma'\to 0$.

4-Acceleration$_{MCRF}$ = $\mathbf{A}_{MCRF}$ = $A_{MCRF}{}^{\mu}$ = $(0,\mathbf{a})_{MCRF}$

Take the Lorentz Scalar Product with the 4-ThermalVector
$\mathbf{A}_{MCRF}\cdot\mathbf{\Theta} = (0,\mathbf{a})_{MCRF}\cdot(c/k_BT,\mathbf{u}/k_BT) = (-\mathbf{a}\cdot\mathbf{u}/k_BT)$ = Lorentz Scalar Invariant

The (**u**) here is part of the 4-ThermalVector: the 3-velocity of the thermal radiation. (not from $\mathbf{A}_{MCRF}$)
Let the thermal radiation be photonic:EM in nature, so $|\mathbf{u}| = c$, and in a direction opposing the acceleration of the “thermal detector”, which removes the minus sign.

(ac/k_BT) = Invariant

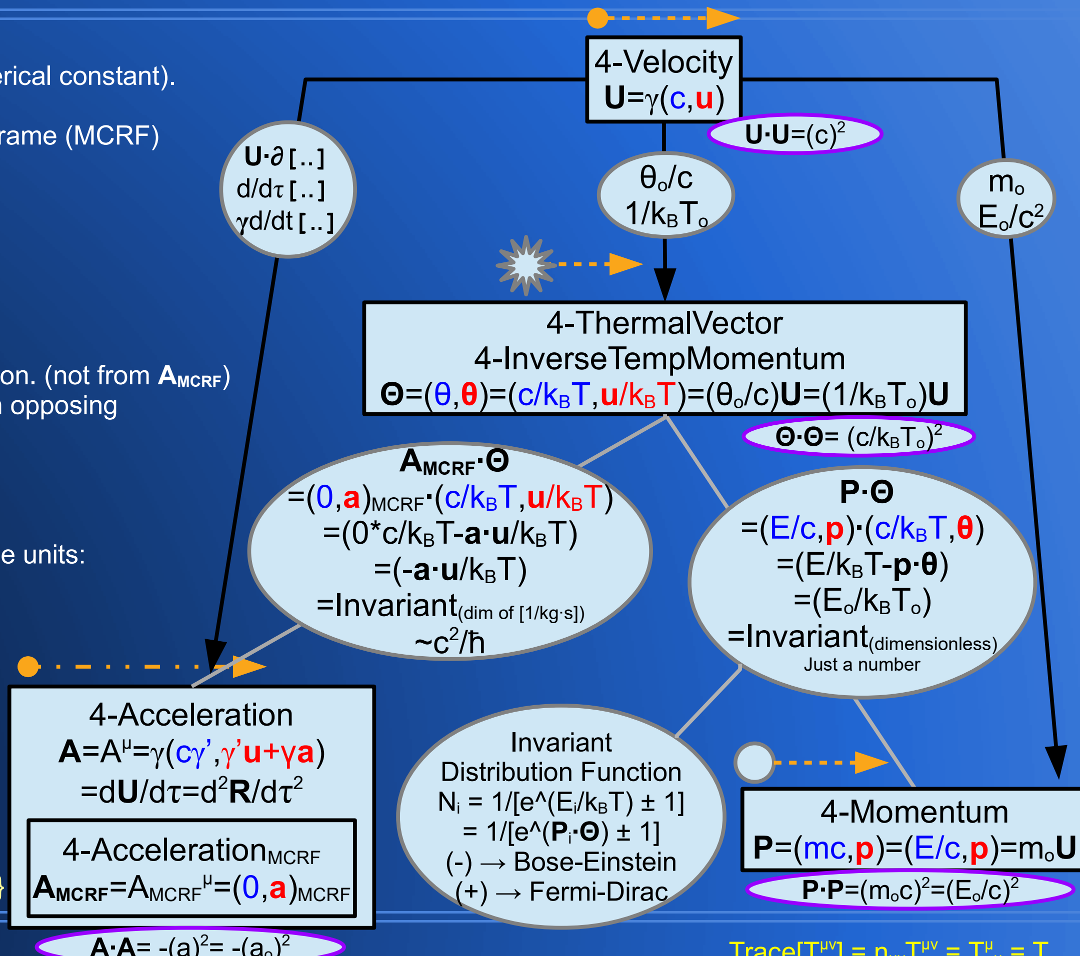
Use Dimensional Analysis to find appropriate Lorentz Scalar Invariant with same units:
[Invariant Units] = $[m/s^2]\cdot[m/s] / [kg\cdot m^2/s^2] = [1/kg\cdot s] \sim c^2/\hbar$

(ac/k_BT) = Invariant $\sim c^2/\hbar$

Temperature $T \sim \hbar a/k_Bc$, {from EM radiation, only from the dir. of acceleration}

Further methods give the constant of proportionality $(1/2\pi)$:
$T_{Unruh} = \hbar a/2\pi k_Bc$ {due to constant Minkowski-hyperbolic acceleration}
$T_{Hawking} = \hbar g/2\pi k_Bc$ {due to gravitational acceleration a=g}

$T_{SR} = -\hbar(\mathbf{a}\cdot\mathbf{u})/2\pi k_Bc^2$ {correct version from 4-Vector derivation $\mathbf{A}_{MCRF}\cdot\mathbf{\Theta} = 2\pi c^2/\hbar$}



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SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Note that the temperature here is relativistically direction-specific, unlike in the classical use of temperature.

Trace$[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^{\mu}{}_{\mu} = T$
$\mathbf{V}\cdot\mathbf{V} = V^{\mu}\eta_{\mu\nu}V^{\nu} = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$ = Lorentz Scalar

SRQM 4-Vector Study:
4-EntropyFlux
Relativistic Thermodynamics

A Tensor Study of Physical 4-Vectors

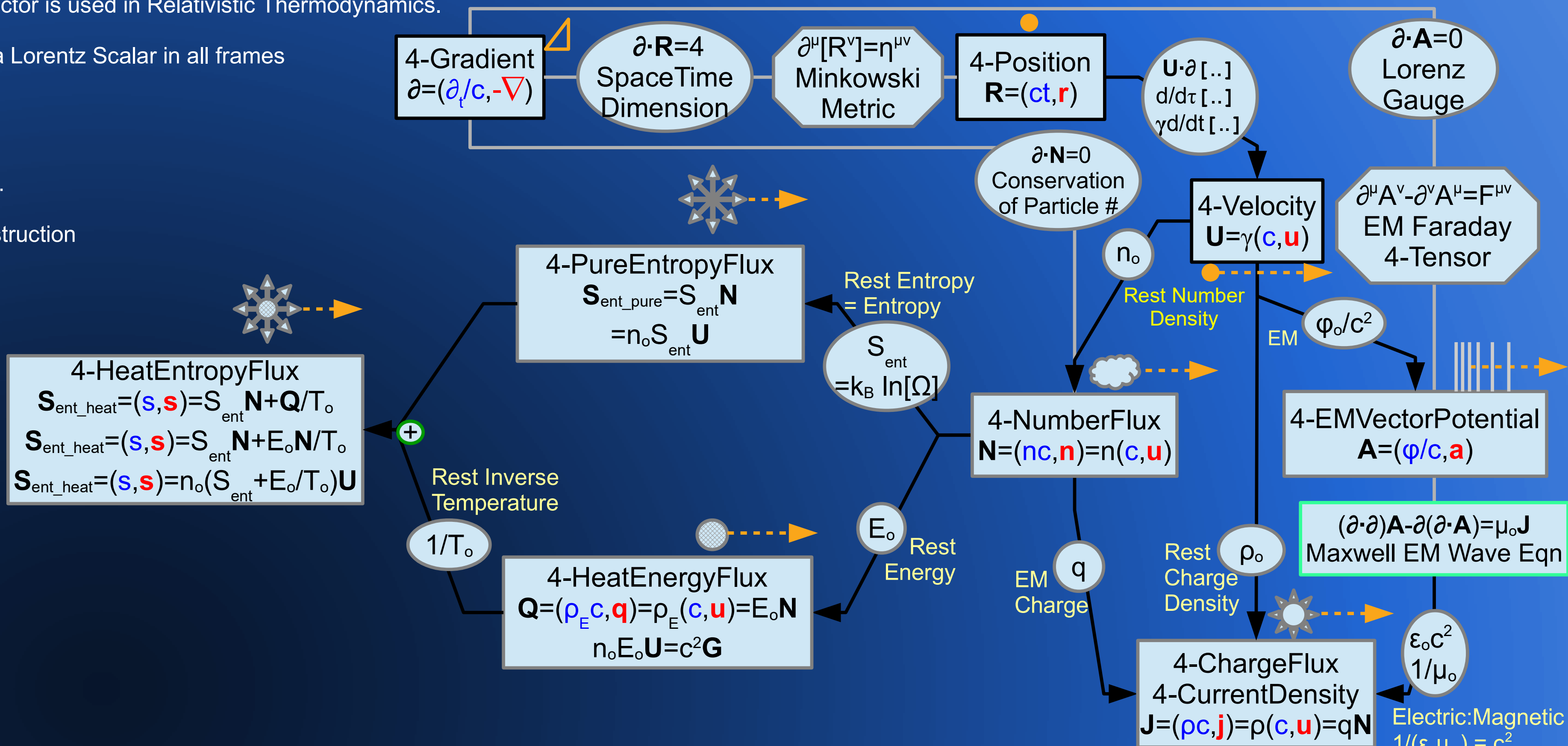
SciRealm.org
John B. Wilson

The 4-EntropyVector is used in Relativistic Thermodynamics.

Pure Entropy is a Lorentz Scalar in all frames

not finished yet…

Page under construction



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SRQM Interpretation:
** Transition to QM **

Up to this point, we have basically been exploring the SR aspects of 4-Vectors.

It is now time to show how RQM and QM fit into the works...

This is SRQM, [SR → QM]

RQM & QM are derivable from SR

SRQM Basic Idea (part 1)
SR → Relativistic Wave Eqn

The basic idea is to show that Special Relativity plus a few empirical facts lead to Relativistic Wave Equations, and thus RQM, without using any assumptions or axioms from Quantum Mechanics.

Start only with the concepts of SR, no concepts from QM
(1) SR provides the ideas of Invariant Intervals and (c) as a Physical Constant, as well as: Poincaré Invariance, Minkowski 4D SpaceTime, ProperTime, and Physical SR 4-Vectors

Note empirical facts which can relate the SR 4-Vectors from the following:
(2a) Elementary matter particles each have RestMass, (m_o), which can be measured by experiment: eg. collision, cyclotrons, Compton Scattering, etc.

(2b) There is a constant, (ħ), which can be measured by classical experiment – eg. the Photoelectric Effect, the inverse Photoelectric Effect, LED's=Injection Electroluminescence, Duane-Hunt Law in Bremsstralung, the Watt/Kibble-Balance, etc. All known particles obey this constant.

(2c) The use of complex numbers (i) and differential operators { ∂_t and $\nabla = (\partial_x,\partial_y,\partial_z)$ } in wave-type equations comes from pure mathematics: not necessary to assume any QM Axioms

*These few things are enough to derive the RQM Klein-Gordon equation, the most basic of the relativistic wave equations. Taking the low-velocity limit { |**v**|<<c }(a standard SR technique) leads to the Schrödinger Equation.*

SRQM Basic Idea (part 2)
Klein-Gordon RWE implies QM

If one has a Relativistic Wave Equation, such as the Klein-Gordon equation, then one has RQM, and thence QM via the low-velocity limit { |**v**|<<c }.

The physical and mathematical properties of QM, usually regarded as axiomatic, are inherent in the Klein-Gordon RWE itself.

QM Principles emerge not from { QM Axioms + SR → RQM },
but from { SR + Empirical Facts → RQM }.

The result is a paradigm shift from the idea of { SR and QM as separate theories }
to { QM derived from SR } – leading to a new interpretation of QM:
The SRQM or [SR→QM] Interpretation.

GR → (low-mass limit = {curvature ~ 0} limit) → SR
SR → (+ a few empirical facts) → RQM
RQM → (low-velocity limit { |**v**|<<c }) → QM

The results of this analysis will be facilitated by the use of SR 4-Vectors

SRQM 4-Vector Path to QM

SR 4-Vector	Definition Component Notation	Unites
4-Position	$\mathbf{R} = R^\mu = (ct,\mathbf{r})$	**Time, Space** *-when & where*
4-Velocity	$\mathbf{U} = U^\mu = \gamma(c,\mathbf{u})$	**Lorentz Gamma * (c, Velocity)** *-nothing faster than c*
4-Momentum	$\mathbf{P} = P^\mu = (E/c,\mathbf{p}) = (mc,\mathbf{p})$	**Mass:Energy, Momentum** *-used in 4-Momenta Conservation* $\Sigma\, \mathbf{P}_{final} = \Sigma\, \mathbf{P}_{initial}$
4-WaveVector	$\mathbf{K} = K^\mu = (\omega/c,\mathbf{k}) = (\omega/c,\omega\hat{\mathbf{n}}/v_{phase})$	**Ang. Frequency, WaveNumber** *-used in Relativistic Doppler Shift* $\omega_{obs}=\omega_{emit}$ / $[\gamma(1 - \beta \cos[\theta])]$, $k=\omega/c$ for photons
4-Gradient	$\partial = \partial^\mu = (\partial_t/c,-\nabla)$ $= (\partial_t/c,-\partial_x,-\partial_y,-\partial_z)$ $= (\partial/\partial ct,-\partial/\partial x,-\partial/\partial y,-\partial/\partial z)$	**Temporal Partial, Spatial Partial** *-used in SR Continuity Eqns., ProperTime* *-eg.* $\partial\cdot\mathbf{A} = 0$ *means* ***A*** *is conserved*

All of these are standard SR 4-Vectors, which can be found and used in a totally relativistic context, with no mention or need of QM.
I want to emphasize that these objects are ALL relativistic in origin.

SRQM 4-Vector Invariants

SR 4-Vector	Lorentz Invariant	What it means in SR...
4-Position	$\mathbf{R}\cdot\mathbf{R} = (ct)^2 - \mathbf{r}\cdot\mathbf{r} = (ct_o)^2 = (c\tau)^2$	SR Invariant Interval
4-Velocity	$\mathbf{U}\cdot\mathbf{U} = \gamma^2(c^2 - \mathbf{u}\cdot\mathbf{u}) = c^2$	Events move into future at magnitude c
4-Momentum	$\mathbf{P}\cdot\mathbf{P} = (E/c)^2 - \mathbf{p}\cdot\mathbf{p} = (E_o/c)^2$	Einstein Mass:Energy Relation
4-WaveVector	$\mathbf{K}\cdot\mathbf{K} = (\omega/c)^2 - \mathbf{k}\cdot\mathbf{k} = (\omega_o/c)^2$	Dispersion Invariance Relation
4-Gradient	$\partial\cdot\partial = (\partial_t/c)^2 - \nabla\cdot\nabla = (\partial_\tau/c)^2$	The d'Alembert Operator

All 4-Vectors have invariant magnitudes, found by taking the scalar product of the 4-Vector with itself. Quite often a simple expression can be found by examining the case when the spatial part is zero. This is usually found when the 3-velocity is zero. The temporal part is then specified by its “rest” value.

For example: $\mathbf{P}\cdot\mathbf{P} = (E/c)^2 - \mathbf{p}\cdot\mathbf{p} = (E_o/c)^2 = (m_oc)^2$
$E = \mathrm{Sqrt}[\ (E_o)^2 + \mathbf{p}\cdot\mathbf{p}\ c^2\ ]$, from above relation
$E = \gamma E_o$, using $\{\gamma = 1/\mathrm{Sqrt}[1-\beta^2] = \mathrm{Sqrt}[1+\gamma^2\beta^2]\}$ and $\{\beta = v/c\}$
meaning the relativistic energy E is equal to the relative gamma factor γ * the rest energy E_o

SR + A few empirical facts: SRQM Overview

SR 4-Vector	Empirical Fact	SI Dimensional Units
4-Position **R** = (ct,**r**); alt. **X** = (ct,**x**)	**R** = <Event>; alt. **X**	[m]
4-Velocity **U** = γ(c,**u**)	**U** = d**R**/dτ	[m/s]
4-Momentum **P** = (E/c,**p**) = (mc,**p**)	**P** = m_o**U**	[kg·m/s]
4-WaveVector **K** = (ω/c,**k**)	**K** = **P**/ħ	[{rad}/m]
4-Gradient **∂** = (∂_t/c,-**∇**)	**∂** = -i**K**	[1/m]

The Axioms of SR, which are actually GR limiting-cases, lead us to the use of Minkowski Space and Physical 4-Vectors, which are elements of Minkowski Space (4D SpaceTime).

Empirical Observation leads us to the transformation relations between the components of these SR 4-Vectors, and to the chain of relations between the 4-Vectors themselves

These relations all turn out to be Lorentz Invariant Constants, whose values are measured empirically.

The combination of these SR objects and their relations is enough to derive RQM.

SRQM:
SR→QM Interpretation Simplified

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http://scirealm.org/SRQM.pdf

SRQM: The [SR→QM] Interpretation of Quantum Mechanics

SR Axioms: Invariant Interval + (c) as Physical Constant lead to SR, although technically SR is itself the low-curvature limiting-case of GR

$\{c,\tau,m_o,\hbar,i\}$: All Empirically Measured SR Lorentz Invariants and/or Mathematical Constants

Standard SR 4-Vectors: Related by these SR Lorentz Invariants

4-Position	$\mathbf{R} = (ct,\mathbf{r})$	= <Event>	$(\mathbf{R}\cdot\mathbf{R}) = (c\tau)^2$
4-Velocity	$\mathbf{U} = \gamma(c,\mathbf{u})$	$= (\mathbf{U}\cdot\partial)\mathbf{R} = (^d/_{d\tau})\mathbf{R} = d\mathbf{R}/d\tau$	$(\mathbf{U}\cdot\mathbf{U}) = (c)^2$
4-Momentum	$\mathbf{P} = (E/c,\mathbf{p})$	$= m_o\mathbf{U}$	$(\mathbf{P}\cdot\mathbf{P}) = (m_oc)^2$
4-WaveVector	$\mathbf{K} = (\omega/c,\mathbf{k})$	$= \mathbf{P}/\hbar$	$(\mathbf{K}\cdot\mathbf{K}) = (m_oc/\hbar)^2$
4-Gradient	$\partial = (\partial_t/c,-\nabla)$	$= -i\mathbf{K}$	$(\partial\cdot\partial) = -(m_oc/\hbar)^2$ = KG Eqn→RQM $\xrightarrow{\lvert\mathbf{v}\rvert<<c}$ QM

SR + Emipirically Measured Physical Constants lead to RQM via the Klein-Gordon Eqn, and thence to QM via the low-velocity limit { |**v**| << c }, giving the Schrödinger Eqn. The relation also leads to the Dirac, Maxwell, Pauli, Proca, Weyl, & Scalar Wave QM Eqns.

SRQM Diagram:
RoadMap of SR (4-Vectors)

4-Position
R=(ct,**r**)
=<Event>

4-Velocity
U=γ(c,**u**)

4-Gradient
∂=(∂_t/c,-∇)

4-WaveVector
K=(ω/c,**k**)

4-Momentum
P=(mc,**p**)=(E/c,**p**)

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(0,0)-Tensor S
Lorentz Scalar

Trace[$T^{\mu\nu}$] = $\eta_{\mu\nu}T^{\mu\nu}$ = $T^{\mu}{}_{\mu}$ = T
V·V = $V^{\mu}\eta_{\mu\nu}V^{\nu}$ = [$(v^0)^2$ - **v·v**] = $(v^{0}{}_{o})^2$
= Lorentz Scalar

SRQM Diagram:
RoadMap of SR (Connections)

A Tensor Study of Physical 4-Vectors

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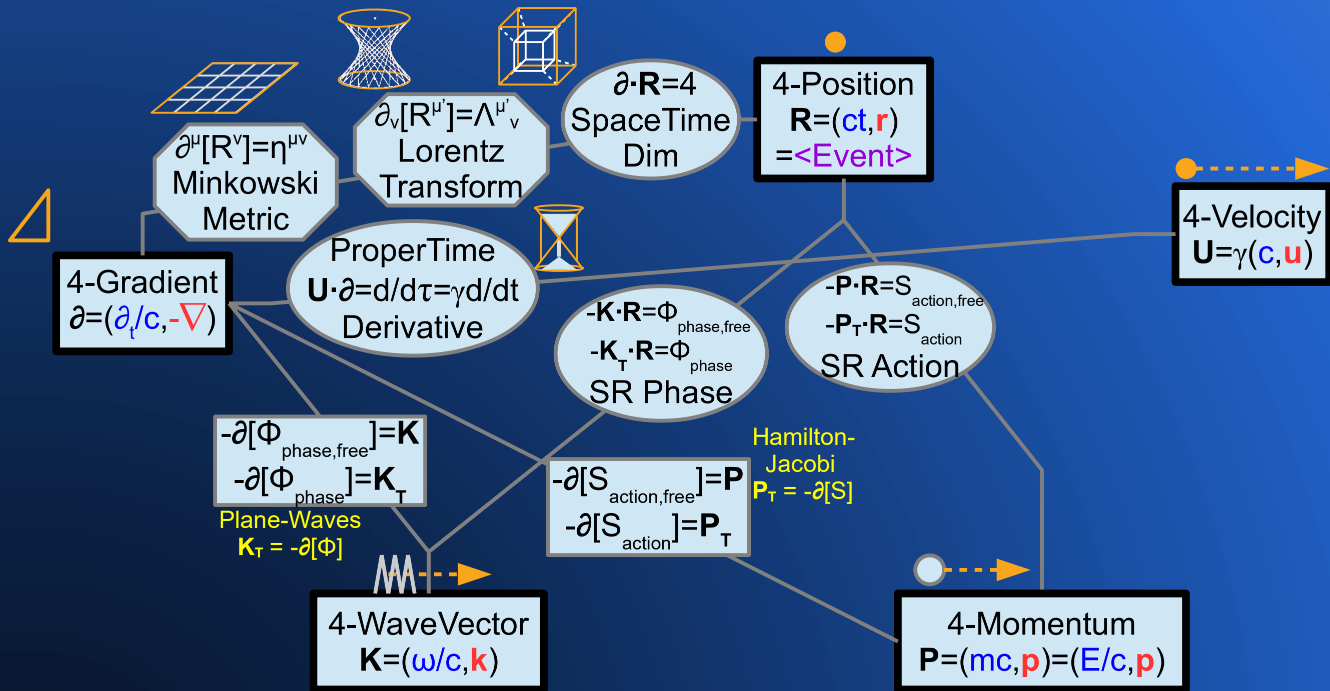

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^\mu{}_\nu$ or $T_\mu{}^\nu$
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(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0,\mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0,-\mathbf{v})$

SR 4-Scalar
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Lorentz Scalar

$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu{}_\mu = T$
$\mathbf{V}\cdot\mathbf{V} = V^\mu\eta_{\mu\nu}V^\nu = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$
= Lorentz Scalar

SRQM Diagram:
RoadMap of SR (Free Particle)

A Tensor Study of Physical 4-Vectors

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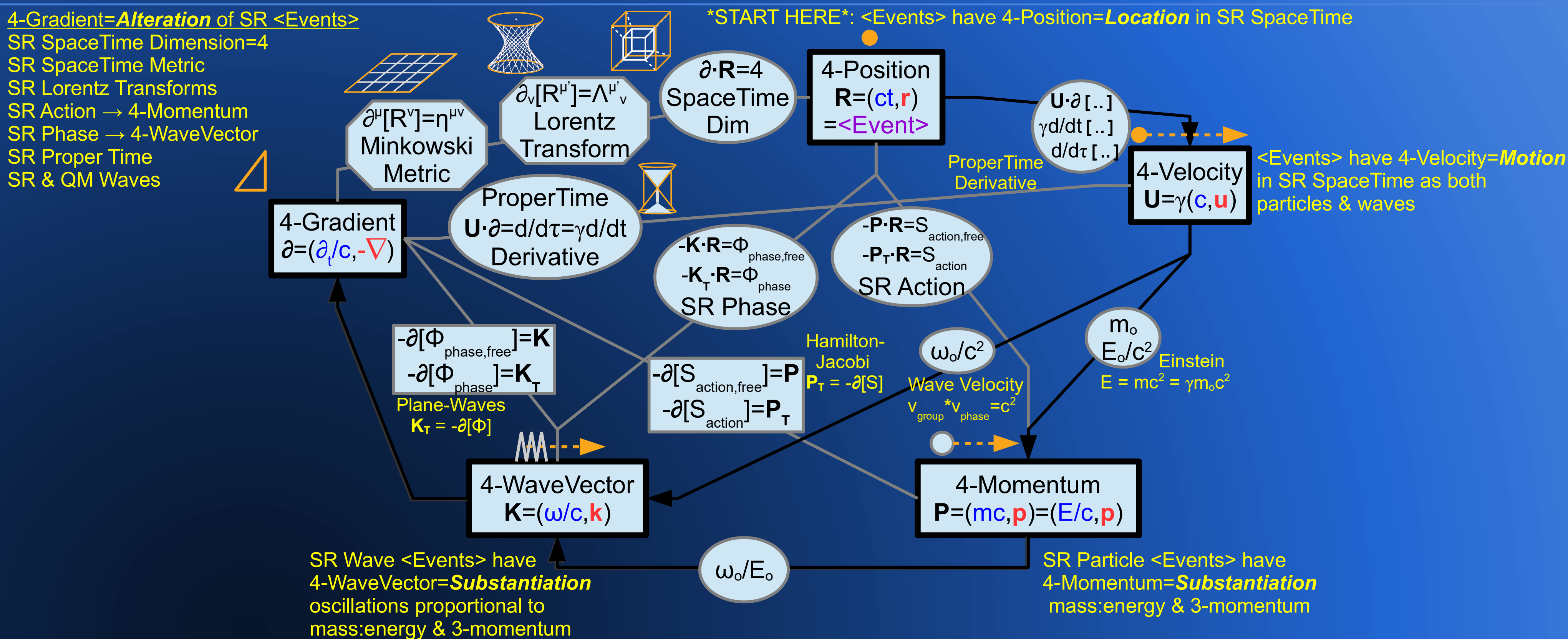

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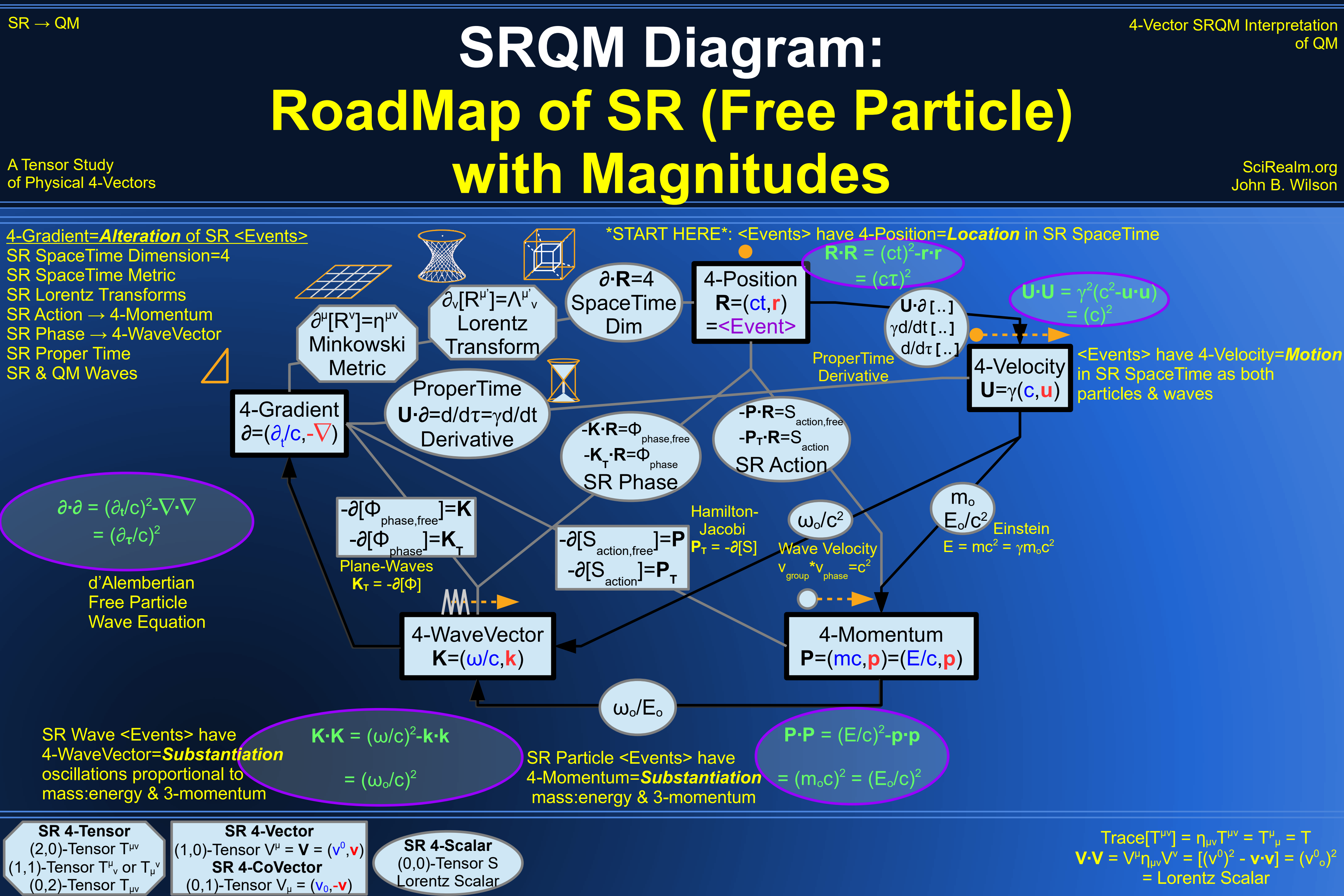
SR → QM
SRQM Diagram:
RoadMap of SR (Free Particle) with Magnitudes
4-Vector SRQM Interpretation of QM
A Tensor Study of Physical 4-Vectors
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4-Gradient=*Alteration* of SR <Events>
SR SpaceTime Dimension=4
SR SpaceTime Metric
SR Lorentz Transforms
SR Action → 4-Momentum
SR Phase → 4-WaveVector
SR Proper Time
SR & QM Waves
START HERE: <Events> have 4-Position=*Location* in SR SpaceTime
$\mathbf{R}\cdot\mathbf{R} = (ct)^2-\mathbf{r}\cdot\mathbf{r} = (c\tau)^2$
4-Position $\mathbf{R}=(ct,\mathbf{r})$ =<Event>
$\partial\cdot\mathbf{R}=4$ SpaceTime Dim
$\partial_\nu[R^{\mu'}]=\Lambda^{\mu'}_\nu$ Lorentz Transform
$\partial^\mu[R^\nu]=\eta^{\mu\nu}$ Minkowski Metric
$\mathbf{U}\cdot\mathbf{U} = \gamma^2(c^2-\mathbf{u}\cdot\mathbf{u}) = (c)^2$
$\mathbf{U}\cdot\partial[..]$ $\gamma d/dt[..]$ $d/d\tau[..]$
ProperTime Derivative
4-Velocity $\mathbf{U}=\gamma(c,\mathbf{u})$
<Events> have 4-Velocity=*Motion* in SR SpaceTime as both particles & waves
4-Gradient $\partial=(\partial_t/c,-\nabla)$
ProperTime $\mathbf{U}\cdot\partial=d/d\tau=\gamma d/dt$ Derivative
$-\mathbf{K}\cdot\mathbf{R}=\Phi_{phase,free}$ $-\mathbf{K_T}\cdot\mathbf{R}=\Phi_{phase}$ SR Phase
$-\mathbf{P}\cdot\mathbf{R}=S_{action,free}$ $-\mathbf{P_T}\cdot\mathbf{R}=S_{action}$ SR Action
$\partial\cdot\partial = (\partial_t/c)^2-\nabla\cdot\nabla = (\partial_\tau/c)^2$
d'Alembertian Free Particle Wave Equation
$-\partial[\Phi_{phase,free}]=\mathbf{K}$ $-\partial[\Phi_{phase}]=\mathbf{K_T}$
Plane-Waves $\mathbf{K_T} = -\partial[\Phi]$
$-\partial[S_{action,free}]=\mathbf{P}$ $-\partial[S_{action}]=\mathbf{P_T}$
Hamilton-Jacobi $\mathbf{P_T} = -\partial[S]$
ω_o/c^2
Wave Velocity $v_{group}*v_{phase}=c^2$
m_o E_o/c^2
Einstein $E = mc^2 = \gamma m_o c^2$
4-WaveVector $\mathbf{K}=(\omega/c,\mathbf{k})$
4-Momentum $\mathbf{P}=(mc,\mathbf{p})=(E/c,\mathbf{p})$
ω_o/E_o
SR Wave <Events> have 4-WaveVector=*Substantiation* oscillations proportional to mass:energy & 3-momentum
$\mathbf{K}\cdot\mathbf{K} = (\omega/c)^2-\mathbf{k}\cdot\mathbf{k} = (\omega_o/c)^2$
SR Particle <Events> have 4-Momentum=*Substantiation* mass:energy & 3-momentum
$\mathbf{P}\cdot\mathbf{P} = (E/c)^2-\mathbf{p}\cdot\mathbf{p} = (m_oc)^2 = (E_o/c)^2$
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SR 4-Scalar (0,0)-Tensor S Lorentz Scalar
$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu{}_\mu = T$
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SRQM Diagram:
RoadMap of SR (EM Potential)

A Tensor Study of Physical 4-Vectors

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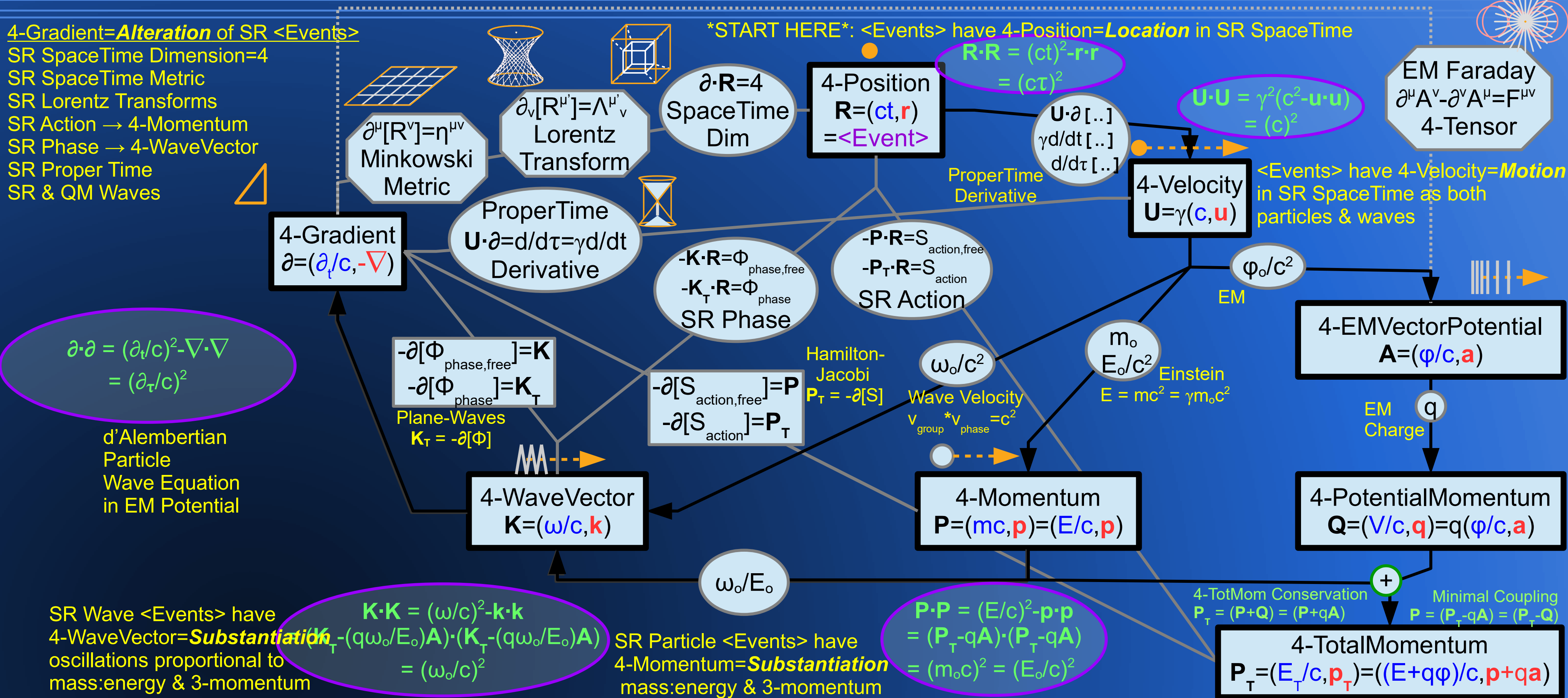

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SRQM Diagram:

Special Relativity → Quantum Mechanics

RoadMap of SR→QM (EM Potential)

A Tensor Study of Physical 4-Vectors

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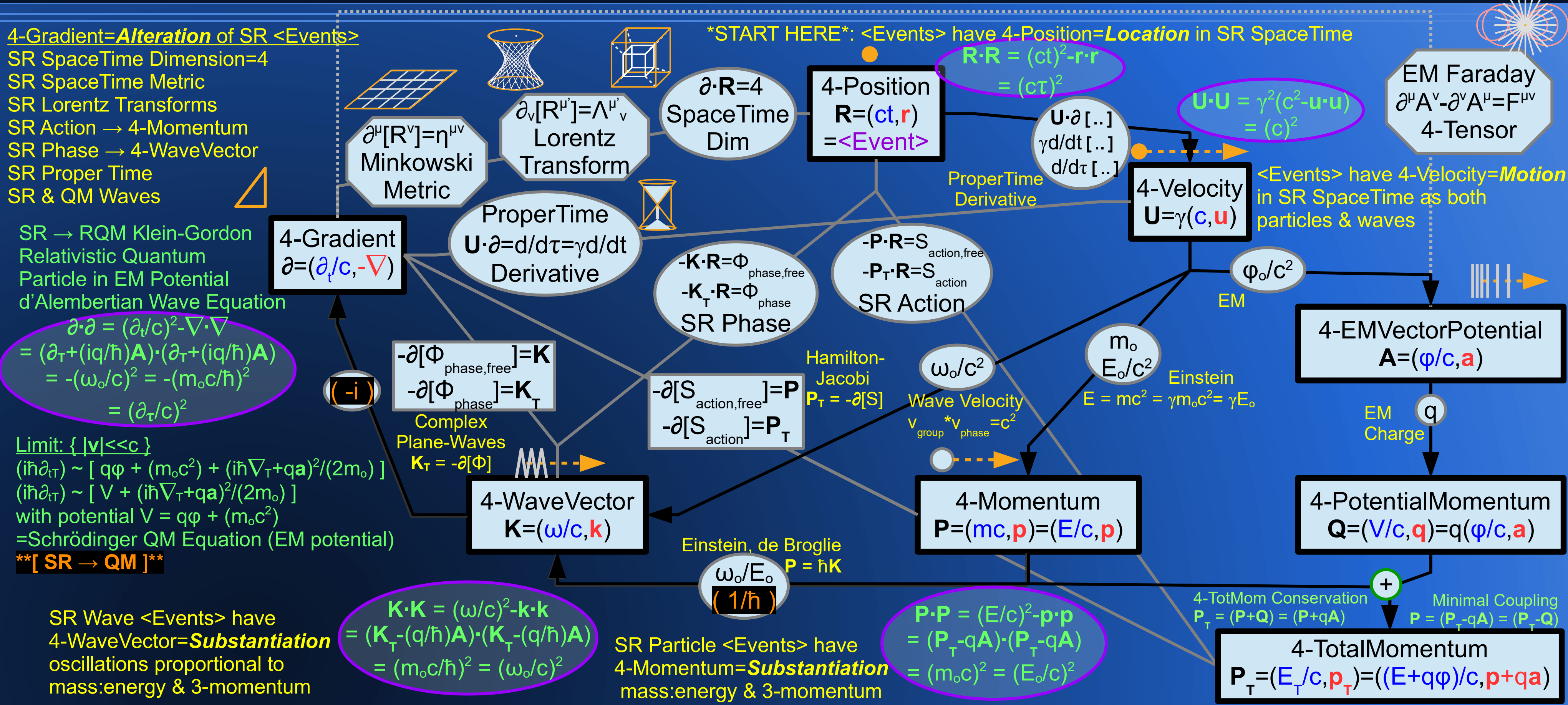

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(0,0)-Tensor S
Lorentz Scalar

Existing SR Rules
Quantum Principles

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SRQM: The Empirical 4-Vector Facts

SR 4-Vector	Empirical Fact	Discoverer	Physics
4-Position	**R** = <Event>	Newton+ Einstein	[t & **r**] Time & Space Dimensions [**R**=(ct,**r**)] SpaceTime
4-Velocity	**U** = d**R**/dτ	Newton Einstein	[**v**=d**r**/dt] Calculus of motion [**U**=γ(c,**u**)=d**R**/dτ] Gamma & Proper Time
4-Momentum	**P** = m_o**U**	Newton Einstein	[**p**=m**v**] Classical Mechanics [**P**=(E/c,**p**)=m_o**U**] SR Mechanics
4-WaveVector	**K** = **P**/ħ	Planck Einstein de Broglie	[h] Thermal Distribution [E=hν=ħω] Photoelectric Effect (ħ=h/2π) [**p**=ħ**k**] Matter Waves
4-Gradient	∂ = -i**K**	Schrödinger	[ω=i∂_t, **k**=-i**∇**] (SR) Wave Mechanics

(1) The SR 4-Vectors and their components are related to each other via constants
(2) We have not taken any 4-vector relation as axiomatic, the constants come from experiment.
(3) c, τ, m_o, ħ come from physical experiments, (-i) comes from the general mathematics of waves

The SRQM 4-Vector Relations Explained

SR 4-Vector	Empirical Fact	What it means in SRQM...	Lorentz Invariant
4-Position **R** = (ct,**r**)	**R** = <Event>	SpaceTime as Unified Concept	c = LightSpeed
4-Velocity **U** = γ(c,**u**)	**U** = d**R**/dτ	Velocity is ProperTime Derivative	$\tau = t_o$ = ProperTime
4-Momentum **P** = (E/c,**p**)	**P** = m_o**U**	Mass:Energy-Momentum Equivalence	m_o = RestMass
4-WaveVector **K** = (ω/c,**k**)	**K** = **P**/ħ	Wave-Particle Duality	ħ = UniversalAction
4-Gradient ∂ = (∂_t/c,-**∇**)	∂ = -i**K**	Unitary Evolution, Operator Formalism	i = ComplexSpace

Three old-paradigm QM Axioms:
Particle-Wave Duality [(**P**)=ħ(**K**)], Unitary Evolution [∂=(-i)**K**], Operator Formalism [(∂)=-i**K**] are actually just empirically-found constant relations between known SR 4-Vectors.
Note that these constants are in fact all Lorentz Scalar Invariants.

Minkowski Space and 4-Vectors also lead to idea of Lorentz Invariance. A Lorentz Invariant is a quantity that always has the same value, independent of the motion of inertial observers.
Lorentz Invariants can typically be derived using the scalar product relation.
$\mathbf{U}\cdot\mathbf{U} = c^2$, $\mathbf{U}\cdot\partial = d/d\tau$, $\mathbf{P}\cdot\mathbf{U} = m_o c^2$, etc.

A very important Lorentz invariant is the Proper Time τ, which is defined as the time displacement between two points on a worldline that is at rest wrt. an observer. It is used in the relations between 4-Position **R**, 4-Velocity **U** = d**R**/dτ, and 4-Acceleration **A** = d**U**/dτ.

SRQM: The SR Path to RQM Follow the Invariants...

SR 4-Vector	Lorentz Invariant	What it means in SRQM...
4-Position	$\mathbf{R}\cdot\mathbf{R} = (ct)^2 - \mathbf{r}\cdot\mathbf{r} = (c\tau)^2$	SR Invariant Interval
4-Velocity	$\mathbf{U}\cdot\mathbf{U} = \gamma^2(c^2 - \mathbf{u}\cdot\mathbf{u}) = c^2$	Events move into future at magnitude c
4-Momentum	$\mathbf{P}\cdot\mathbf{P} = (m_o c)^2$	Einstein Mass:Energy Relation
4-WaveVector	$\mathbf{K}\cdot\mathbf{K} = (m_o c/\hbar)^2 = (\omega_o/c)^2$	Matter-Wave Dispersion Relation
4-Gradient	$\partial\cdot\partial = (-im_o c/\hbar)^2 = -(m_o c/\hbar)^2$	The Klein-Gordon Equation → RQM!

$\mathbf{U} = d\mathbf{R}/d\tau$
Remember, everything after 4-Velocity was just a constant times the last 4-vector, and the Invariant Magnitude of the 4-Velocity is itself a constant
$\mathbf{P} = m_o\mathbf{U}$, $\mathbf{K} = \mathbf{P}/\hbar$, $\partial = -i\mathbf{K}$, so e.g. $\mathbf{P}\cdot\mathbf{P} = m_o\mathbf{U}\cdot m_o\mathbf{U} = m_o{}^2\mathbf{U}\cdot\mathbf{U} = (m_o c)^2$

The last equation is the Klein-Gordon RQM Equation, which we have just derived without invoking any QM axioms, only SR plus a few empirical facts

SR → QM

4-Vector SRQM Interpretation of QM

SRQM: Some Basic 4-Vectors 4-Momentum, 4-WaveVector, 4-Position, 4-Velocity, 4-Gradient, Wave-Particle

A Tensor Study of Physical 4-Vectors

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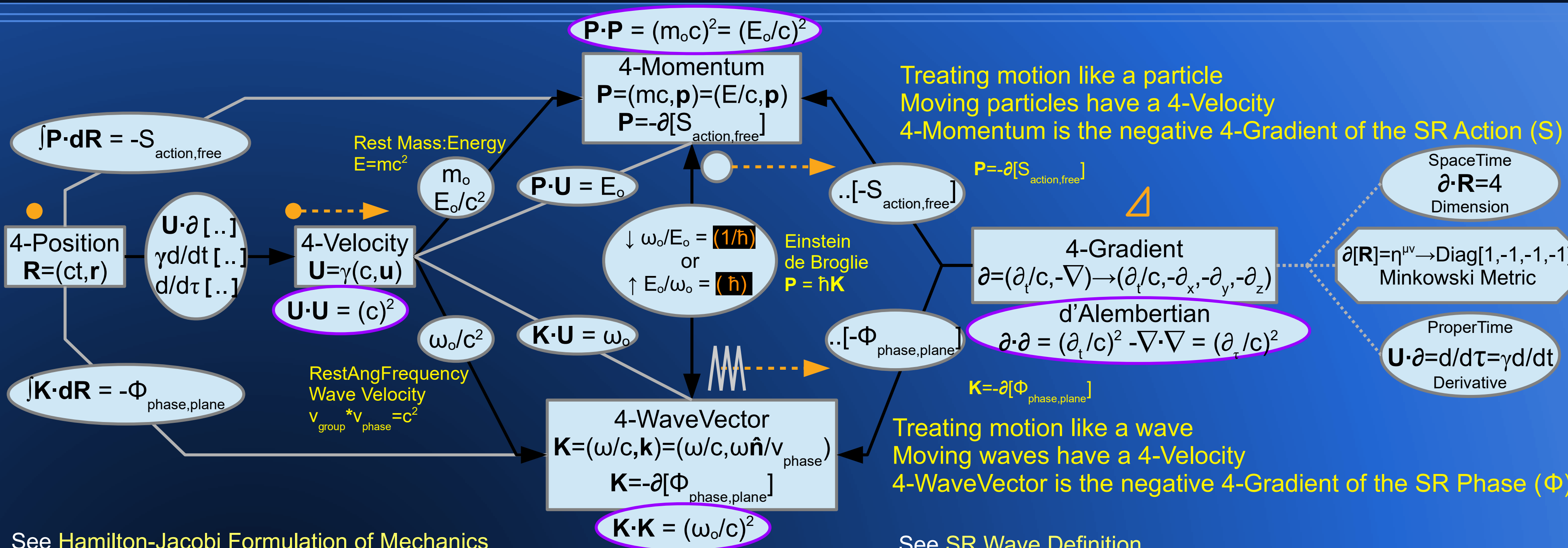

See Hamilton-Jacobi Formulation of Mechanics
for info on the Lorentz Scalar Invariant SR Action.
{ $\mathbf{P} = (E/c,\mathbf{p}) = -\partial[S] = (-\partial/c\partial t[S],\nabla[S])$ }
{temporal component} $E = -\partial/\partial t[S] = -\partial_t[S]$
{spatial component} $\mathbf{p} = \nabla[S]$
Note This is the Action (S_{action}) for a free particle.
Generally Action is for the 4-TotalMomentum $\mathbf{P_T}$ of a system.

See SR Wave Definition
for info on the Lorentz Scalar Invariant SR WavePhase.
{ $\mathbf{K} = (\omega/c,\mathbf{k}) = -\partial[\Phi] = (-\partial/c\partial t[\Phi],\nabla[\Phi])$ }
{temporal component} $\omega = -\partial/\partial t[\Phi] = -\partial_t[\Phi]$
{spatial component} $\mathbf{k} = \nabla[\Phi]$
Note This is the Phase (Φ) for a single plane-wave.
Generally WavePhase is for the 4-TotalWaveVector $\mathbf{K_T}$ of a system.

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SR 4-Scalar
(0,0)-Tensor S
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Existing SR Rules
Quantum Principles

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= Lorentz Scalar

SRQM: Wave-Particle Diffraction/Interference Types

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The 4-Vector Wave-Particle relation is inherent in all particle types: Einstein-de Broglie $\mathbf{P} = (E/c,\mathbf{p}) = \hbar\mathbf{K} = \hbar(\omega/c,\mathbf{k})$.

All waves can diffract: Water waves, gravitational waves, photonic waves of all frequencies, etc.
In all cases: experiments using single particles build the diffraction/interference pattern over the course many iterations.

Photon/light Diffraction: Photonic particles diffracted by matter particles.
Photons of any frequency encounter a “solid” object or grating.
Most often encountered are diffraction gratings and the famous double-slit experiment

Matter Diffraction: Matter particles diffracted by matter particles.
Electrons, neutrons, atoms, small molecules, buckyballs (fullerenes), macromolecules, etc.
have been shown to diffract through crystals.
Crystals may be solid single pieces or in powder form.

Kapitsa-Dirac Diffraction: Matter particles diffracted by photonic standing waves.
Electrons, atoms, super-sonic atom beams have been diffracted from resonant standing waves of light.

Photonic-Photonic Diffraction?: Delbruck scattering
Light-by-light scattering/two-photon physics/gamma-gamma physics.
Normally, photons do not interact, but at high enough relative energy, virtual particles can form which allow interaction.

$\mathbf{P}\cdot\mathbf{P} = (m_o c)^2 = (E_o/c)^2$

4-Momentum
$\mathbf{P}=(mc,\mathbf{p})=(E/c,\mathbf{p})$
$\mathbf{P}=-\partial[S_{action,free}]$

$\downarrow \omega_o/E_o = (1/\hbar)$
or
$\uparrow E_o/\omega_o = (\hbar)$

Einstein
de Broglie
$\mathbf{P} = \hbar\mathbf{K}$

4-WaveVector
$\mathbf{K}=(\omega/c,\mathbf{k})=(\omega/c,\omega\hat{\mathbf{n}}/v_{phase})$
$\mathbf{K}=-\partial[\Phi_{phase,plane}]$

$\mathbf{K}\cdot\mathbf{K} = (\omega_o/c)^2$

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= Lorentz Scalar

Hold on, aren't you getting the “ħ” from a QM Axiom?

SR 4-Vector	SR Empirical Fact	What it means...
4-WaveVector	$\mathbf{K} = (\omega/c,\mathbf{k}) = (\omega/c,\omega\hat{\mathbf{n}}/v_{phase}) = (\omega_o/c^2)\mathbf{U}$	Wave-Particle Duality

ħ is actually an empirically measurable quantity, just like e or c. It can be measured classically from the photoelectric effect, the inverse photoelectric effect, from LED's (injection electroluminescence), from the Duane-Hunt Law in Bremsstrahlung, Electron Diffraction in crystals, the Watt/Kibble-Balance, etc.

For the LED experiment, one uses several different LED's, each with its own characteristic wavelength.
One then makes a chart of wavelength (λ) vs threshold voltage (V) needed to make each individual LED emit.
One finds that: {$\lambda = h*c/(eV)$}, where e=ElectronCharge and c=LightSpeed. h is found by measuring the slope.
Consider this as a blackbox where no assumption about QM is made. However, we know the SR relations {$E = eV$}, and {$\lambda f = c$}.
The data force one to conclude that {$E = hf = \hbar\omega$}.
Applying our 4-Vector knowledge, we recognize this as the temporal components of a 4-Vector relation. $(E/c,\ldots) = \hbar(\omega/c,\ldots)$
*Due to manifest tensor invariance, this means that 4-Momentum $\mathbf{P} = (E/c,\mathbf{p}) = \hbar\mathbf{K} = \hbar(\omega/c,\mathbf{k}) = \hbar$*4-WaveVector $\mathbf{K}$.*

The spatial component (due to De Broglie) follows naturally from the temporal component (due to Einstein) via to the nature of 4-Vector mathematics.

This is also derivable from pure SR 4-Vector (Tensor) arguments: $\mathbf{P} = m_o\mathbf{U} = (E_o/c^2)\mathbf{U}$ and $\mathbf{K} = (\omega_o/c^2)\mathbf{U}$
Since **P** and **K** are both Lorentz Scalar proportional to **U**, then by the rules of tensor mathematics, **P** must also be Lorentz Scalar proportional to **K**.
i.e. Tensors obey certain mathematical structures:
Transitivity{if a~b and b~c, then a~c} & Euclideaness: {if a~c and b~c, then a~b} **Not to be confused with the Euclidean Metric**

This invariant proportional constant is empirically measured to be (ħ) for each known particle type, massive ($m_o>0$) or massless ($m_o=0$):
$\mathbf{P} = m_o\mathbf{U} = (E_o/c^2)\mathbf{U} = (E_o/c^2)/(\omega_o/c^2)\mathbf{K} = (E_o/\omega_o)\mathbf{K} = (\gamma E_o/\gamma\omega_o)\mathbf{K} = (E/\omega)\mathbf{K} = (\hbar)\mathbf{K}$

Hold on, aren't you getting the "K" from a QM Axiom?

SR 4-Vector	SR Empirical Fact	What it means...
4-WaveVector	$\mathbf{K} = (\omega/c,\mathbf{k}) = (\omega/c,\omega\hat{\mathbf{n}}/v_{phase}) = (\omega_o/c^2)\mathbf{U}$	Wave-Particle Duality

K is a standard SR 4-Vector, used in generating the SR formulae:

Relativistic Doppler Effect:

$\omega_{obs} = \omega_{emit} / [\gamma(1 - \beta\cos[\theta])]$, $\quad k = \omega/c$ for photons

Relativistic Aberration Effect:

$\cos[\theta_{obs}] = (\cos[\theta_{emit}] + |\boldsymbol{\beta}|) / (1 + |\boldsymbol{\beta}|\cos[\theta_{emit}])$

The 4-WaveVector **K** can be derived in terms of periodic motion, where families of surfaces move through space as time increases, or alternately, as families of hypersurfaces in SpaceTime, formed by all events passed by the wave surface. The 4-WaveVector is everywhere in the direction of propagation of the wave surfaces.

$\mathbf{K} = -\partial[\Phi_{phase}]$

From this structure, one obtains relativistic/wave optics without ever mentioning QM.

Hold on, aren't you getting the "-i" from a QM Axiom?

SR 4-Vector	SR Empirical Fact	What it means...
4-Gradient	$\partial = (\partial_t/c, -\nabla) = -i\mathbf{K}$	Unitary Evolution of States Operator Formalism

[$\partial = -i\mathbf{K}$] gives the sub-equations [$\partial_t = -i\omega$] and [$\nabla = i\mathbf{k}$], and is certainly the main equation that relates QM and SR by allowing Operator Formalism. But, this is a basic equation regarding the general mathematics of plane-waves; not just quantum-waves, but anything that can be mathematically described by plane-waves and superpositions of plane-waves…
This includes purely SR waves, an example of which would be EM plane-waves (i.e. photons)...

$\psi(t,\mathbf{r}) = ae^{\wedge}[i(\mathbf{k}\cdot\mathbf{r}-\omega t)]$: Standard mathematical plane-wave equation

$\partial_t[\psi(t,\mathbf{r})] = \partial_t[ae^{\wedge}[i(\mathbf{k}\cdot\mathbf{r}-\omega t)]\ ] = (-i\omega)[ae^{\wedge}[i(\mathbf{k}\cdot\mathbf{r}-\omega t)]\ ] = (-i\omega)\psi(t,\mathbf{r})$, or [$\partial_t = -i\omega$]
$\nabla[\psi(t,\mathbf{r})] = \nabla[ae^{\wedge}[i(\mathbf{k}\cdot\mathbf{r}-\omega t)]\ ] = (i\mathbf{k})[ae^{\wedge}[i(\mathbf{k}\cdot\mathbf{r}-\omega t)]\ ] = (i\mathbf{k})\psi(t,\mathbf{r})$, or [$\nabla = i\mathbf{k}$]

In the more economical SR notation:
$\partial[\psi(\mathbf{R})] = \partial[ae^{\wedge}(-i\mathbf{K}\cdot\mathbf{R})] = (-i\mathbf{K})[ae^{\wedge}(-i\mathbf{K}\cdot\mathbf{R})] = (-i\mathbf{K})\psi(\mathbf{R})$, or [$\partial = -i\mathbf{K}$]

This one is more of a mathematical empirical fact, but regardless, it is not axiomatic.
It can describe purely SR waves, again without any mention of QM.

Hold on, aren't you getting the "∂" from a QM Axiom?

SR 4-Vector	SR Empirical Fact	What it means...
4-Gradient	$\partial = (\partial_t/c, -\nabla) = -i\mathbf{K}$	4D Gradient Operator

[$\partial = (\partial_t/c,-\nabla)$] is the SR 4-Vector Gradient Operator. It occurs in a purely relativistic context without ever mentioning QM.

$\partial \cdot \mathbf{X} = (\partial_t/c,-\nabla)\cdot(ct,\mathbf{x}) = (\partial_t/c[ct] - (-\nabla\cdot\mathbf{x})) = (\partial_t[t] + \nabla\cdot\mathbf{x})\ (1)+(3) = 4$
The 4-Divergence of the 4-Position ($\partial\cdot\mathbf{X} = \partial^\mu \eta_{\mu\nu} X^\nu$)gives the dimensionality of SpaceTime.

$\partial[\mathbf{X}] = (\partial_t/c,-\nabla)(ct,\mathbf{x}) = (\partial_t/c[ct],-\nabla[\mathbf{x}]) = \mathrm{Diag}[1,-\mathbf{1}] = \eta^{\mu\nu}$
The 4-Gradient acting on the 4-Position ($\partial[\mathbf{X}] = \partial^\mu[X^\nu]$) gives the Minkowski Metric Tensor

$\partial\cdot\mathbf{J} = (\partial_t/c,-\nabla)\cdot(\rho c,\mathbf{j}) = (\partial_t/c[\rho c]- (-\nabla\cdot\mathbf{j})) = (\partial_t[\rho] + \nabla\cdot\mathbf{j}) = 0$
The 4-Divergence of the 4-CurrentDensity is equal to 0 for a conserved current. It can be rewritten as ($\partial_t[\rho] = -\nabla\cdot\mathbf{j}$), which means that the time change of ChargeDensity is balanced by the space change or divergence of CurrentDensity. It is a Continuity Equation, giving local conservation of ChargeDensity. It is related to Noether's Theorem.

Hold on, doesn't using "∂" in an Equation of Motion presume a QM Axiom?

SR 4-Vector	SR Empirical Fact	What it means...
4-(Position)Gradient	$\partial_{\mathbf{R}} = \partial = (\partial_t/c, -\nabla) = -i\mathbf{K}$	4D Gradient Operator

Klein-Gordon Relativistic Quantum Wave Equation
$\partial \cdot \partial[\Psi] = -(m_o c/\hbar)^2[\Psi] = -(\omega_o/c)^2[\Psi]$

Relativistic Euler-Lagrange Equations
$\partial_{\mathbf{R}}[L] = (d/d\tau)\partial_{\mathbf{U}}[L]$: {particle format}
$\partial_{[\Phi]}[\mathcal{L}] = (\partial_{\mathbf{R}})\, \partial_{[\partial_{\mathbf{R}}(\Phi)]}[\mathcal{L}]$: {density format}

[$\partial = (\partial_t/c, -\nabla)$] is the SR 4-Vector (Position)Gradient Operator.
It occurs in a purely relativistic context without ever mentioning QM.
There is a long history of using the gradient operator on classical physics functions, in this case the Lagrangian. And, in fact, it is another area where the same mathematics is used in both classical and quantum contexts.

SRQM Diagram: RoadMap of SR→QM
QM Schrödinger Relation

A Tensor Study of Physical 4-Vectors

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The QM Schrödinger Relation
P = iħ**∂**

This is derived from the combination of:

The Einstein-de Broglie Relation
P = ħ**K**

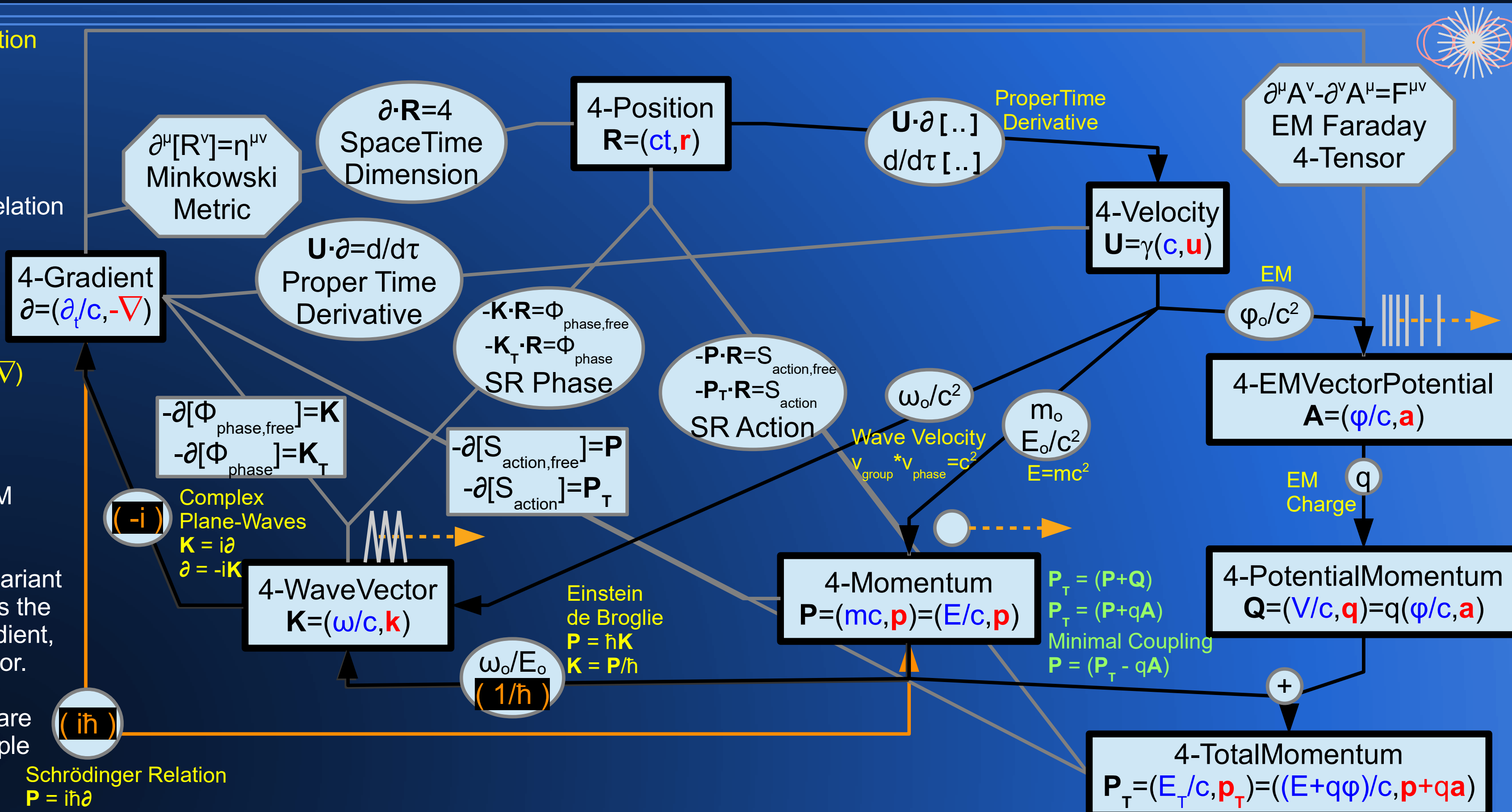
Complex Plane-Waves
K = i**∂**

P = (E/c,**p**) = iħ**∂** = iħ(∂_t/c,-∇)
{temporal} E = iħ∂_t
{spatial} **p** = -iħ∇

These are the standard QM Schrödinger Relations.

It is this Lorentz Scalar Invariant relation (iħ) which connects the 4-Momentum to the 4-Gradient, making it into a QM operator.

Note that these 4-Vectors are already connected in multiple ways in standard SR.



SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^{\mu}{}_{\nu}$ or $T_{\mu}{}^{\nu}$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor V^{μ} = **V** = (v^0,**v**)
SR 4-CoVector
(0,1)-Tensor V_{μ} = (v_0,-**v**)

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Existing SR Rules
Quantum Principles

Trace[$T^{\mu\nu}$] = $\eta_{\mu\nu}T^{\mu\nu}$ = $T^{\mu}{}_{\mu}$ = T
V·V = $V^{\mu}\eta_{\mu\nu}V^{\nu}$ = [$(v^0)^2$ - **v·v**] = $(v^0{}_o)^2$
= Lorentz Scalar

Review of SR 4-Vector Mathematics

4-Gradient $\partial = (\partial_t/c, -\nabla)$ — $\partial\cdot\partial = (\partial_t/c)^2 - \nabla\cdot\nabla = -(\omega_o/c)^2$

4-Position $\mathbf{X} = (ct, \mathbf{x})$ — $\mathbf{X}\cdot\mathbf{X} = ((ct)^2 - \mathbf{x}\cdot\mathbf{x}) = (ct_o)^2 = (c\tau)^2$: Invariant Interval Measure

4-Velocity $\mathbf{U} = \gamma(c, \mathbf{u})$ — $\mathbf{U}\cdot\mathbf{U} = \gamma^2(c^2 - \mathbf{u}\cdot\mathbf{u}) = (c)^2$

4-Momentum $\mathbf{P} = (E/c, \mathbf{p}) = (E_o/c^2)\mathbf{U}$ — $\mathbf{P}\cdot\mathbf{P} = (E/c)^2 - \mathbf{p}\cdot\mathbf{p} = (E_o/c)^2$

4-WaveVector $\mathbf{K} = (\omega/c, \mathbf{k}) = (\omega_o/c^2)\mathbf{U}$ — $\mathbf{K}\cdot\mathbf{K} = (\omega/c)^2 - \mathbf{k}\cdot\mathbf{k} = (\omega_o/c)^2$

$\partial\cdot\mathbf{X} = (\partial_t/c, -\nabla)\cdot(ct, \mathbf{x}) = (\partial_t/c[ct] - (-\nabla\cdot\mathbf{x})) = 1-(-3) = 4$: Dimensionality of SpaceTime

$\mathbf{U}\cdot\partial = \gamma(c, \mathbf{u})\cdot(\partial_t/c, -\nabla) = \gamma(\partial_t + \mathbf{u}\cdot\nabla) = \gamma(d/dt) = d/d\tau$: Derivative wrt. ProperTime is Lorentz Scalar

$\partial[\mathbf{X}] = (\partial_t/c, -\nabla)(ct, \mathbf{x}) = (\partial_t/c[ct], -\nabla[\mathbf{x}]) = \mathrm{Diag}[1, -\mathbf{1}] = \eta^{\mu\nu}$: The Minkowski Metric

$\partial[\mathbf{K}] = (\partial_t/c, -\nabla)(\omega/c, \mathbf{k}) = (\partial_t/c[\omega/c], -\nabla[\mathbf{k}]) = [[\mathbf{0}]]$

$\mathbf{K}\cdot\mathbf{X} = (\omega/c, \mathbf{k})\cdot(ct, \mathbf{x}) = (\omega t - \mathbf{k}\cdot\mathbf{x}) = \varphi$: Phase of SR Wave

$\partial[\mathbf{K}\cdot\mathbf{X}] = \partial[\mathbf{K}]\cdot\mathbf{X} + \mathbf{K}\cdot\partial[\mathbf{X}] = \mathbf{K} = -\partial[\varphi]$: Neg 4-Gradient of Phase gives 4-WaveVector

$(\partial\cdot\partial)[\mathbf{K}\cdot\mathbf{X}] = ((\partial_t/c)^2 - \nabla\cdot\nabla)(\omega t - \mathbf{k}\cdot\mathbf{x}) = 0$

$(\partial\cdot\partial)[\mathbf{K}\cdot\mathbf{X}] = \partial\cdot(\partial[\mathbf{K}\cdot\mathbf{X}]) = \partial\cdot\mathbf{K} = 0$: Wave Continuity Equation, No sources or sinks

let $f = ae\text{^}b(\mathbf{K}\cdot\mathbf{X})$: Standard mathematical plane-waves if { b = -i }

then $\partial[f] = (-i\mathbf{K})ae\text{^}{-i}(\mathbf{K}\cdot\mathbf{X}) = (-i\mathbf{K})f$: $(\partial = -i\mathbf{K})$: Unitary Evolution, Operator Formalism

and $\partial\cdot\partial[f] = (-i)^2(\mathbf{K}\cdot\mathbf{K})f = -(\omega_o/c)^2 f$:

$(\partial\cdot\partial) = (\partial_t/c)^2 - \nabla\cdot\nabla = -(\omega_o/c)^2$: The Klein-Gordon Equation → RQM

Note that no QM Axioms are assumed: This is all just pure SR 4-vector (tensor) manipulation

Review of SR 4-Vector Mathematics

Klein-Gordon Equation: $\partial\cdot\partial = (\partial_t/c)^2 - \nabla\cdot\nabla = -(m_o c/\hbar)^2 = -(\omega_o/c)^2 = -(1/\lambda\!\!\!^{-}_C)^2$

Let $\mathbf{X}_T = (ct+c\Delta t,\mathbf{x})$, then $\partial[\mathbf{X}_T] = (\partial_t/c,-\nabla)(ct+c\Delta t,\mathbf{x}) = \mathrm{Diag}[1,-\mathbf{I}_{(3)}] = \partial[\mathbf{X}] = \eta^{\mu\nu}$
so $\partial[\mathbf{X}_T] = \partial[\mathbf{X}]$ and $\partial[\mathbf{K}] = [[\mathbf{0}]]$
let f = ae^-i($\mathbf{K}\cdot\mathbf{X}_T$), the time translated version
$(\partial\cdot\partial)[f]$
$\partial\cdot(\partial[f])$
$\partial\cdot(\partial[e\hat{\ }{-i}(\mathbf{K}\cdot\mathbf{X}_T)])$
$\partial\cdot(e\hat{\ }{-i}(\mathbf{K}\cdot\mathbf{X}_T)\partial[-i(\mathbf{K}\cdot\mathbf{X}_T)])$
$-i\partial\cdot(f\partial[\mathbf{K}\cdot\mathbf{X}_T])$
$-i\partial[f]\partial[\mathbf{K}\cdot\mathbf{X}_T])+\Psi(\partial\cdot\partial)[\mathbf{K}\cdot\mathbf{X}_T])$
$(-i)^2f(\partial[\mathbf{K}\cdot\mathbf{X}_T])^2 + 0$
$(-i)^2f(\partial[\mathbf{K}]\cdot\mathbf{X}_T + \mathbf{K}\cdot\partial[\mathbf{X}_T])^2$
$(-i)^2f(0+\mathbf{K}\cdot\partial[\mathbf{X}])^2$
$(-i)^2f(\mathbf{K})^2$
$-(\mathbf{K}\cdot\mathbf{K})f$
$-(\omega_o/c)^2f$

What does the Klein-Gordon Equation give us?... A lot of RQM!

Relativistic Quantum Wave Equation: $\partial\cdot\partial = (\partial_t/c)^2 - \nabla\cdot\nabla = -(m_o c/\hbar)^2 = (im_o c/\hbar)^2 = -(\omega_o/c)^2$

The Klein-Gordon Eqn is itself the Relativistic Quantum Equation for spin=0 particles (Scalars)
Factoring the KG Eqn leads to the RQM Dirac Equation for spin=1/2 particles (Spinors)
Applying the KG Eqn to a SR 4-Vector field leads to the RQM Proca Equation for spin=1 particles (4-Vectors)

Taking the low-velocity-limit of the KG leads to the standard QM non-relativistic Schrödinger Eqn, for spin=0
Taking the low-velocity-limit of the Dirac leads to the standard QM non-relativistic Pauli Eqn, for spin=1/2

Setting RestMass $\{m_o \to 0\}$ leads to the RQM Free Wave, Weyl, and Free Maxwell Eqns

In all of these cases, the equations can be modified to work with various potentials by using more SR 4-Vectors, and more empirically found relations between them, e.g. the Minimal Coupling Relations:
4-TotalMomentum $\mathbf{P}_{tot} = \mathbf{P} + q\mathbf{A}$, where **P** is the particle 4-Momentum, (q) is a charge, and **A** is a 4-VectorPotential, typically the 4-EMVectorPotential.

Also note that generating QM from RQM (via a low-energy limit) is much more natural than attempting to “relativize or generalize” a given NRQM equation. Facts assumed from a non-relativistic equation may or may not be applicable to a relativistic one, whereas the relativistic facts are still true in the low-velocity limiting-cases. This leads to the idea that QM is an approximation only of a more general RQM, just as SR is an approximation only of GR.

Relativistic Quantum Wave Eqns.

Spin-(Statistics) **Bose-Einstein=n** **Fermi-Dirac=n/2**	**Relativistic Light-like** **Mass = 0**	**Relativistic Matter-like** **Mass > 0**	**Non-Relativistic Limit $(\lvert v \rvert << c)$** **Mass >0**	**Field** **Representation**
0-(Boson)	**Free Wave** N-G Bosons $(\partial\cdot\partial)\Psi = 0$	**Klein-Gordon** Higgs Bosons, maybe Axions $(\partial\cdot\partial + (m_o c/\hbar)^2)\Psi = [\partial_\mu + i m_o c/\hbar][\partial^\mu - i m_o c/\hbar]\Psi = 0$ with minimal coupling $((i\hbar\partial_t - q\varphi)^2 - (m_o c^2)^2 - c^2(-i\hbar\nabla - qa)^2)\Psi = 0$?Axions? are KG with EM invariant src term $(\partial\cdot\partial + (m_{ao})^2)\Psi = -\kappa \mathbf{e}\cdot\mathbf{b} = -\kappa c\mathrm{Sqrt}[\mathrm{Det}[F^{\mu\nu}]]$ $L = (-\hbar^2/m_o)\partial^\mu\Psi^*\partial_\nu\Psi - m_o c^2\Psi^*\Psi$	**Schrödinger** Common NRQM Systems $(i\hbar\partial_t + [\hbar^2\nabla^2/2m_o - V])\Psi = 0$ with minimal coupling $(i\hbar\partial_t - q\varphi - [(\mathbf{p} - q\mathbf{a})^2]/2m_o)\Psi = 0$	Scalar (0-Tensor) $\Psi = \Psi[K_\mu X^\mu]$ $= \Psi[\Phi]$
1/2-(Fermion)	**Weyl** Idealized Matter Neutinos $(\boldsymbol{\sigma}\cdot\partial)\boldsymbol{\Psi} = 0$ factored to Right & Left Spinors $(\boldsymbol{\sigma}\cdot\partial)\boldsymbol{\Psi}_R = 0$, $(\overline{\boldsymbol{\sigma}}\cdot\partial)\boldsymbol{\Psi}_L = 0$ $L = i\boldsymbol{\Psi}^\dagger{}_R\sigma^\mu\partial_\mu\boldsymbol{\Psi}_R$, $L = i\boldsymbol{\Psi}^\dagger{}_L\overline{\sigma}^\mu\partial_\mu\boldsymbol{\Psi}_L$	**Dirac** Matter Leptons/Quarks $(i\boldsymbol{\gamma}\cdot\partial - m_o c/\hbar)\boldsymbol{\Psi} = 0$ $(\boldsymbol{\gamma}\cdot\partial + i m_o c/\hbar)\boldsymbol{\Psi} = 0$ with minimal coupling $(i\boldsymbol{\gamma}\cdot(\partial + iq\mathbf{A}) - m_o c/\hbar)\boldsymbol{\Psi} = 0$ $L = i\hbar c\overline{\boldsymbol{\Psi}}\gamma^\mu\partial_\mu\boldsymbol{\Psi} - m_o c^2\overline{\boldsymbol{\Psi}}\boldsymbol{\Psi}$	**Pauli** Common NRQM Systems w Spin $(i\hbar\partial_t - [(\boldsymbol{\sigma}\cdot\mathbf{p})^2]/2m_o)\boldsymbol{\Psi} = 0$ with minimal coupling $(i\hbar\partial_t - q\varphi - [(\boldsymbol{\sigma}\cdot(\mathbf{p} - q\mathbf{a}))^2]/2m_o)\boldsymbol{\Psi} = 0$	Spinor $\boldsymbol{\Psi} = \boldsymbol{\Psi}[K_\mu X^\mu]$ $= \boldsymbol{\Psi}[\Phi]$
1-(Boson)	**Maxwell** Photons/Gluons $(\partial\cdot\partial)\mathbf{A} = 0$ free $(\partial\cdot\partial)\mathbf{A} = \mu_o\mathbf{J}$ w current src where $\partial\cdot\mathbf{A} = 0$ $(\partial\cdot\partial)\mathbf{A} = \mu_o e\overline{\Psi}\gamma^\nu\Psi$ QED	**Proca** Force Bosons $(\partial\cdot\partial + (m_o c/\hbar)^2)\mathbf{A} = 0$ where $\partial\cdot\mathbf{A} = 0$ $\partial^\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) + (m_o c/\hbar)^2 A^\nu = 0$		4-Vector (1-Tensor) $\mathbf{A} = A^\nu = A^\nu[K_\mu X^\mu]$ $= A^\nu[\Phi]$

Factoring the KG Equation → Dirac Eqn

Klein-Gordon Equation: $\partial\cdot\partial = (\partial_t/c)^2 - \nabla\cdot\nabla = -(m_o c/\hbar)^2$

Since the 4-vectors are related by constants, we can go back to the 4-Momentum description:

$(\partial_t/c)^2 - \nabla\cdot\nabla = -(m_o c/\hbar)^2$
$(E/c)^2 - \mathbf{p}\cdot\mathbf{p} = (m_o c)^2$
$E^2 - c^2\mathbf{p}\cdot\mathbf{p} - (m_o c^2)^2 = 0$

Factoring: $[\, E - c\,\boldsymbol{\alpha}\cdot\mathbf{p} - \beta(m_o c^2)\,]\,[\, E + c\,\boldsymbol{\alpha}\cdot\mathbf{p} + \beta(m_o c^2)\,] = 0$

E & **p** are quantum operators,
α & β are matrices which must obey $\boldsymbol{\alpha}_i\beta = -\beta\boldsymbol{\alpha}_i$, $\boldsymbol{\alpha}_i\boldsymbol{\alpha}_j = -\boldsymbol{\alpha}_j\boldsymbol{\alpha}_i$, $\boldsymbol{\alpha}_i^2 = \beta^2 = \mathbf{I}$
The left hand term can be set to 0 by itself, giving...
$[\, E - c\,\boldsymbol{\alpha}\cdot\mathbf{p} - \beta(m_o c^2)\,] = 0$, which is one form of the Dirac equation

Remember: $P^\mu = (p^0,\mathbf{p}) = (E/c,\mathbf{p})$ and $\alpha^\mu = (\alpha^0,\boldsymbol{\alpha})$ where $\alpha^0 = I_{(2)}$

$[\, E - c\,\boldsymbol{\alpha}\cdot\mathbf{p} - \beta(m_o c^2)\,] = [\, c\alpha^0 p^0 - c\,\boldsymbol{\alpha}\cdot\mathbf{p} - \beta(m_o c^2)\,] = [\, c\alpha^\mu P_\mu - \beta(m_o c^2)\,] = 0$
$[\, \alpha^\mu P_\mu - \beta(m_o c)\,] = [i\hbar\, \alpha^\mu \partial_\mu - \beta(m_o c)\,] = 0$
$\alpha^\mu \partial_\mu = -\,\beta(i m_o c/\hbar)$

Transforming from Pauli Spinor (2 component) to Dirac Spinor (4 component) form:
Dirac Equation: $(\gamma^\mu\partial_\mu)[\psi] = -(i m_o c/\hbar)\psi$

Thus, the Dirac Eqn is guaranteed by construction to be one solution of the KG Eqn

The KG Equation is at the heart of all the various relativistic wave equations, which differ based on mass and spin values, but all of them respect $E^2 - c^2\mathbf{p}\cdot\mathbf{p} - (m_o c^2)^2 = 0$

SRQM Study: Lots of Relativistic Quantum Wave Equations: A lot of RQM!

Relativistic Quantum Wave Equation: $\partial\cdot\partial = (\partial_t/c)^2 - \nabla\cdot\nabla = -(m_oc/\hbar)^2 = (im_oc/\hbar)^2 = -(\omega_o/c)^2$
$\partial\cdot\partial = -(m_oc/\hbar)^2$

The Klein-Gordon Eqn is itself the Relativistic Quantum Equation for spin=0 particles {Higgs} (4-Scalars)
Factoring the KG Eqn leads to the RQM Dirac Equation for spin=1/2 particles (4-Spinors)
Applying the KG Eqn to a SR 4-Vector field leads to the RQM Proca Equation for spin=1 particles (4-Vectors)

<u>Setting RestMass {$m_o \to 0$} leads to the:</u>
RQM Free Wave (4-Scalar massless)
RQM Weyl (4-Spinor massless)
Free Maxwell Eqns (4-Vector massless)

So, the same Relativistic Quantum Wave Equation is simply applied to different SR Tensorial Quantum Fields
See Mathematical_formulation_of_the_Standard_Model at Wikipedia:

4-Scalar (massive)	Higgs Field φ	$[\partial\cdot\partial = -(m_oc/\hbar)^2]\varphi$	Free Field Eqn→Klein-Gordon Eqn	$\partial\cdot\partial[\varphi] = -(m_oc/\hbar)^2\varphi$
4-Vector (massive)	Weak Field Z^μ,$W^{\pm\mu}$	$[\partial\cdot\partial = -(m_oc/\hbar)^2]Z^\mu$	Free Field Eqn→Proca Eqn	$\partial\cdot\partial[Z^\mu]= -(m_oc/\hbar)^2Z^\mu$
4-Vector (massless m_o=0)	Photon Field A^μ	$[\partial\cdot\partial = 0]A^\mu$	Free Field Eqn→EM Wave Eqn	$\partial\cdot\partial[A^\mu]= 0^\mu$
4-Spinor (massive)	Fermion Field ψ	$[\boldsymbol{\gamma}\cdot\partial = -im_oc/\hbar]\Psi$	Free Field Eqn→Dirac Eqn	$\boldsymbol{\gamma}\cdot\partial[\Psi]= -(im_oc/\hbar)\Psi$

*The Fermion field is a special case, the Dirac Gamma Matrices γ^μ and 4-Spinor field Ψ work together to preserve Lorentz Invariance.

SRQM Study: Lots of Relativistic Quantum Wave Equations: A lot of RQM!

In relativistic quantum mechanics and quantum field theory, the Bargmann–Wigner equations describe free particles of arbitrary spin j, an integer for bosons (j = 1, 2, 3 ...) or half-integer for fermions (j = ½, 3⁄2, 5⁄2 ...). The solutions to the equations are wavefunctions, mathematically in the form of multi-component spinor fields.

Bargmann–Wigner equations: $(-\gamma^{\mu}P_{\mu} + mc)_{\alpha r\, \alpha' r}\, \psi_{\alpha 1 \ldots \alpha' r \ldots \alpha 2j} = 0$

In relativistic quantum mechanics and quantum field theory, the Joos–Weinberg equation is a relativistic wave equations applicable to free particles of arbitrary spin j, an integer for bosons (j = 1, 2, 3 ...) or half-integer for fermions (j = ½, 3⁄2, 5⁄2 ...). The solutions to the equations are wavefunctions, mathematically in the form of multi-component spinor fields. The spin quantum number is usually denoted by s in quantum mechanics, however in this context j is more typical in the literature.

Joos–Weinberg equation: $[\gamma^{\mu 1 \mu 2 \ldots \mu 2j} P_{\mu 1} P_{\mu 2} \ldots P_{\mu 2j} + (mc)^{2j}]\, \Psi = 0$

The primary difference appears to be the expansion in either the wavefunctions for (BW) or the Dirac Gamma’s for (JW)

For both of these: A state or quantum field in such a representation would satisfy no field equation except the Klein-Gordon equation.

Yet another form is the Duffin-Kemmer-Petiau Equation vs Dirac Equation
DKP Eqn {spin 0 or 1}: $(i\hbar\beta^{\alpha}\partial_{\alpha} - m_{o}c)\Psi = 0$, with β^{α} as the DKP matrices
Dirac Eqn (spin ½}: $(i\hbar\gamma^{\alpha}\partial_{\alpha} - m_{o}c)\Psi = 0$, with γ^{α} as the Dirac Gamma matrices

A few more SR 4-Vectors

SR 4-Vector	Definition	Unites
4-Position	$\mathbf{R} = (ct,\mathbf{r})$; alt. $\mathbf{X} = (ct,\mathbf{x})$	Time, Space
4-Velocity	$\mathbf{U} = \gamma(c,\mathbf{u})$	Gamma, Velocity
4-Momentum	$\mathbf{P} = (E/c,\mathbf{p}) = (mc,\mathbf{p})$	Energy:Mass, Momentum
4-WaveVector	$\mathbf{K} = (\omega/c,\mathbf{k}) = (\omega/c,\omega\hat{\mathbf{n}}/v_{phase})$	Frequency, WaveNumber
4-Gradient	$\partial = (\partial_t/c,-\nabla)$	Temporal Partial, Space Partial
4-VectorPotential	$\mathbf{A} = (\varphi/c,\mathbf{a})$	Scalar Potential, Vector Potential
4-TotalMomentum	$\mathbf{P}_{\mathbf{tot}} = (E/c+q\varphi/c,\mathbf{p}+q\mathbf{a})$	Energy-Momentum inc. EM fields
4-TotalWaveVector	$\mathbf{K}_{\mathbf{tot}} = (\omega/c+(q/\hbar)\varphi/c,\mathbf{k}+(q/\hbar)\mathbf{a})$	Freq-WaveNum inc. EM fields
4-CurrentDensity	$\mathbf{J} = (c\rho,\mathbf{j}) = q\mathbf{J}_{\mathbf{prob}}$	Charge Density, Current Density
4-ProbabiltyCurrentDensity can have complex values	$\mathbf{J}_{\mathbf{prob}} = (c\rho_{prob},\mathbf{j}_{\mathbf{prob}})$	QM Probability (Density, Current Density)

More SR 4-Vectors Explained

SR 4-Vector	Empirical Fact	What it means...
4-Position	$\mathbf{R} = (ct,\mathbf{r})$	SpaceTime as Single United Concept
4-Velocity	$\mathbf{U} = d\mathbf{R}/d\tau$	Velocity is Proper Time Derivative
4-Momentum	$\mathbf{P} = m_o\mathbf{U} = (E_o/c^2)\mathbf{U}$	Mass-Energy-Momentum Equivalence
4-WaveVector	$\mathbf{K} = \mathbf{P}/\hbar = (\omega_o/c^2)\mathbf{U}$	Wave-Particle Duality
4-Gradient	$\partial = -i\mathbf{K}$	Unitary Evolution of States Operator Formalism, Complex Waves
4-VectorPotential	$\mathbf{A} = (\varphi/c,\mathbf{a}) = (\varphi_o/c^2)\mathbf{U}$	Potential Fields...
4-TotalMomentum	$\mathbf{P}_{\mathbf{tot}} = \mathbf{P} + q\mathbf{A}$	Energy-Momentum inc. Potential Fields
4-TotalWaveVector	$\mathbf{K}_{\mathbf{tot}} = \mathbf{K} + (q/\hbar)\mathbf{A}$	Freq-WaveNum inc. Potential Fields
4-CurrentDensity	$\mathbf{J} = \rho_o\mathbf{U} = q\mathbf{J}_{\mathbf{prob}}$ $\partial\cdot\mathbf{J} = 0$	ChargeDensity-CurrentDensity Equivalence CurrentDensity is conserved
4-Probability CurrentDensity	$\mathbf{J}_{\mathbf{prob}} = (c\rho_{prob},\mathbf{j}_{\mathbf{prob}})$ $\partial\cdot\mathbf{J}_{\mathbf{prob}} = 0$	QM Probability from SR Probability Worldlines are conserved

Minimal Coupling = Potential Interaction

Klein-Gordon Eqn → Schrödinger Eqn

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$\mathbf{P_T} = \mathbf{P} + \mathbf{Q} = \mathbf{P} + q\mathbf{A}$ — Minimal Coupling: Total = Dynamic + Charge_Coupled to 4-(EM)VectorPotential

$\mathbf{K} = i\partial$ — Complex Plane-Waves

$\mathbf{P} = \hbar\mathbf{K}$ — Einstein-de Broglie QM Relations

$\mathbf{P} = i\hbar\partial$ — Schrödinger Relations

$\mathbf{P} = (E/c,\mathbf{p}) = \mathbf{P_T} - q\mathbf{A} = (E_T/c - q\varphi/c, \mathbf{p_T} - q\mathbf{a}) = \hbar\mathbf{K} = i\hbar\partial$

$\partial = (\partial_t/c, -\nabla) = \partial_T + (iq/\hbar)\mathbf{A} = (\partial_{tT}/c + (iq/\hbar)\varphi/c, -\nabla_T + (iq/\hbar)\mathbf{a}) = -i\mathbf{K} = (-i/\hbar)\mathbf{P}$

$\partial\cdot\partial = (\partial_t/c)^2 - \nabla^2 = -(m_o c/\hbar)^2$: — The Klein-Gordon RQM Wave Equation (relativistic QM)

$\mathbf{P}\cdot\mathbf{P} = (E/c)^2 - \mathbf{p}^2 = (m_o c)^2$: — Einstein Mass:Energy:Momentum Equivalence

$E^2 = (m_o c^2)^2 + c^2\mathbf{p}^2$: — Relativistic

$E \sim [\, (m_o c^2) + \mathbf{p}^2/2m_o \,]$: — Low velocity limit { $|\mathbf{v}| << c$ } from $(1+x)^n \sim [1 + nx + O(x^2)]$ for $|x|<<1$

$(E_T - q\varphi)^2 = (m_o c^2)^2 + c^2(\mathbf{p_T} - q\mathbf{a})^2$: — Relativistic with Minimal Coupling

$(E_T - q\varphi) \sim [\, (m_o c^2) + (\mathbf{p_T} - q\mathbf{a})^2/2m_o \,]$: — Low velocity with Minimal Coupling

$(i\hbar\partial_{tT} - q\varphi)^2 = (m_o c^2)^2 + c^2(-i\hbar\nabla_T - q\mathbf{a})^2$: — Relativistic with Minimal Coupling

$(i\hbar\partial_{tT} - q\varphi) \sim [\, (m_o c^2) + (-i\hbar\nabla_T - q\mathbf{a})^2/2m_o \,]$: — Low velocity with Minimal Coupling

The better statement is that the Schrödinger Eqn is the limiting low-velocity case of the more general KG Egn, not that the KG Eqn is the relativistic generalization of the Schrödinger Eqn

$(i\hbar\partial_{tT}) \sim [\, q\varphi + (m_o c^2) + (i\hbar\nabla_T + q\mathbf{a})^2/2m_o \,]$: — Low velocity with Minimal Coupling

$(i\hbar\partial_{tT}) \sim [\, V + (i\hbar\nabla_T + q\mathbf{a})^2/2m_o \,]$: — $V = q\varphi + (m_o c^2)$

$(i\hbar\partial_{tT}) \sim [\, V - (\hbar\nabla_T)^2/2m_o \,]$: — Typically the 3-vector_potential $\mathbf{a} \sim 0$ in many situations

$(i\hbar\partial_{tT})|\Psi\rangle \sim [\, V - (\hbar\nabla_T)^2/2m_o \,]|\Psi\rangle$: — The Schrödinger NRQM Wave Equation (non-relativistic QM)

Once one has a Relativistic Wave Eqn...

Klein-Gordon Equation: $\partial\cdot\partial = (\partial_t/c)^2 - \nabla\cdot\nabla = (-im_o c/\hbar)^2 = -(m_o c/\hbar)^2$

Once we have derived a RWE, what does it imply?

The KG Eqn. was derived from the physics of SR plus a few empirical facts. It is a 2nd order, linear, wave PDE that pertains to physical objects of reality from SR.

Just being a linear wave PDE implies all the mathematical techniques that have been discovered to solve such equations generally: Hilbert Space, Superpositions, <Bra|,|Ket> notation, wavevectors, wavefunctions, etc. These things are from mathematics in general, not only and specifically from an Axiom of QM.

Therefore, if one has a physical RWE, it implies the mathematics of waves, the formalism of the mathematics, and thus the mathematical Principles and Formalism of QM. Again, QM Axioms are not required – they emerge from the physics and math...

Once one has a Relativistic Wave Eqn… Examine Photon Polarization

From the Wikipedia page on [Photon Polarization]

Photon polarization is the quantum mechanical description of the classical polarized sinusoidal plane electromagnetic wave. An individual photon can be described as having right or left circular polarization, or a superposition of the two. Equivalently, a photon can be described as having horizontal or vertical linear polarization, or a superposition of the two.

The description of photon polarization contains many of the physical concepts and much of the mathematical machinery of more involved quantum descriptions and forms a fundamental basis for an understanding of more complicated quantum phenomena. Much of the mathematical machinery of quantum mechanics, such as state vectors, probability amplitudes, unitary operators, and Hermitian operators, emerge naturally from the classical Maxwell's equations in the description. The quantum polarization state vector for the photon, for instance, is identical with the Jones vector, usually used to describe the polarization of a classical wave. Unitary operators emerge from the classical requirement of the conservation of energy of a classical wave propagating through lossless media that alter the polarization state of the wave. Hermitian operators then follow for infinitesimal transformations of a classical polarization state.

Many of the implications of the mathematical machinery are easily verified experimentally. In fact, many of the experiments can be performed with two pairs (or one broken pair) of polaroid sunglasses.

The connection with quantum mechanics is made through the identification of a minimum packet size, called a photon, for energy in the electromagnetic field. The identification is based on the theories of Planck and the interpretation of those theories by Einstein. The correspondence principle then allows the identification of momentum and angular momentum (called spin), as well as energy, with the photon.

Principle of Superposition: From the mathematics of waves

Klein-Gordon Equation: $\partial \cdot \partial = (\partial_t/c)^2 - \nabla \cdot \nabla = -(m_o c/\hbar)^2 = -(\omega_o/c)^2$

The Extended Superposition Principle for Linear Equations

==

Suppose that the non-homogeneous equation, where L is linear, is solved by some particular u_p
Suppose that the associated homogeneous problem is solved by a sequence of u_i.
$L(u_p) = C$; $L(u_0) = 0$, $L(u_1) = 0$, $L(u_2) = 0$ ...
Then u_p plus any linear combination of the u_n satisfies the original non-homogeneous equation:
$L(u_p + \Sigma\, a_n u_n) = C$,
where a_n is a sequence of (possibly complex) constants and the sum is arbitrary.

==

Note that there is no mention of partial differentiation. Indeed, it's true for any linear equation, algebraic or integro-partial differential-whatever.

QM superposition is not axiomatic, it emerges from the mathematics of the Linear PDE

Klein-Gordon obeys Principle of Superposition

Klein-Gordon Equation: $\partial\cdot\partial = (\partial_t/c)^2 - \nabla\cdot\nabla = -(m_o c/\hbar)^2 = -(\omega_o/c)^2$

$\mathbf{K}\cdot\mathbf{K} = (\omega/c)^2 - \mathbf{k}\cdot\mathbf{k} = (\omega_o/c)^2$: The particular solution (w rest mass)
$\mathbf{K_n}\cdot\mathbf{K_n} = (\omega_n/c)^2 - \mathbf{k_n}\cdot\mathbf{k_n} = 0$: The homogenous solution for a (virtual photon?) microstate n
Note that $\mathbf{K_n}\cdot\mathbf{K_n} = 0$ is a null 4-vector (photonic)

Let $\Psi_p = Ae$^$-i(\mathbf{K}\cdot\mathbf{X})$, then $\partial\cdot\partial[\Psi_p] = (-i)^2(\mathbf{K}\cdot\mathbf{K})\Psi_p = -(\omega_o/c)^2\Psi_p$
which is the Klein-Gordon Equation, the particular solution...

Let $\Psi_n = A_n e$^$-i(\mathbf{K_n}\cdot\mathbf{X})$, then $\partial\cdot\partial[\Psi_n] = (-i)^2(\mathbf{K_n}\cdot\mathbf{K_n})\Psi_n = (0)\Psi_n$
which is the Klein-Gordon Equation homogeneous solution for a microstate n

We may take $\Psi = \Psi_p + \Sigma_n \Psi_n$

Hence, the Principle of Superposition is not required as an QM Axiom, it follows from SR and our empirical facts which lead to the Klein-Gordon Equation. The Klein-Gordon equation is a linear wave PDE, which has overall solutions which can be the complex linear sums of individual solutions – i.e. it obeys the Principle of Superposition.
This is not an axiom – it is a general mathematical property of linear PDE's.
This property continues over as well to the limiting case { $|\mathbf{v}| \ll c$ } of the Schrödinger Equation.

QM Hilbert Space:
From the mathematics of waves

Klein-Gordon Equation: $\partial\cdot\partial = (\partial_t/c)^2 - \nabla\cdot\nabla = -(m_o c/\hbar)^2$

Hilbert Space (HS) representation:
if $|\Psi\rangle$ ε HS, then $c|\Psi\rangle$ ε HS, where c is complex number
if $|\Psi_1\rangle$ and $|\Psi_2\rangle$ ε HS, then $|\Psi_1\rangle+|\Psi_2\rangle$ ε HS
if $|\Psi\rangle = c_1|\Psi_1\rangle+c_2|\Psi_2\rangle$, then $\langle\Phi|\Psi\rangle = c_1\langle\Phi|\Psi_1\rangle+c_2\langle\Phi|\Psi_2\rangle$ and $\langle\Psi| = c_1^*\langle\Psi_1|+c_2^*\langle\Psi_2|$
$\langle\Phi|\Psi\rangle = \overline{\langle\Psi|\Phi\rangle}$
$\langle\Psi|\Psi\rangle >= 0$
if $\langle\Psi|\Psi\rangle = 0$, then $|\Psi\rangle = \mathbf{0}$
etc.

Hilbert spaces arise naturally and frequently in mathematics, physics, and engineering, typically as infinite-dimensional function spaces. They are indispensable tools in the theories of partial differential equations, Fourier analysis, signal processing, heat transfer, ergodic theory, and Quantum Mechanics.

The QM Hilbert Space emerges from the fact that the KG Equation is a linear wave PDE – Hilbert spaces as solutions to PDE's are a purely mathematical phenomenon – no QM Axiom is required.

Likewise, this introduces the <bra|,|ket> notation, wavevectors, wavefunctions, etc.

Note:

One can use Hilbert Space descriptions of Classical Mechanics using the Koopman-von Neumann formulation.
One can not use Hilbert Space descriptions of Quantum Mechanics by using the Phase Space formulation of QM.

Canonical Commutation Relation: Viewed from standard QM

Standard QM Canonical Commutation Relation: $[\mathbf{x}^j,\mathbf{p}^k] = i\hbar\delta^{jk}$

The Standard QM Canonical Commutation Relation is simply an axiom in standard QM.
It is just given, with no explanation. You just had to accept it.

I always found that unsatisfactory.

There are at least 4 parts to it:

Where does the commutation ([,]) come from?
Where does the imaginary constant (i) come from?
Where does the Planck constant (ħ) come from?
Where does the Kronecker Delta (δ^{jk}) come from?

See the next page for SR enlightenment...

SR → QM

4-Vector SRQM Interpretation of QM

SRQM Diagram:
Canonical QM Commutation Relation Derived from SR

A Tensor Study of Physical 4-Vectors

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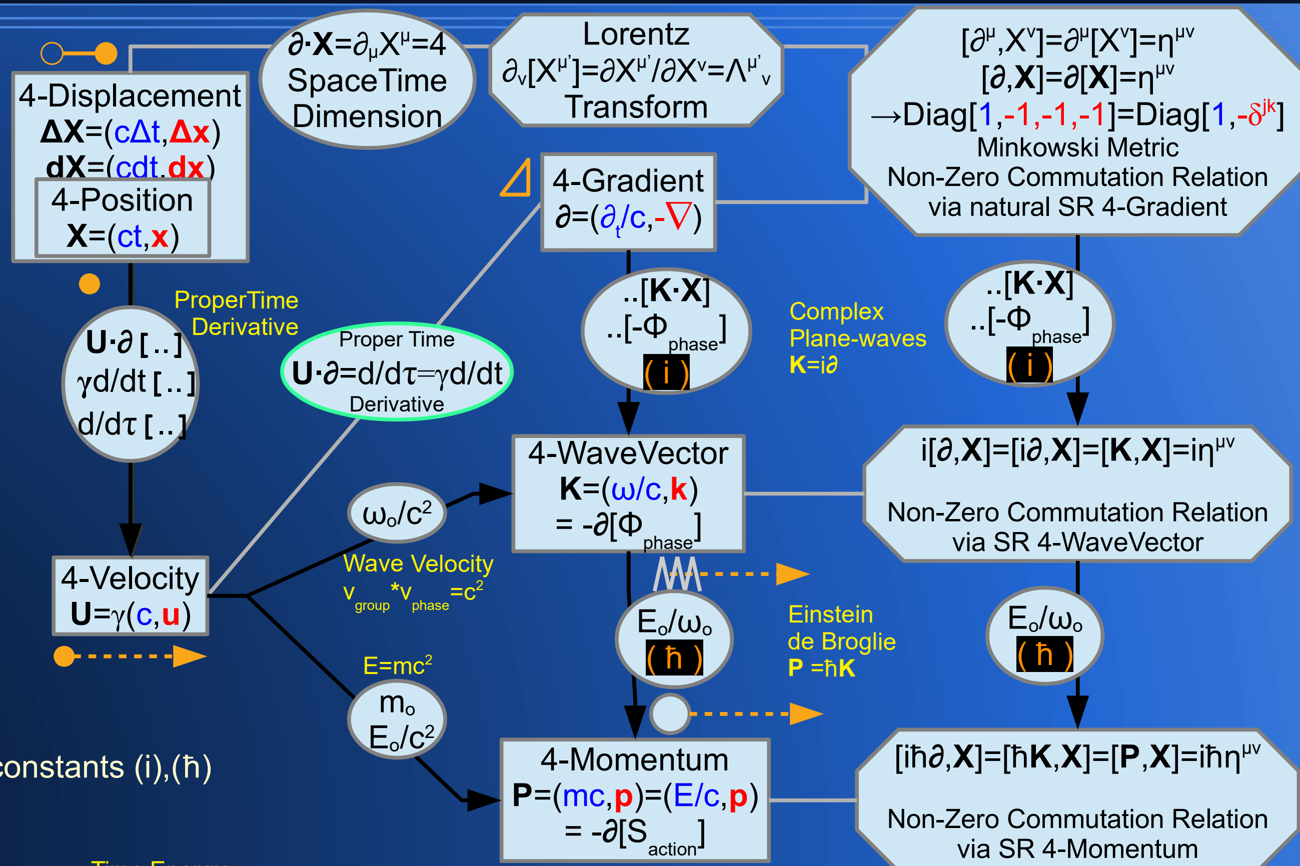
Let (f) be an arbitrary SR function
$\mathbf{X}[f] = \mathbf{X}f$, $\partial[f] = \partial f$
X, function or not, has no effect on (f)
$\partial=\partial[\ ]$ is definitely an SR function:operator

$\mathbf{X}[\partial[f]] = \mathbf{X}\partial[f]$
$\partial[\mathbf{X}f] = \partial[\mathbf{X}]f + \mathbf{X}\partial[f]$
$\partial[\mathbf{X}f] - \mathbf{X}\partial[f] = \partial[\mathbf{X}]f$
$\partial[\mathbf{X}[f]] - \mathbf{X}[\partial[f]] = \partial[\mathbf{X}]f$
Recognize this as a commutation relation
$[\ \partial\ ,\ \mathbf{X}\ ]f = \partial[\mathbf{X}]f$

$[\ \partial\ ,\ \mathbf{X}\ ] = \partial[\mathbf{X}]$
$= \partial^\mu[X^\nu]$
$= (\partial_t/c,-\nabla)[(ct,\mathbf{x})]$
$= (\partial_t/c,-\partial_x,-\partial_y,-\partial_z)[(ct,x,y,z)]$
$= \mathrm{Diag}\{1,-1,-1,-1\} = \mathrm{Diag}[1,-\delta^{jk}]$
$= \eta^{\mu\nu}$ = Minkowski Metric

$[\partial^\mu,X^\nu] = \eta^{\mu\nu}$ Tensor form:true for all observers
$[P^\mu,X^\nu] = i\hbar\eta^{\mu\nu}$ Independently true from empirical constants (i),(ħ)
$[p^k,x^j] = -i\hbar\delta^{kj}$ $[p^0,x^0] = [E/c,ct] = [E,t] = i\hbar$

$[x^j,p^k] = i\hbar\delta^{jk}$ Position:Momentum QM Commutation Relation $[t,E] = -i\hbar$ Time:Energy QM Commutation Relation



{**P** = ħ**K**} and {**K** = i∂} are empirical SR relations

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^\mu{}_\nu$ or $T_\mu{}^\nu$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0,\mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0,-\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Existing SR Rules
Quantum Principles

$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu{}_\mu = T$
$\mathbf{V}\cdot\mathbf{V} = V^\mu\eta_{\mu\nu}V^\nu = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$ = Lorentz Scalar

SRQM Study:
4-Position and 4-Gradient

A Tensor Study of Physical 4-Vectors

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Invariant Interval
$\mathbf{R}\cdot\mathbf{R}=(ct)^2-\mathbf{r}\cdot\mathbf{r} = (c\tau)^2$

4-Displacement
$\Delta\mathbf{R}=(c\Delta t,\Delta\mathbf{r})$
$d\mathbf{R}=(cdt,d\mathbf{r})$

4-Position
$\mathbf{R}=(ct,\mathbf{r})$

Invariant d'Alembertian Wave Equation
$\partial\cdot\partial=(\partial_t/c)^2-\nabla\cdot\nabla=(\partial_\tau/c)^2$

4-Gradient
$\partial=(\partial_t/c,-\nabla)$
$=(\partial_t/c,-\partial_x,-\partial_y,-\partial_z)$

SR: Minkowski Metric
$\partial[\mathbf{R}] = \partial^\mu R^\nu = \eta^{\mu\nu}$

$\rightarrow \mathrm{Diag}[1,-1,-1,-1]$
$= \mathrm{Diag}[1,-I_{(3)}]$
$= \mathrm{Diag}[1,-\delta^{jk}]$
{in Cartesian form}
"Particle Physics" Convention
$\{\eta_{\mu\mu}\} = 1/\{\eta^{\mu\mu}\}$
$\mathrm{Tr}[\eta^{\mu\nu}] = 4$
$\eta_\mu{}^\nu = \delta_\mu{}^\nu$

SR: Lorentz Transform
$\partial_\nu[R^{\mu'}] = \partial R^{\mu'}/\partial R^\nu = \Lambda^{\mu'}{}_\nu$

$\Lambda^\mu{}_\alpha\Lambda^\alpha{}_\nu = \eta^\mu{}_\nu = \delta^\mu{}_\nu$
$\eta_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta = \eta_{\alpha\beta}$
$(\mathrm{Det}[\Lambda])^2 = 1$
$\mathrm{Det}[\Lambda] = \pm 1$
$\Lambda^\mu{}_\nu = (\Lambda^{-1})_\nu{}^\mu$
$\Lambda_{\mu\nu}\Lambda^{\mu\nu} = 4$

Rotations
Boosts
CPT

SpaceTime
$\partial\cdot\mathbf{R} = \partial_\mu R^\mu = 4$
Dimension

SRQM: Tensor Zero Exterior Product
$\partial\wedge\mathbf{R} = \partial^\mu R^\nu - \partial^\nu R^\mu$
$= \eta^{\mu\nu} - \eta^{\nu\mu} = 0^{\mu\nu}$

SRQM: Non-Zero Commutation
$[\partial,\mathbf{R}] = [\partial^\mu,R^\nu]$
$=\partial^\mu R^\nu - R^\nu\partial^\mu$
$= \eta^{\mu\nu}$

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^\mu{}_\nu$ or $T_\mu{}^\nu$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0,\mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0,-\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu{}_\mu = T$
$\mathbf{V}\cdot\mathbf{V} = V^\mu\eta_{\mu\nu}V^\nu = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$
= Lorentz Scalar

Heisenberg Uncertainty Principle: Viewed from SRQM

Heisenberg Uncertainty $\{ \sigma^2_A \sigma^2_B \} >= (1/2)|\langle[A,B]\rangle| \}$
arises from the non-commuting nature of certain operators.

The commutator is $[A,B] = AB-BA$, where A & B are functional “measurement” operators.
The Operator Formalism arose naturally from our SR → QM path: [$\partial = -i\mathbf{K}$].

The Generalized Uncertainty Relation: $\sigma_f^2\sigma_g^2 = (\Delta F) * (\Delta G) >= (1/2)|\langle i[F,G] \rangle|$

The uncertainty relation is a very general mathematical property, which applies to both classical or quantum systems. From Wikipedia: Photon Polarization: "This is a purely mathematical result. No reference to a physical quantity or principle is required.”

The Cauchy–Schwarz inequality asserts that (for all vectors f and g of an inner product space, with either real or complex numbers):
$\sigma_f^2\sigma_g^2 = [\langle f | f \rangle \cdot \langle g | g \rangle] >= |\langle f | g \rangle|^2$

But first, let's back up a bit; Using standard complex number math, we have:
$z = a + ib$
$z^* = a - ib$
$Re(z) = a = (z + z^*)/(2)$
$Im(z) = b = (z - z^*)/(2i)$
$z^*z = |z|^2 = a^2 + b^2 = [Re(z)]^2 + [Im(z)]^2 = [(z + z^*)/(2)]^2 + [(z - z^*)/(2i)]^2$
or
$|z|^2 = [(z + z^*)/(2)]^2 + [(z - z^*)/(2i)]^2$

Now, generically, based on the rules of a complex inner product space we can arbitrarily assign:
$z = \langle f | g \rangle$, $z^* = \langle g | f \rangle$

Which allows us to write:
$|\langle f | g \rangle|^2 = [(\langle f | g \rangle + \langle g | f \rangle)/(2)]^2 + [(\langle f | g \rangle - \langle g | f \rangle)/(2i)]^2$

We can also note that:
$| f \rangle = F| \Psi \rangle$ and $| g \rangle = G| \Psi \rangle$

Thus,
$|\langle f | g \rangle|^2 = [(\langle \Psi |F^* G| \Psi \rangle + \langle \Psi |G^* F| \Psi \rangle)/(2)]^2 + [(\langle \Psi |F^* G| \Psi \rangle - \langle \Psi |G^* F| \Psi \rangle)/(2i)]^2$

For Hermetian Operators…
$F^* = +F$, $G^* = +G$

For Anti-Hermetian (Skew-Hermetian) Operators…
$F^* = -F$, $G^* = -G$

Assuming that F and G are either both Hermetian, or both anti-Hermetian…
$|\langle f | g \rangle|^2 = [(\langle \Psi |(\pm)FG| \Psi \rangle + \langle \Psi |(\pm)GF| \Psi \rangle)/(2)]^2 + [(\langle \Psi |(\pm)FG| \Psi \rangle - \langle \Psi |(\pm)GF| \Psi \rangle)/(2i)]^2$
$|\langle f | g \rangle|^2 = [(\pm)(\langle \Psi |FG| \Psi \rangle + \langle \Psi |GF| \Psi \rangle)/(2)]^2 + [(\pm)(\langle \Psi |FG| \Psi> - \langle \Psi |GF| \Psi \rangle)/(2i)]^2$

We can write this in commutator and anti-commutator notation…
$|\langle f | g \rangle|^2 = [(\pm)(\langle \Psi |\{F,G\}| \Psi \rangle)/(2)]^2 + [(\pm)(\langle \Psi |[F,G]| \Psi \rangle)/(2i)]^2$

Due to the squares, the (±)'s go away, and we can also multiply the commutator by an (i^2)

$|\langle f | g \rangle|^2 = [(\langle \Psi |\{F,G\}| \Psi \rangle)/2]^2 + [(\langle \Psi |i[F,G]| \Psi \rangle)/2]^2$

$|\langle f | g \rangle|^2 = [(\langle \{F,G\} \rangle)/2]^2 + [(\langle i[F,G] \rangle)/2]^2$

The Cauchy–Schwarz inequality again…
$\sigma_f^2\sigma_g^2 = [\langle f | f \rangle \cdot \langle g | g \rangle] >= |\langle f | g \rangle|^2 = [(\langle \{F,G\} \rangle)/2]^2 + [(\langle i[F,G] \rangle)/2]^2$

Taking the root:
$\sigma_f^2\sigma_g^2 >= (1/2)|\langle i[F,G] \rangle|$

Which is what we had for the generalized Uncertainty Relation.

Note This is not a QM axiom - This is just pure math. At this stage we already see the hints of commutation and anti-commutation.
It is true generally, whether applying to a physical or purely mathematical situation.

Heisenberg Uncertainty Principle: Simultaneous vs Sequential

Heisenberg Uncertainty { $\sigma^2{}_A\sigma^2{}_B >= (1/2)|<[A,B]>|$ } arises from the non-commuting nature of certain operators.
$[\partial^\mu,X^\nu] = \partial[\mathbf{X}] = \eta^{\mu\nu}$ = Minkowski Metric
$[P^\mu,X^\nu] = [i\hbar\partial^\mu,X^\nu] = i\hbar[\partial^\mu,X^\nu] = i\hbar\eta^{\mu\nu}$

Consider the following:
Operator A acts on System |Ψ> at SR Event A: A|Ψ> →|Ψ'>
Operator B acts on System |Ψ'> at SR Event B: B|Ψ'> →|Ψ''>
or BA|Ψ> = B|Ψ'> = |Ψ''>

If measurement Events A & B are space-like separated, then there are observers who can see {A before B, A simultaneous with B, A after B}, which of course does not match the quantum description of how Operators act on Kets

If Events A & B are time-like separated, then all observers will always see A before B. This does match how the operators act on Kets, and also matches how |Ψ> would be evolving along its worldline, starting out as |Ψ>, getting hit with operator A at Event A to become |Ψ'>, then getting hit with operator B at Event B to become |Ψ''>.

The Uncertainty Relation here does NOT refer to simultaneous (space-like separated) measurements, it refers to sequential (time-like separated) measurements. This removes the need for ideas about the particles not having simultaneous properties. There are simply no “simultaneous measurements” of non-zero commuting properties on an individual system, a single worldline – they are sequential, and the first measurement places the system in such a state that the outcome of the second measurement will be altered wrt. if the order of the operations had been reversed.

Pauli Exclusion Principle: Requires SR for the detailed explanation

The Pauli Exclusion Principle is a result of the empirical fact that nature uses identical particles, and this combined with the Spin-Statistics theorem from SR, leads to an exclusion principle for fermions (anti-symmetric, Fermi-Dirac statistics) and an aggregation principle for bosons (symmetric, Bose-Einstein statistics). The Spin-Statistics Theorem is related as well to the CPT Theorem.

For large numbers and/or mixed states these both tend to the Maxwell-Boltzmann statistics. In the $\{kT>>(\varepsilon_i-\mu)\}$ limit, Bose-Einstein reduces to Rayleigh-Jeans. The commutation relations here are based on space-like separation particle exchanges, unlike the time-like separation for measurement operator exchanges in the Uncertainty Principle.

Spin	Particle Type	Quantum Statistics	Classical $\{ kT>>(\varepsilon_i-\mu) \}$
spin:(0,1,...,N)	Indistinguishable, Commutation relation (ab = ba)	Bose-Einstein: $n_i = g_i / [e^{(\varepsilon_i-\mu)/kT} -1]$ aggregation principle	Rayleigh-Jeans: from $e^x \sim (1 + x +...)$ $n_i = g_i / [(\varepsilon_i-\mu)/kT]$
		↓ Limit as $e^{(\varepsilon_i-\mu)/kT} >>1$ ↓	
Multi-particle Mixed	Distinguishable, or high temp, or low density	Maxwell-Boltzmann: $n_i = g_i / [e^{(\varepsilon_i-\mu)/kT} +0]$	Maxwell-Boltzmann: $n_i = g_i / [e^{(\varepsilon_i-\mu)/kT}]$
		↑ Limit as $e^{(\varepsilon_i-\mu)/kT} >>1$ ↑	
spin:(1/2,3/2,...,N/2)	Indistinguishable, Anti-commutation relation (ab = - ba)	Fermi-Dirac: $n_i = g_i / [e^{(\varepsilon_i-\mu)/kT} +1]$ exclusion principle	

4-Vectors & Minkowski Space Review
Complex 4-Vectors

Complex 4-vectors are simply 4-Vectors where the components may be complex-valued

$\mathbf{A} = A^\mu = (a^0,\mathbf{a}) = (a^0,a^1,a^2,a^3) \rightarrow (a^t,a^x,a^y,a^z)$
$\mathbf{B} = B^\mu = (b^0,\mathbf{b}) = (b^0,b^1,b^2,b^3) \rightarrow (b^t,b^x,b^y,b^z)$

Examples of 4-Vectors with complex components are the 4-Polarization and the 4-ProbabilityCurrentDensity

Minkowski Metric $g^{\mu\nu} \rightarrow \eta^{\mu\nu} = \eta_{\mu\nu} \rightarrow \mathrm{Diag}[1,-1,-1,-1] = \mathrm{Diag}[1,-\mathbf{I}_{(3)}]$,
which is the {curvature~0 limit = low-mass limit} of the GR metric $g^{\mu\nu}$.

Applying the Metric to raise or lower an index also applies a complex-conjugation *

Scalar Product = Lorentz Invariant → Same value for all inertial observers
$\mathbf{A}\cdot\mathbf{B} = \eta_{\mu\nu} A^\mu B^\nu = A_\nu{}^* B^\nu = A^\mu B_\mu{}^* = (a^{0*} b^0 - \mathbf{a}^*\cdot\mathbf{b})$ using the Einstein summation convention

This reverts to the usual rules for real components
However, it does imply that $\mathbf{A}\cdot\mathbf{B} = \overline{\mathbf{B}\cdot\mathbf{A}}$

SR → QM

4-Vector SRQM Interpretation of QM

SRQM: CPT Theorem
Phase Connection, Lorentz Invariance

A Tensor Study of Physical 4-Vectors

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John B. Wilson

The Phase is a Lorentz Scalar Invariant – all observers must agree on its value.
$\mathbf{K}\cdot\mathbf{X} = (\omega/c,\mathbf{k})\cdot(ct,\mathbf{x}) = (\omega t - \mathbf{k}\cdot\mathbf{x}) = -\Phi$: Phase of SR Wave

We take the point of view of an observer operating on a particle at 4-Position **X**, which has an initial 4-WaveVector **K**. The 4-Position **X** of the particle, the operation's event, will not change: we are applying the various operations only to the particle's 4-Momentum **K**.

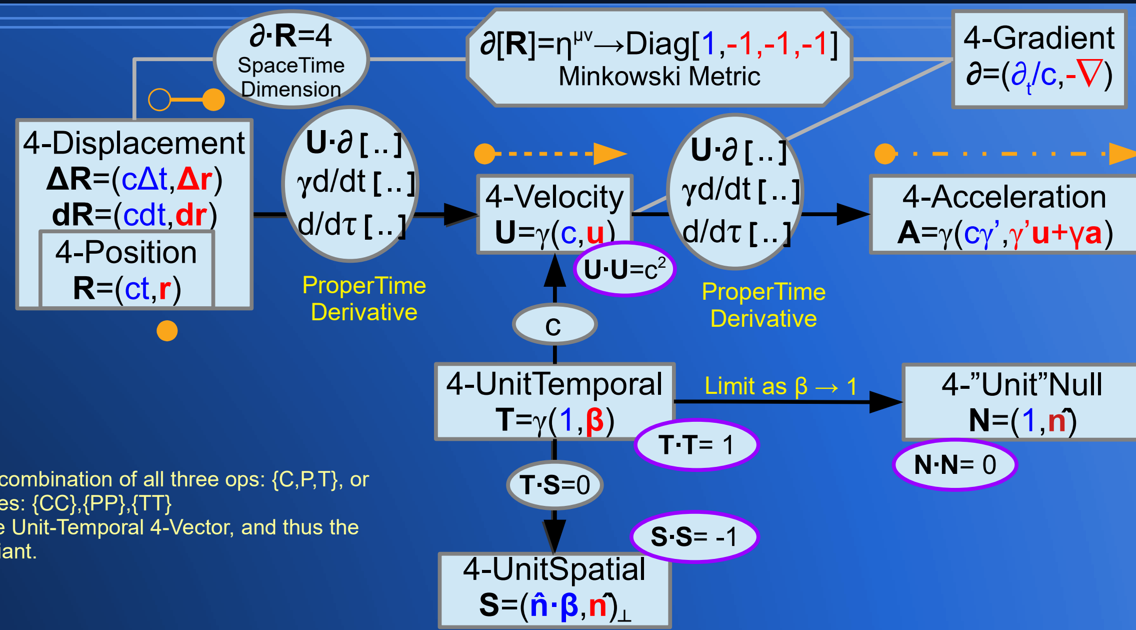
Note that for matter particles $\mathbf{K} = (\omega_o/c)\mathbf{T}$,
where **T** is the Unit-Temporal 4-Vector $\mathbf{T} = \gamma(1,\boldsymbol{\beta})$,
which defines the particle's worldline at each point.
The gamma factor (γ) will be unaffected in the following operations,
since it uses the square of $\boldsymbol{\beta}$: $\gamma=1/\mathrm{Sqrt}(1-\boldsymbol{\beta}\cdot\boldsymbol{\beta})$.
For photonic particles, $\mathbf{K} = (\omega/c)\mathbf{N}$,
where **N** is the "Unit"-Null 4-Vector $\mathbf{N} = (1,\mathbf{n})$ and **n** is a unit-spatial 3-vector.
All operations listed below work similarly on the Null 4-Vector.

Do a Time Reversal Operation: T
The particle's temporal direction is reversed & complex-conjugated:
$\mathbf{T}_T = -\mathbf{T}^* = \gamma(-1,\boldsymbol{\beta})^*$

Do a Parity Operation (Space Reflection): P
Only the spatial directions are reversed:
$\mathbf{T}_P = \gamma(1,-\boldsymbol{\beta})$

Do a Charge Conjugation Operation: C
Charge Conjugation actually changes all internal quantum #'s:
charge, lepton #, etc.
Feynman showed this is the equivalent of
a world-line reversal & complex-conjugation:
$\mathbf{T}_C = \gamma(-1,-\boldsymbol{\beta})^*$

Pairwise combinations:
$\mathbf{T}_{TP} = \mathbf{T}_{PT} = \mathbf{T}_C = \gamma(-1,-\boldsymbol{\beta})^*$
$\mathbf{T}_{TC} = \mathbf{T}_{CT} = \mathbf{T}_P = \gamma(1,-\boldsymbol{\beta})$
$\mathbf{T}_{PC} = \mathbf{T}_{CP} = \mathbf{T}_T = \gamma(-1,\boldsymbol{\beta})^*$, a CP event is mathematically the same as a T event
$\mathbf{T}_{CPT} = \mathbf{T} = \gamma(1,\boldsymbol{\beta})$ $\quad$ $\mathbf{T}_{CC} = \mathbf{T} = \gamma(1,\boldsymbol{\beta})$ $\quad$ $\mathbf{T}_{PP} = \mathbf{T} = \gamma(1,\boldsymbol{\beta})$ $\quad$ $\mathbf{T}_{TT} = \mathbf{T} = \gamma(1,\boldsymbol{\beta})$



It is only the combination of all three ops: {C,P,T}, or pairs of singles: {CC},{PP},{TT} that leave the Unit-Temporal 4-Vector, and thus the Phase, Invariant.

Matter-like
$\mathbf{T} = \gamma(1,\boldsymbol{\beta})$
$\mathbf{T}\cdot\mathbf{T} = \gamma(1,\boldsymbol{\beta})^*\cdot\gamma(1,\boldsymbol{\beta}) = \gamma^2(1^2 - \boldsymbol{\beta}\cdot\boldsymbol{\beta}) = 1$: It's a temporal 4-vector

$\mathbf{T}_C\cdot\mathbf{T}_C = \gamma(-1,-\boldsymbol{\beta})\cdot\gamma(-1,-\boldsymbol{\beta})^* = \gamma^2((-1)^2 - (-\boldsymbol{\beta})\cdot(-\boldsymbol{\beta})) = \gamma^2(1^2 - \boldsymbol{\beta}\cdot\boldsymbol{\beta}) = 1$
$\mathbf{T}_P\cdot\mathbf{T}_P = \gamma(1,-\boldsymbol{\beta})^*\cdot\gamma(1,-\boldsymbol{\beta}) = \gamma^2(1^2 - (-\boldsymbol{\beta})\cdot(-\boldsymbol{\beta})) = \gamma^2(1^2 - \boldsymbol{\beta}\cdot\boldsymbol{\beta}) = 1$
$\mathbf{T}_T\cdot\mathbf{T}_T = \gamma(-1,\boldsymbol{\beta})\cdot\gamma(-1,\boldsymbol{\beta})^* = \gamma^2((-1)^2 - (\boldsymbol{\beta})\cdot(\boldsymbol{\beta})) = \gamma^2(1^2 - \boldsymbol{\beta}\cdot\boldsymbol{\beta}) = 1$
They all remain temporal 4-vectors

$\mathbf{T}_{CPT} = \mathbf{T} = \gamma(1,\boldsymbol{\beta})$
$\mathbf{T}_{CPT}\cdot\mathbf{T}_{CPT} = \mathbf{T}\cdot\mathbf{T} = 1$

Light-like/Photonic
$\mathbf{N} = (1,\mathbf{n})$
$\mathbf{N}\cdot\mathbf{N} = (1,\mathbf{n})^*\cdot(1,\mathbf{n}) = (1^2 - \mathbf{n}\cdot\mathbf{n}) = (1-1) = 0$: It's a null 4-vector

$\mathbf{N}_C\cdot\mathbf{N}_C = (-1,-\mathbf{n})\cdot(-1,-\mathbf{n})^* = ((-1)^2 - (-\mathbf{n})\cdot(-\mathbf{n})) = (1^2 - \mathbf{n}\cdot\mathbf{n}) = (1-1) = 0$
$\mathbf{N}_P\cdot\mathbf{N}_P = (1,-\mathbf{n})^*\cdot(1,-\mathbf{n}) = (1^2 - (-\mathbf{n})\cdot(-\mathbf{n})) = (1^2 - \mathbf{n}\cdot\mathbf{n}) = (1-1) = 0$
$\mathbf{N}_T\cdot\mathbf{N}_T = (-1,\mathbf{n})\cdot(-1,\mathbf{n})^* = ((-1)^2 - (\mathbf{n})\cdot(\mathbf{n})) = (1^2 - \mathbf{n}\cdot\mathbf{n}) = (1-1) = 0$
They all remain null 4-vectors

$\mathbf{N}_{CPT} = \mathbf{N} = (1,\mathbf{n})$
$\mathbf{N}_{CPT}\cdot\mathbf{N}_{CPT} = \mathbf{N}\cdot\mathbf{N} = 0$

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^{\mu}{}_{\nu}$ or $T_{\mu}{}^{\nu}$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^{\mu} = \mathbf{V} = (v^0,\mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_{\mu} = (v_0,-\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^{\mu}{}_{\mu} = T$
$\mathbf{V}\cdot\mathbf{V} = V^{\mu}\eta_{\mu\nu}V^{\nu} = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$
= Lorentz Scalar

SRQM: CPT Theorem (Charge) vs (Parity) vs (Time)

A Tensor Study of Physical 4-Vectors

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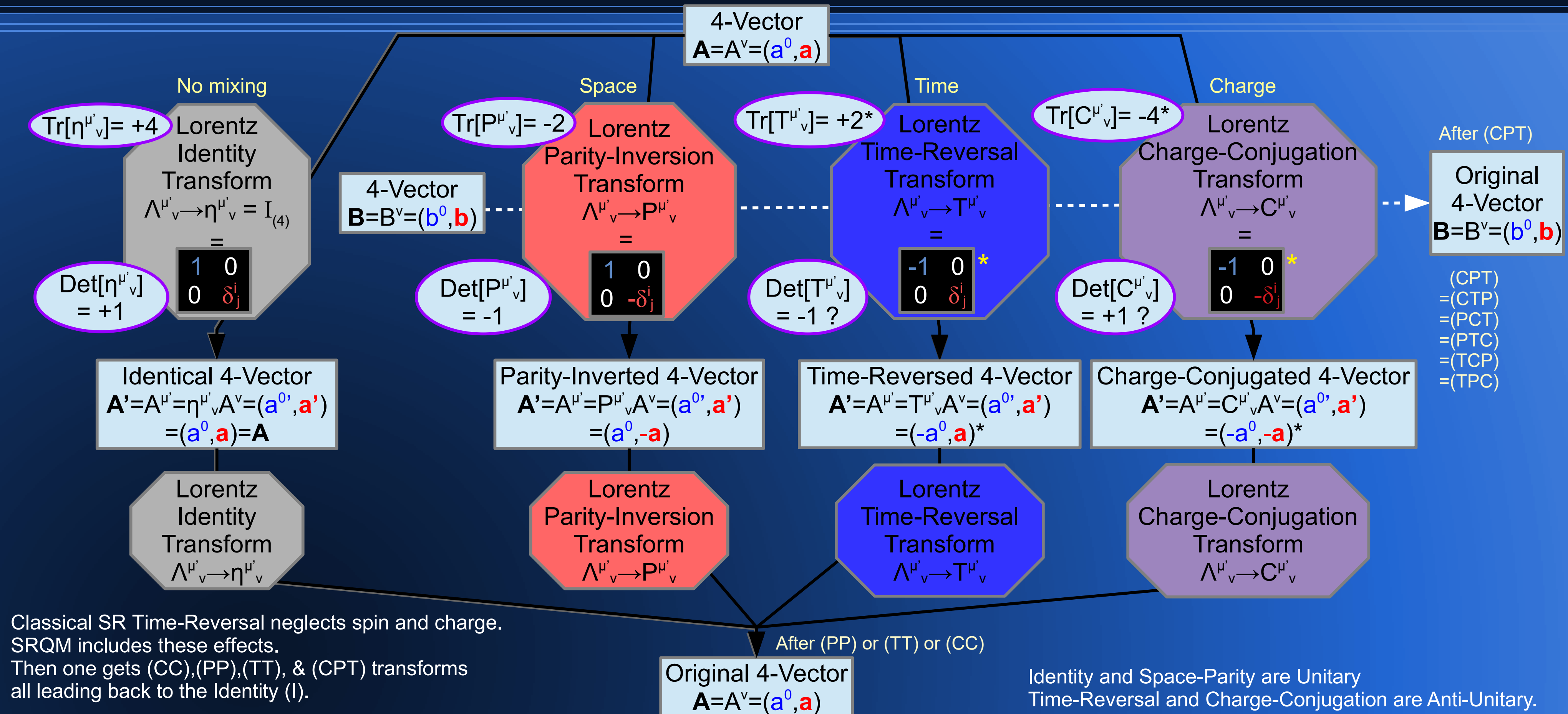

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(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor V^{μ} = **V** = (v^0,**v**)
SR 4-CoVector
(0,1)-Tensor V_{μ} = (v_0,-**v**)

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Trace[$T^{\mu\nu}$] = $\eta_{\mu\nu}T^{\mu\nu}$ = $T^{\mu}{}_{\mu}$ = T
V·V = $V^{\mu}\eta_{\mu\nu}V^{\nu}$ = [$(v^0)^2$ - **v·v**] = $(v^0{}_o)^2$
= Lorentz Scalar

SRQM Transforms: Venn Diagram

Poincaré = Lorentz + Translations

(10) (6) (4)

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Transformations

(# of independent parameters = # continuous symmetries = # Lie Dimensions)

Poincaré Transformation Group aka. Inhomogeneous Lorentz Transformation
Lie group of all affine isometries of SR:Minkowski TimeSpace (preserve quadratic form $\eta_{\mu\nu}$)
General Linear,Affine Transform $X^{\mu'} = \Lambda^{\mu'}{}_{\nu} X^{\nu} + \Delta X^{\mu'}$ with $\mathrm{Det}[\Lambda^{\mu'}{}_{\nu}] = \pm 1$
(6+4=10)

Lorentz Transform $\Lambda^{\mu'}{}_{\nu}$

(3+3=6) 4-Tensor {mixed type-(1,1)}

Discrete

Time-reversal
$\Lambda^{\mu'}{}_{\nu} \to T^{\mu'}{}_{\nu}$
(0)
t → -t*
time parity
anti-unitary

SpatialFlipCombos
$\Lambda^{\mu'}{}_{\nu} \to F^{\mu'}{}_{\nu}$
(0)
{x|y|z} → -{x|y|z}
unitary

Parity-Inversion
$\Lambda^{\mu'}{}_{\nu} \to P^{\mu'}{}_{\nu}$
(0)
r → -**r**
space parity
unitary

Identity $I_{(4)}$
$\Lambda^{\mu'}{}_{\nu} \to \eta^{\mu'}{}_{\nu} = \delta^{\mu'}{}_{\nu}$
(0)
no mixing
unitary

Charge-Conjugation
$\Lambda^{\mu'}{}_{\nu} \to C^{\mu'}{}_{\nu}$
(0)
R → -**R***, q → -q
charge parity
anti-unitary

CPT Symmetry
{Charge}
{Partiy}
{Time}

Continuous

Rotation
$\Lambda^{\mu'}{}_{\nu} \to R^{\mu'}{}_{\nu}$
(3)
x:y | x:z | y:z

Boost
$\Lambda^{\mu'}{}_{\nu} \to B^{\mu'}{}_{\nu}$
(3)
t:x | t:y | t:z

Isotropy
{same all directions}

Translation Transform $\Delta X^{\mu'}$

(1+3=4) 4-Vector

Discrete

4-Zero
$\Delta X^{\mu'} \to (0,\mathbf{0})$
(0)
no motion

Continuous

Temporal
$\Delta X^{\mu'} \to (c\Delta t,\mathbf{0})$
(1)
Δt

Spatial
$\Delta X^{\mu'} \to (0,\Delta\mathbf{x})$
(3)
Δx | Δy | Δz

Homogeneity
{same all points}

	M^{01}	M^{02}	M^{03}
M^{10}		M^{12}	M^{13}
M^{20}	M^{21}		M^{23}
M^{30}	M^{31}	M^{32}	

P^0
P^1
P^2
P^3

4-AngularMomentum $M^{\mu\nu} = X^{\mu} \wedge P^{\nu} = X^{\mu}P^{\nu} - X^{\nu}P^{\mu}$
= Generator of Lorentz Transformations (6)
= { $\Lambda^{\mu'}{}_{\nu} \to R^{\mu'}{}_{\nu}$ Rotations (3) + $\Lambda^{\mu'}{}_{\nu} \to B^{\mu'}{}_{\nu}$ Boosts (3) }

4-LinearMomentum P^{μ}
= Generator of Translation Transformations (4)
= { $\Delta X^{\mu'} \to (c\Delta t,\mathbf{0})$ Time (1) + $\Delta X^{\mu'} \to (0,\Delta\mathbf{x})$ Space (3) }

$\mathrm{Det}[\Lambda^{\mu'}{}_{\nu}] = +1$ for Proper Lorentz Transforms
$\mathrm{Det}[\Lambda^{\mu'}{}_{\nu}] = -1$ for Improper Lorentz Transforms

Lorentz Matrices can be generated by a matrix M with Tr[M]=0 which gives:
{ $\Lambda = e \wedge M = e \wedge (+\boldsymbol{\theta}\cdot\mathbf{J} - \boldsymbol{\zeta}\cdot\mathbf{K})$ }
{ $\Lambda^{T} = (e \wedge M)^{T} = e \wedge M^{T}$ }
{ $\Lambda^{-1} = (e \wedge M)^{-1} = e \wedge -M$ }

$M = +\boldsymbol{\theta}\cdot\mathbf{J} - \boldsymbol{\zeta}\cdot\mathbf{K}$
$B[\boldsymbol{\zeta}] = e\wedge(-\boldsymbol{\zeta}\cdot\mathbf{K})$
$R[\boldsymbol{\theta}] = e\wedge(+\boldsymbol{\theta}\cdot\mathbf{J})$
$\Lambda = e \wedge M = e \wedge (+\boldsymbol{\theta}\cdot\mathbf{J} - \boldsymbol{\zeta}\cdot\mathbf{K})$

SR:Lorentz Transform
$\partial_{\nu}[R^{\mu'}] = \partial R^{\mu'}/\partial R^{\nu} = \Lambda^{\mu'}{}_{\nu}$
$\Lambda^{\mu}{}_{\nu} = (\Lambda^{-1})_{\nu}{}^{\mu} : \Lambda^{\mu}{}_{\alpha}\Lambda^{\alpha}{}_{\nu} = \eta^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu}$
$\eta_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta} = \eta_{\alpha\beta}$
$\mathrm{Det}[\Lambda^{\mu}{}_{\nu}] = \pm 1$ $\Lambda_{\mu\nu}\Lambda^{\mu\nu} = 4$

Rotations $J_i = -\varepsilon_{imn}M^{mn}/2$, Boosts $K_i = M_{i0}$

[(**R** → -**R***)] or [(t → -t*) & (**r** → -**r**)] imply q → -q
Feynman-Stueckelberg Interpretation
Amusingly, Inhomogeneous Lorentz adds homogeneity.

Hermitian Generators

Noether's Theorem - Continuity

The Hermitian Generators that lead to translations and rotations via unitary operators in QM...

These all ultimately come from the Poincaré Invariance → Lorentz Invariance that is at the heart of SR and Minkowski Space.

Infintesimal Unitary Transformation
$\hat{\mathbf{U}}_\varepsilon(\hat{\mathbf{G}}) = \mathbf{I} + i\varepsilon\hat{\mathbf{G}}$

Finite Unitary Transformation
$\hat{\mathbf{U}}_\alpha(\hat{\mathbf{G}}) = e^\wedge(i\alpha\hat{\mathbf{G}})$

let $\hat{\mathbf{G}} = \mathbf{P}/\hbar = \mathbf{K}$
let $\alpha = \mathbf{\Delta x}$

$\hat{\mathbf{U}}_{\mathbf{\Delta x}}(\mathbf{P}/\hbar)\Psi(\mathbf{X}) = e^\wedge(i\mathbf{\Delta x}\cdot\mathbf{P}/\hbar)\Psi(\mathbf{X}) = e^\wedge(-\mathbf{\Delta x}\cdot\partial)\Psi(\mathbf{X}) = \Psi(\mathbf{X} - \mathbf{\Delta x})$

Time component: $\hat{\mathbf{U}}_{\mathbf{\Delta ct}}(\mathbf{P}/\hbar)\Psi(ct) = e^\wedge(i\Delta tE/\hbar)\Psi(ct) = e^\wedge(-\Delta t\,\partial_t)\Psi(ct) = \Psi(ct - c\Delta t) = c\Psi(t - \Delta t)$
Space component: $\hat{\mathbf{U}}_{\mathbf{\Delta x}}(\mathbf{p}/\hbar)\Psi(\mathbf{x}) = e^\wedge(i\mathbf{\Delta x}\cdot\mathbf{p}/\hbar)\Psi(\mathbf{x}) = e^\wedge(\mathbf{\Delta x}\cdot\nabla)\Psi(\mathbf{x}) = \Psi(\mathbf{x} + \mathbf{\Delta x})$

By Noether's Theorem, this leads to $\partial\cdot\mathbf{K} = 0$

We had already calculated
$(\partial\cdot\partial)[\mathbf{K}\cdot\mathbf{X}] = ((\partial_t/c)^2 - \nabla\cdot\nabla)(\omega t - \mathbf{k}\cdot\mathbf{x}) = 0$
$(\partial\cdot\partial)[\mathbf{K}\cdot\mathbf{X}] = \partial\cdot(\partial[\mathbf{K}\cdot\mathbf{X}]) = \partial\cdot\mathbf{K} = 0$

Poincaré Invariance also gives the Casimir invariants of mass and spin, and ultimately leads to the spin-statistics theorem of RQM.

QM Correspondence Principle: Analogous to the GR and SR limits

Basically, the old school QM Correspondence Principle says that QM should give the same results as classical physics in the realm of large quantum systems, i.e. where macroscopic behavior overwhelms quantum effects. Perhaps a better way to state it is when the change of system by a single quantum has a negligible effect on the overall state.

There is a way to derive this limit, by using Hamilton-Jacobi Theory:
$(i\hbar\partial_{tT})|\Psi\rangle \sim [\ V - (\hbar\nabla_T)^2/2m_o\ ]|\Psi\rangle$: The Schrödinger NRQM Equation for a point particle (non-relativistic QM)

Examine solutions of form $\Psi = \Psi_o e^{\wedge}(i\Phi) = \Psi_o e^{\wedge}(iS/\hbar)$, where S is the QM Action
$\partial_t[\Psi] = (i/\hbar)\Psi\partial_t[S]$ and $\partial_x[\Psi] = (i/\hbar)\Psi\partial_x[S]$ and $\nabla^2[\Psi] = (i/\hbar)\Psi\nabla^2[S] - (\Psi/\hbar^2)(\nabla[S])^2$

$(i\hbar)(i/\hbar)\Psi\partial_t[S] = V\Psi - (\hbar^2/2m_o)((i/\hbar)\Psi\nabla^2[S] - (\Psi/\hbar^2)(\nabla[S])^2)$

$(i)(i)\Psi\partial_t[S] = V\Psi - ((i\hbar/2m_o)\Psi\nabla^2[S] - (\Psi/2m_o)(\nabla[S])^2)$

$\partial_t[S] = -V + (i\hbar/2m_o)\nabla^2[S] - (1/2m_o)(\nabla[S])^2$

$\partial_t[S] + [V+(1/2m_o)(\nabla[S])^2\ ] = (i\hbar/2m_o)\nabla^2[S]$: Quantum Single Particle Hamilton-Jacobi
$\partial_t[S] + [V+(1/2m_o)(\nabla[S])^2\ ] = 0$: Classical Single Particle Hamilton-Jacobi

Thus, the classical limiting case is:
$\nabla^2[\Phi] \ll (\nabla[\Phi])^2$
$\hbar\nabla^2[S] \ll (\nabla[S])^2$
$\hbar\nabla\cdot\mathbf{p} \ll (\mathbf{p}\cdot\mathbf{p})$
$(p\lambda\!\!\!^{-})\nabla\cdot\mathbf{p} \ll (\mathbf{p}\cdot\mathbf{p})$

QM Correspondence Principle: Analogous to the GR and SR limits

$\partial_t[S] + [V+(1/2m_o)(\nabla[S])^2\,] = (i\hbar/2m_o)\nabla^2[S]$: Quantum Single Particle Hamilton-Jacobi
$\partial_t[S] + [V+(1/2m_o)(\nabla[S])^2\,] = 0$: Classical Single Particle Hamilton-Jacobi

Thus, the quantum→classical limiting-case is: {all equivalent representations}

$\hbar\nabla^2[S_{action}] \ll (\nabla[S_{action}])^2$	$\nabla^2[\Phi_{phase}] \ll (\nabla[\Phi_{phase}])^2$
$\hbar\nabla\cdot\nabla[S_{action}] \ll (\nabla[S_{action}])^2$	$\nabla\cdot\nabla[\Phi_{phase}] \ll (\nabla[\Phi_{phase}])^2$
$\hbar\nabla\cdot\mathbf{p} \ll (\mathbf{p}\cdot\mathbf{p})$	$\nabla\cdot\mathbf{k} \ll (\mathbf{k}\cdot\mathbf{k})$
$(p\bar{\lambda})\nabla\cdot\mathbf{p} \ll (\mathbf{p}\cdot\mathbf{p})$	

This page needs some work. Source was from Goldstein

with
$\mathbf{P} = (E/c,\mathbf{p}) = -\partial[S_{action}] = -(\partial_t/c,-\nabla)[S_{action}] = (-\partial_t/c,\nabla)[S_{action}]$
$\mathbf{K} = (\omega/c,\mathbf{k}) = -\partial[\Phi_{phase}] = -(\partial_t/c,-\nabla)[\Phi_{phase}] = (-\partial_t/c,\nabla)[\Phi_{phase}]$

It is analogous to GR → SR in limit of low curvature (low mass), or SR → CM in limit of low velocity { $|v| \ll c$ }.
It still applies, but is now understood as the same type of limiting-case as these others.

Note The commonly seen form of ($c\to\infty,\hbar\to 0$) as limits are incorrect!
c and ħ are universal constants – they never change.
If $c\to\infty$, then photons (light-waves) would have infinite energy { $E = pc$ }. This is not true classically.
If $\hbar\to 0$, then photons (light-waves) would have zero energy { $E = \hbar\omega$ }. This is not true classically.
Always better to write the SR Classical limit as { $|\mathbf{v}| \ll c$ }, the QM Classical limit as { $\nabla^2[\Phi_{phase}] \ll (\nabla[\Phi_{phase}])^2$ }

Again, it is more natural to find a limiting-case of a more general system than to try to unite two separate theories which may or may not ultimately be compatible. From logic, there is always the possibility to have a paradox result from combination of arbitrary axioms, whereas deductions from a single true axiom will always give true results.

SRQM: 4-Vector Quantum Probability

Conservation of ProbabilityDensity

Conservation of Probability : Probability Current : Charge Current
Consider the following purely mathematical argument
(based on Green's Vector Identity):

$\partial\cdot(f\,\partial[g] - \partial[f]\,g) = f\,\partial\cdot\partial[g] - \partial\cdot\partial[f]\,g$
with (f) and (g) as SR Lorentz Scalar functions

Proof:
$\partial\cdot(f\,\partial[g] - \partial[f]\,g)$
$= \partial\cdot(f\,\partial[g]) - \partial\cdot(\partial[f]\,g)$
$= (f\,\partial\cdot\partial[g] + \partial[f]\cdot\partial[g]) - (\partial[f]\cdot\partial[g] + \partial\cdot\partial[f]\,g)$
$= f\,\partial\cdot\partial[g] - \partial\cdot\partial[f]\,g$

We can also multiply this by a Lorentz Invariant Scalar Constant s
$s\,(f\,\partial\cdot\partial[g] - \partial\cdot\partial[f]\,g) = s\,\partial\cdot(f\,\partial[g] - \partial[f]\,g) = \partial\cdot s(f\,\partial[g] - \partial[f]\,g)$

Ok, so we have the math that we need…

Now, on to the physics… Start with the Klein-Gordon Eqn.
$\partial\cdot\partial = (-im_oc/\hbar)^2 = -(m_oc/\hbar)^2$
$\partial\cdot\partial + (m_oc/\hbar)^2 = 0$

Let it act on SR Lorentz Invariant function g
$\partial\cdot\partial[g] + (m_oc/\hbar)^2[g] = 0\,[g]$
Then pre-multiply by f
$[f]\partial\cdot\partial[g] + [f]\,(m_oc/\hbar)^2[g] = [f]\,0\,[g]$
$[f]\partial\cdot\partial[g] + (m_oc/\hbar)^2[f][g] = 0$

Do similarly with SR Lorentz Invariant function f
$\partial\cdot\partial[f] + (m_oc/\hbar)^2[f] = 0\,[f]$
Then post-multiply by g
$\partial\cdot\partial[f][g] + (m_oc/\hbar)^2[f][g] = 0\,[f][g]$
$\partial\cdot\partial[f][g] + (m_oc/\hbar)^2[f][g] = 0$

Now, subtract the two equations
$\{[f]\,\partial\cdot\partial[g] + (m_oc/\hbar)^2[f][g] = 0\} - \{\,\partial\cdot\partial[f][g] + (m_oc/\hbar)^2[f][g] = 0\}$
$[f]\,\partial\cdot\partial[g] + (m_oc/\hbar)^2[f][g] - \partial\cdot\partial[f][g\,] - (m_oc/\hbar)^2[f][g] = 0$
$[f]\,\partial\cdot\partial[g] - \partial\cdot\partial[f][g] = 0$

And as we noted from the mathematical Green's Vector identity at the start…
$[f]\,\partial\cdot\partial[g] - \partial\cdot\partial[f][g] = \partial\cdot(f\,\partial[g] - \partial[f]\,g) = 0$

Therefore,
$s\,\partial\cdot(f\,\partial[g] - \partial[f]\,g) = 0$
$\partial\cdot s(f\,\partial[g] - \partial[f]\,g) = 0$

Thus, there is a conserved current 4-Vector, $\mathbf{J}_{prob} = s(f\,\partial[g] - \partial[f]\,g)$, for which $\partial\cdot\mathbf{J}_{prob} = 0$,
and which also solves the Klein-Gordon equation.

Let's choose as before ($\partial = -i\mathbf{K}$) with a plane wave function $f = ae^{-i(\mathbf{K}\cdot\mathbf{X})} = \psi$,
and choose $g = f^* = ae^{i(\mathbf{K}\cdot\mathbf{X})} = \psi^*$ as its complex conjugate.

At this point, I am going to choose $s = (i\hbar/2m_o)$, which is Lorentz Scalar Invariant, in order to make the probability have dimensionless units and be normalized to unity in the rest case.

SR → QM

4-Vector SRQM Interpretation of QM

4-Vector Quantum Probability

4-ProbabilityFlux, Klein-Gordon RQM Eqn

A Tensor Study of Physical 4-Vectors

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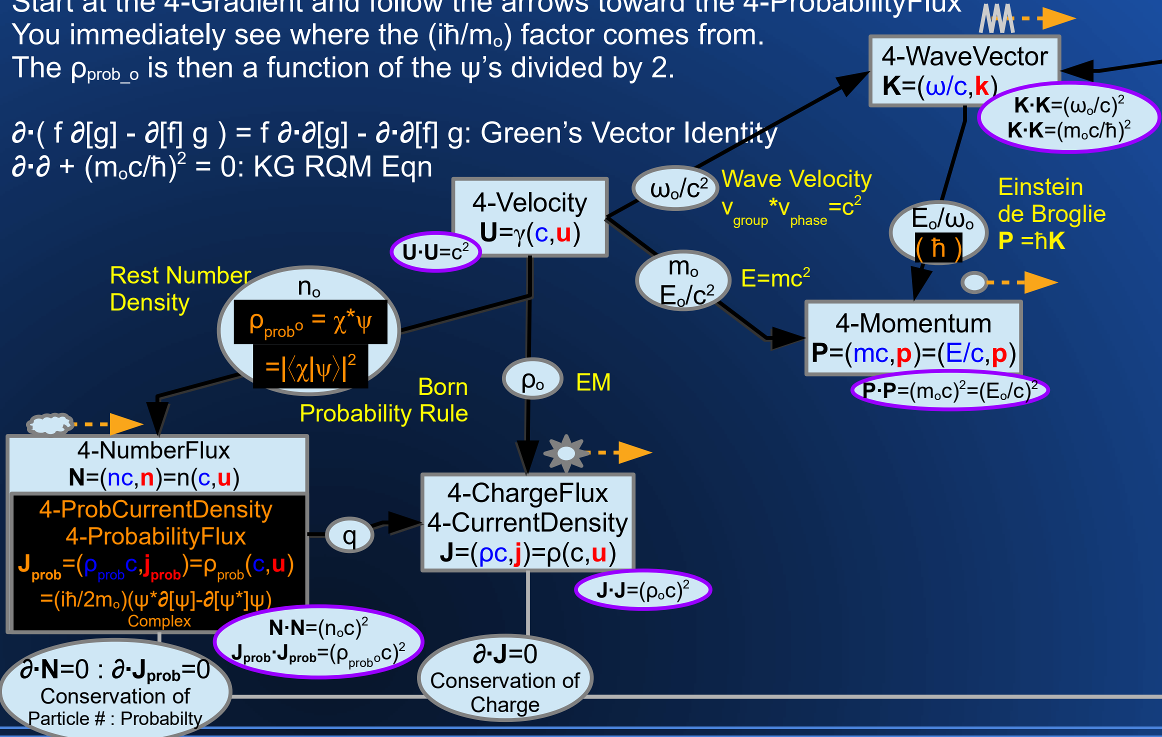
4-ProbabilityCurrentDensity, a.k.a. 4-ProbabilityFlux

$\mathbf{J}_{prob} = (c\rho_{prob},\mathbf{j}_{prob}) = (i\hbar/2m_o)(\psi^*\partial[\psi]-\partial[\psi^*]\psi) = (\rho_{prob_o})\mathbf{U} = (\rho_{prob_o})\gamma(c,\mathbf{u}) = (\gamma\rho_{prob_o})(c,\mathbf{u}) = (\rho_{prob})(c,\mathbf{u})$

with 4-Divergence of Probability { $\partial\cdot\mathbf{J}_{prob} = 0$ } by construction via Green's Vector Identity and the Klein-Gordon RQM Eqn.

The reason for $s = (i\hbar/2m_o)$ becomes more clear by examining our diagram:
Start at the 4-Gradient and follow the arrows toward the 4-ProbabilityFlux
You immediately see where the $(i\hbar/m_o)$ factor comes from.
The ρ_{prob_o} is then a function of the ψ's divided by 2.

$\partial\cdot(\ f\ \partial[g]\ -\ \partial[f]\ g\ ) = f\ \partial\cdot\partial[g]\ -\ \partial\cdot\partial[f]\ g$: Green's Vector Identity
$\partial\cdot\partial + (m_oc/\hbar)^2 = 0$: KG RQM Eqn



Examine the temporal component, the Relativistic Probability Density

$\rho_{prob} = (i\hbar/2m_oc^2)(\psi^*\ \partial_t[\psi]-\partial_t[\psi^*]\ \psi)$

Assume wave solution in following general form:

{ $\psi = A\ f\ [k]\ e(-i\omega t)$ }
{ $\psi^* = A^*\ f\ [k]^*\ e(+i\omega t)$ }

then

{ $\partial_t[\psi] = (-i\omega)A\ f\ [k]\ e(-i\omega t) = (-i\omega)\psi$ }
{ $\partial_t[\psi^*] = (+i\omega)A^*\ f\ [k]^*\ e(+i\omega t) = (+i\omega)\psi^*$ }

then

$\rho_{prob} = (i\hbar/2m_oc^2)(\psi^*\ \partial_t[\psi]\ -\ \partial_t[\psi^*]\ \psi)$
$\rho_{prob} = (i\hbar/2m_oc^2)((-i\omega)\psi^*\psi\ -\ (+i\omega)\psi^*\psi)$
$\rho_{prob} = (i\hbar/2m_oc^2)((-2i\omega)\psi^*\psi)$
$\rho_{prob} = (\hbar\omega/m_oc^2)(\psi^*\psi)$
$\rho_{prob} = (\hbar\gamma\omega_o/m_oc^2)(\psi^*\psi)$
$\rho_{prob} = (\gamma)(\psi^*\psi) = (\gamma)(\rho_{prob^o})$

Finally, multiply by charge (q) to get standard SR EM
4-CurrentDensity = 4-ChargeFlux = $\mathbf{J} = (c\rho,\mathbf{j}) = q\mathbf{J}_{prob} = q(c\rho_{prob},\mathbf{j}_{prob})$

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Existing SR Rules
Quantum Principles

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4-ProbabilityFlux, Klein-Gordon RQM Eqn with Minimal Coupling

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with 4-Divergence of Probability { $\partial\cdot\mathbf{J}_{prob} = 0$ } by construction via Green's Vector Identity and the Klein-Gordon RQM Eqn.

If we include minimal coupling:

$\mathbf{J}_{prob} = (c\rho_{prob},\mathbf{j}_{prob}) = (i\hbar/2m_o)(\psi^*\partial[\psi]-\partial[\psi^*]\psi) + (q/m_o)(\psi^*\psi)\mathbf{A}$

Start at **A** on the chart

Follow past (q) factor to get to **Q** = q**A**

Minimal Coupling allows passage back to **P** with no factors

Follow back past ($1/m_o$) to get to **U**

Follow past Born Rule ($\psi^*\psi$)

Now have the additional factor:

$+ (q/m_o)(\psi^*\psi)\mathbf{A}$

An alternate way would be to take **A** to **U** via the direct route:

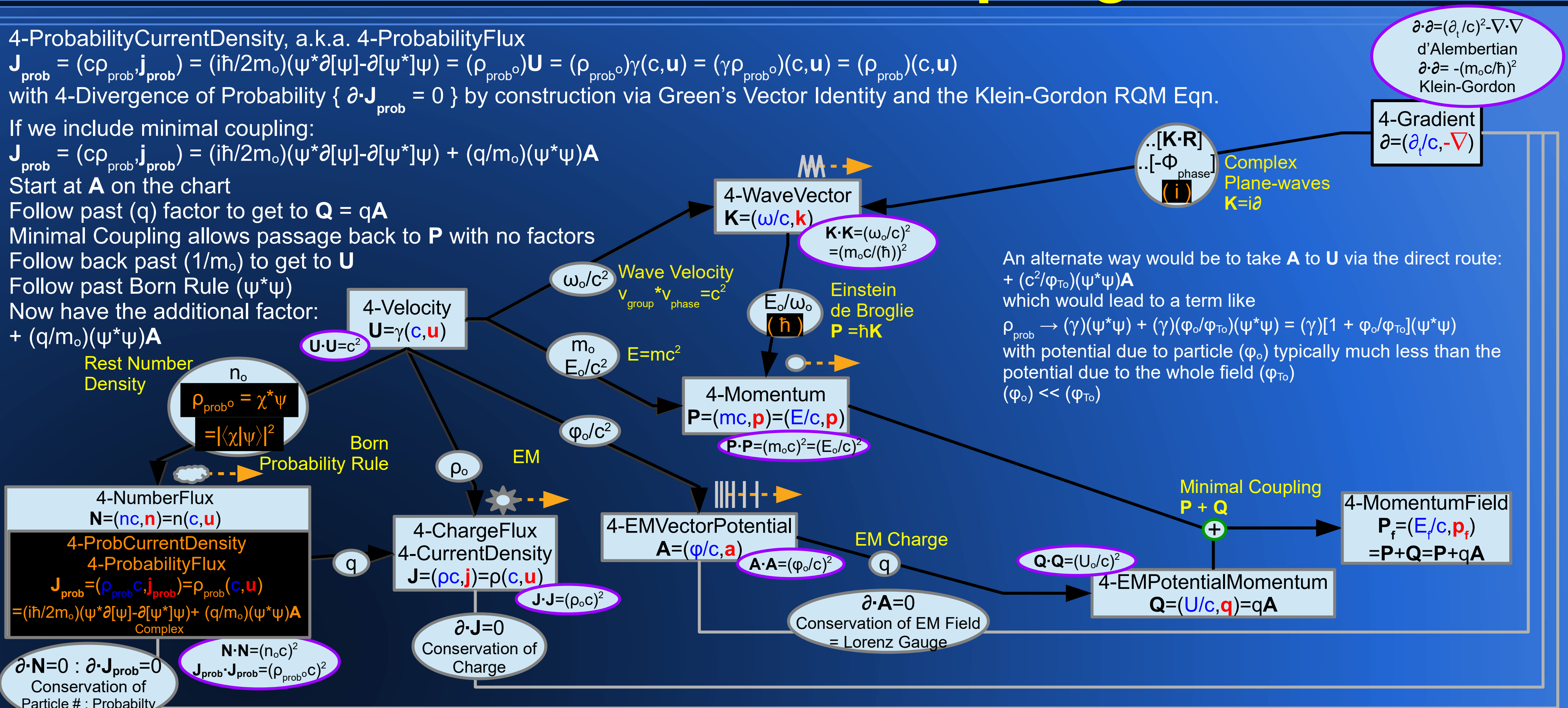
$+ (c^2/\varphi_{To})(\psi^*\psi)\mathbf{A}$

which would lead to a term like

$\rho_{prob} \rightarrow (\gamma)(\psi^*\psi) + (\gamma)(\varphi_o/\varphi_{To})(\psi^*\psi) = (\gamma)[1 + \varphi_o/\varphi_{To}](\psi^*\psi)$

with potential due to particle (φ_o) typically much less than the potential due to the whole field (φ_{To})

(φ_o) << (φ_{To})



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= Lorentz Scalar

4-Vector Quantum Probability
Newtonian Limit

4-ProbabilityCurrentDensity $\mathbf{J}_{\mathbf{prob}} = (c\rho_{prob}, \mathbf{j}_{\mathbf{prob}}) = (i\hbar/2m_o)(\psi^*\partial[\psi]-\partial[\psi^*]\psi) + (q/m_o)(\psi^*\psi)\mathbf{A}$

Examine the temporal component:
$\rho_{prob} = (i\hbar/2m_oc^2)(\psi^*\ \partial_t[\psi]-\partial_t[\psi^*]\ \psi) + (q/m_o)(\psi^*\psi)(\varphi/c^2)$
$\rho_{prob} \rightarrow (\gamma)(\psi^*\psi) + (\gamma)(q\varphi_o/m_oc^2)(\psi^*\psi) = (\gamma)[1 + q\varphi_o/E_o](\psi^*\psi)$

Typically, the particle EM potential energy ($q\varphi_o$) is much less than the particle rest energy (E_o), else it could generate new particles.
So, take ($q\varphi_o << E_o$), which gives the EM factor ($q\varphi_o/E_o$) ~ 0

Now, taking the low-velocity limit ($\gamma \rightarrow 1$), $\rho_{prob} = \gamma[1 + \sim 0](\psi^*\psi)$, $\rho_{prob} \rightarrow (\psi^*\psi) = (\rho_{prob^o})$ for $|\mathbf{v}|<<c$

The Standard Born Probability Interpretation, $(\psi^*\psi) = (\rho_{prob})$, only applies in the low-potential-energy & low-velocity limit

This is why the {non-positive-definite} probabilities and {|probabilities| > 1} in the RQM Klein-Gordon equation gave physicists fits, and is the reason why one must regard the probabilities as charge conservation instead.

The original definition from SR is Continuity of Worldlines, $\partial\cdot\mathbf{J}_{\mathbf{prob}} = 0$, for which all is good and well in the RQM version.
The definition says there are no external sources or sinks of probability = conservation of probability.

The Born idea that $(\rho_{prob}) \rightarrow \mathrm{Sum}[(\psi^*\psi)] = 1$ is just the Low-Velocity QM limit.
Only the non-EM rest version $(\rho_{prob^o}) = \mathrm{Sum}[(\psi^*\psi)] = 1$ is true.
It is not a fundamental axiom, it is an emergent property which is valid only in the NRQM limit

We now multiply by charge (q) to instead get a
4-"Charge"CurrentDensity $\mathbf{J} = (c\rho, \mathbf{j}) = q\mathbf{J}_{\mathbf{prob}} = q(c\rho_{prob}, \mathbf{j}_{\mathbf{prob}})$, which is the standard SR EM 4-CurrentDensity

SR → QM

4-Vector SRQM Interpretation of QM

SRQM 4-Vector Study:
The QM Compton Effect
Compton Scattering

A Tensor Study of Physical 4-Vectors

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Electron e^- initial | Photon γ initial

4-Momentum e^-: $\mathbf{P}_{ei}=(m_{ei}c,\mathbf{p}_{ei})=(E_{ei}/c,\mathbf{p}_{ei})$

4-WaveVector γ: $\mathbf{K}_{pi}=(\omega_{pi}/c,\mathbf{k}_{pi})$

$E/\omega = E_o/\omega_o = (\hbar)$

4-Momentum γ: $\mathbf{P}_{pi}=(m_{pi}c,\mathbf{p}_{pi})=(E_{pi}/c,\mathbf{p}_{pi})=(\hbar)\mathbf{K}_{pi}$

4-TotalMomentum e^-+γ: $\mathbf{P}_{ti}=(E_T/c,\mathbf{p}_T)=(H/c,\mathbf{p}_T) =\mathbf{P}_{ei}+\mathbf{P}_{pi}$

Electron:Photon Interaction

4-TotalMomentum e^-+γ: $\mathbf{P}_{tf}=(E_T/c,\mathbf{p}_T)=(H/c,\mathbf{p}_T) =\mathbf{P}_{ef}+\mathbf{P}_{pf}$

4-Momentum γ: $\mathbf{P}_{pf}=(m_{pf}c,\mathbf{p}_{pf})=(E_{pf}/c,\mathbf{p}_{pf})=(\hbar)\mathbf{K}_{pf}$

$\omega/E = \omega_o/E_o = (1/\hbar)$

4-Momentum e^-: $\mathbf{P}_{ef}=(m_{ef}c,\mathbf{p}_{ef})=(E_{ef}/c,\mathbf{p}_{ef})$

4-WaveVector γ: $\mathbf{K}_{pf}=(\omega_{pf}/c,\mathbf{k}_{pf})$

Electron e^- final | Photon γ final

Compton Scattering Derivation : Compton Effect

$\mathbf{P}\cdot\mathbf{P} = (m_oc)^2$ generally → 0 for photons (m_o=0)

$\mathbf{P}_{phot1}\cdot\mathbf{P}_{phot2} = \hbar^2\mathbf{K}_1\cdot\mathbf{K}_2 = (\hbar^2\omega_1\omega_2/c^2)(1-\hat{\mathbf{n}}_1\cdot\hat{\mathbf{n}}_2) = (\hbar^2\omega_1\omega_2/c^2)(1-\cos[\theta])$

$\mathbf{P}_{phot}\cdot\mathbf{P}_{mass} = \hbar\mathbf{K}\cdot\mathbf{P} = (\hbar\omega/c)(1,\hat{\mathbf{n}})\cdot(E/c,\mathbf{p}) = (\hbar\omega/c)(E/c - \hat{\mathbf{n}}\cdot\mathbf{p}) = (\hbar\omega E_o/c^2) = (\hbar\omega m_o)$

$\mathbf{P}_{phot} + \mathbf{P}_{mass} = \mathbf{P'}_{phot} + \mathbf{P'}_{mass}$:4-MomentumConservation in Photon·Mass Interaction

===

$\mathbf{P}_{phot} + \mathbf{P}_{mass} - \mathbf{P'}_{phot} = \mathbf{P'}_{mass}$:rearrange

$(\mathbf{P}_{phot} + \mathbf{P}_{mass} - \mathbf{P'}_{phot})^2 = (\mathbf{P'}_{mass})^2$:square to get scalars

$(\mathbf{P}_{phot}\cdot\mathbf{P}_{phot} + 2\mathbf{P}_{phot}\cdot\mathbf{P}_{mass} - 2\mathbf{P}_{phot}\cdot\mathbf{P'}_{phot} + \mathbf{P}_{mass}\cdot\mathbf{P}_{mass} - 2\mathbf{P}_{mass}\cdot\mathbf{P'}_{phot} + \mathbf{P'}_{phot}\cdot\mathbf{P'}_{phot}) = (\mathbf{P'}_{mass})^2$

$(0 + 2\mathbf{P}_{phot}\cdot\mathbf{P}_{mass} - 2\mathbf{P}_{phot}\cdot\mathbf{P'}_{phot} + (m_oc)^2 - 2\mathbf{P}_{mass}\cdot\mathbf{P'}_{phot} + 0) = (m_oc)^2$

$\mathbf{P}_{phot}\cdot\mathbf{P}_{mass} - \mathbf{P}_{mass}\cdot\mathbf{P'}_{phot} = \mathbf{P}_{phot}\cdot\mathbf{P'}_{phot}$

$(\hbar\omega m_o) - (\hbar\omega' m_o) = (\hbar^2\omega\omega'/c^2)(1 - \cos[\theta])$

$(\omega-\omega')/(\omega\omega') = (\hbar/m_oc^2)(1 - \cos[\theta])$

$(1/\omega' - 1/\omega) = (\hbar/m_oc^2)(1 - \cos[\theta])$

$\Delta\lambda = (\lambda' - \lambda) = (\hbar/m_oc)(1 - \cos[\theta]) = \lambda_C(1 - \cos[\theta])$

The Compton Effect:Compton Scattering

with

$\lambda_C = \lambda_C/2\pi = (\hbar/m_oc)$ = Reduced Compton Wavelength

$\lambda_C = (h/m_oc)$ = Compton Wavelength (not a rest-wavelength, but the wavelength of a photon with the energy equivalent to a massive particle of rest-mass m_o)

Calculates the wavelength shift of a photon scattering from an electron (*ignoring spin*)
Proves that light does not have a "wave-only" description, photon 4-Momentum required
$E/\omega = \gamma E_o/\gamma\omega_o = E_o/\omega_o = \hbar$ $\quad \mathbf{K}_{photon} = (\omega/c)(1,\mathbf{n})$ = null $\quad \{\omega\lambda = \nu\lambda = c\}$ for photons

$\partial^\mu[R^\nu]=\eta^{\mu\nu}$ Minkowski Metric

4-Gradient $\partial=(\partial_t/c,-\nabla)$

SpaceTime $\partial\cdot\mathbf{R}$=4 Dimension

4-Position $\mathbf{R}=(ct,\mathbf{r})$

$\mathbf{U}\cdot\partial$[..] $d/d\tau$[..]

4-Velocity $\mathbf{U}=\gamma(c,\mathbf{u})$

Wave Velocity $v_{group}*v_{phase} = c^2$

ω_o/c^2

$\mathbf{K}\cdot\mathbf{K}=(\omega_o/c)^2 =(m_oc/(\hbar))^2$

4-WaveVector $\mathbf{K}=(\omega/c,\mathbf{k})=(1/cT,\hat{\mathbf{n}}/\lambda)$

$\{\omega_o=0\} \leftrightarrow \{\mathbf{K}\cdot\mathbf{U}=0\} \leftrightarrow \{\mathbf{K}$ is null$\}$

Energy:Mass $E = mc^2$

m_o E_o/c^2

$\mathbf{P}\cdot\mathbf{P}=(m_oc)^2 =(E_o/c)^2$

Einstein de Broglie $\mathbf{P} =\hbar\mathbf{K}$

E_o/ω_o ($\hbar$)

4-Momentum $\mathbf{P}=(mc,\mathbf{p})=(E/c,\mathbf{p})$

$\{m_o=0\} \leftrightarrow \{\mathbf{P}\cdot\mathbf{U}=0\} \leftrightarrow \{\mathbf{P}$ is null$\}$

Conservation of 4-TotalMomentum

Σ_n[..]

4-TotalMomentum $\mathbf{P}_T=(E_T/c,\mathbf{p}_T)=(H/c,\mathbf{p}_T)$

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(0,0)-Tensor S
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Existing SR Rules
Quantum Principles

Trace$[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu{}_\mu = T$
$\mathbf{V}\cdot\mathbf{V} = V^\mu\eta_{\mu\nu}V^\nu = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0_o)^2$ = Lorentz Scalar

SRQM 4-Vector Study:
The QM Aharonov-Bohm Effect
QM Potential $\Delta\Phi_{pot} = -(q/\hbar)\int_{path} \mathbf{A}\cdot d\mathbf{X}$

A Tensor Study of Physical 4-Vectors

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Aharonov-Bohm Effect

The EM 4-VectorPotential gives the Aharonov-Bohm Effect.
$\Phi_{pot} = -(q/\hbar)\mathbf{A}\cdot\mathbf{X} = -\mathbf{K}_{pot}\cdot\mathbf{X}$

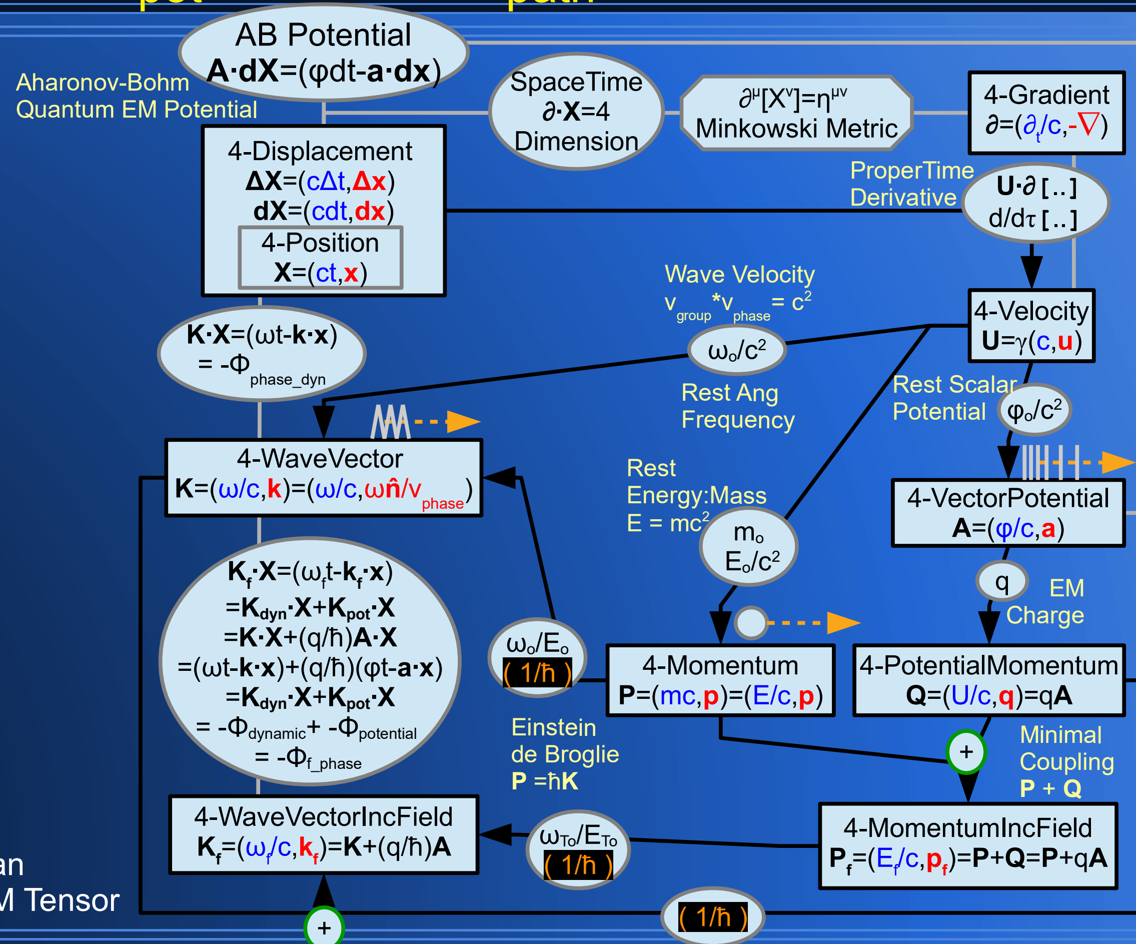
or taking the differential...
$d\Phi_{pot} = -(q/\hbar)\mathbf{A}\cdot d\mathbf{X}$

over a path...
$\Delta\Phi_{pot} = \int_{path} d\Phi_{pot}$
$\Delta\Phi_{pot} = -(q/\hbar)\int_{path} \mathbf{A}\cdot d\mathbf{X}$
$\Delta\Phi_{pot} = -(q/\hbar)\int_{path} [(\varphi/c)(cdt) - \mathbf{a}\cdot d\mathbf{x}]$
$\Delta\Phi_{pot} = -(q/\hbar)\int_{path} (\varphi dt - \mathbf{a}\cdot d\mathbf{x})$

Note that both the Electric and Magnetic effects come out by using the 4-Vector notation.

Electric AB effect: $\Delta\Phi_{pot\_Elec} = -(q/\hbar)\int_{path} (\varphi dt)$
Magnetic AB effect: $\Delta\Phi_{pot\_Mag} = +(q/\hbar)\int_{path} (\mathbf{a}\cdot d\mathbf{x})$

Proves that the 4-VectorPotential **A** is more fundamental than **e** and **b** fields, which are just components of the Faraday EM Tensor



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SR → QM

4-Vector SRQM Interpretation of QM

SRQM 4-Vector Study:

The QM Josephson Junction Effect = SuperCurrent

EM 4-VectorPotential $\mathbf{A} = -(\hbar/q)\partial[\Delta\Phi_{pot}]$

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Josephson Effect

The EM 4-VectorPotential gives the Aharonov-Bohm Effect.
Phase $\Phi_{pot} = -(q/\hbar)\mathbf{A}\cdot\mathbf{X} = -\mathbf{K}_{pot}\cdot\mathbf{X}$

Rearrange the equation a bit:
$-(\hbar/q)\Delta\Phi_{pot} = \mathbf{A}\cdot\Delta\mathbf{X}$
$\mathbf{A}\cdot\Delta\mathbf{X} = -(\hbar/q)\Delta\Phi_{pot}$
$d/d\tau[\mathbf{A}\cdot\Delta\mathbf{X}] = d/d\tau[-(\hbar/q)\Delta\Phi_{pot}] = d/d\tau[\mathbf{A}]\cdot\Delta\mathbf{X} + \mathbf{A}\cdot d/d\tau[\Delta\mathbf{X}] = d/d\tau[\mathbf{A}]\cdot\Delta\mathbf{X} + \mathbf{A}\cdot\mathbf{U}$

Assume that ($d/d\tau[\mathbf{A}]\cdot\Delta\mathbf{X} \sim 0$)
$[\mathbf{A}\cdot\mathbf{U}] = d/d\tau[-(\hbar/q)\Delta\Phi_{pot}]$
$[\mathbf{U}\cdot\mathbf{A}] = (\mathbf{U}\cdot\partial)[-(\hbar/q)\Delta\Phi_{pot}]$
$[\mathbf{A}] = -(\hbar/q)(\partial)[\Delta\Phi_{pot}]$
$\mathbf{A} = -(\hbar/q)\partial[\Delta\Phi_{pot}]$
$(\varphi/c,\mathbf{a}) = -(\hbar/q)(\partial_t/c,-\nabla)[\Delta\Phi_{pot}]$

Which explains Josephson Effect criteria :
$\Delta\mathbf{X} \sim 0$: small gap
$d/d\tau[\mathbf{A}] \sim 0$: “critical current” & no voltage
$d/d\tau[\mathbf{A}]\cdot\Delta\mathbf{X} \sim$ orthogonal: ??

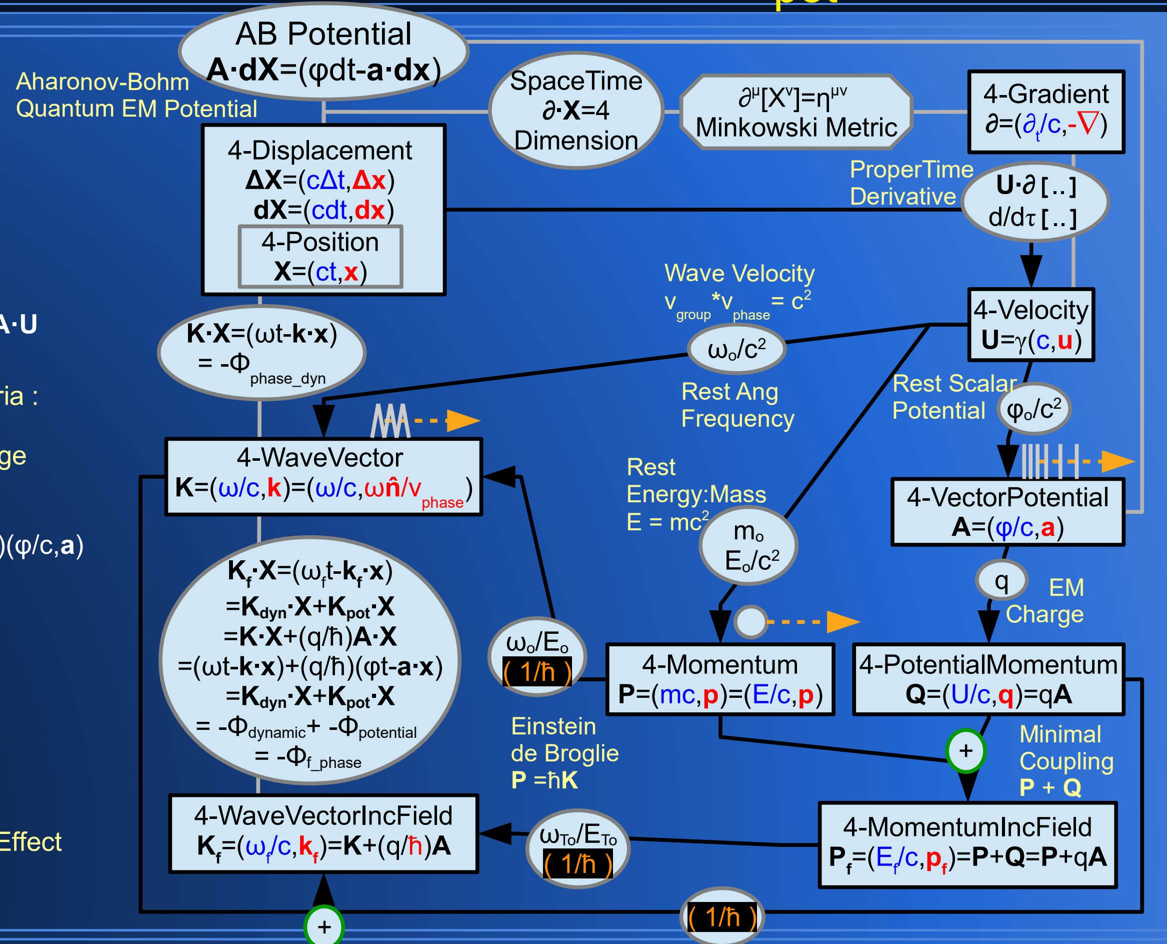
$\mathbf{A} = (\hbar/q)\mathbf{K}$; $\mathbf{K} = (\omega/c,\mathbf{k}) = (q/\hbar)\mathbf{A} = (q/\hbar)(\varphi/c,\mathbf{a})$

Take the temporal part:
EM ScalarPotential $\varphi = -(\hbar/q)(\partial_t)[\Delta\Phi_{pot}]$; $\omega = (q/\hbar)\varphi$

If the charge (q) is a Cooper-electron-pair: { $q = -2e$ }

Voltage $V(t) = \varphi(t) = (\hbar/2e)(\partial/\partial t)[\Delta\Phi_{pot}]$; AngFreq $\omega = -2eV/\hbar$
This is the superconducting phase evolution equation of the Josephson Effect

$(\hbar/2e)$ is defined to be the Magnetic Flux Quantum Φ_o



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Existing SR Rules
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SRQM Symmetries:
Hamilton-Jacobi vs Relativistic Action
Josephson vs Aharonov-Bohm
Differential (4-Vector) vs Integral (4-Scalar)

Differential Formats : 4-Vectors — **Notice the Symmetry:** — **Integral Formats : 4-Scalars**

SR Hamilton-Jacobi Equation
$\mathbf{P_T} = -\partial[\Delta S_{action}]$

$\mathbf{P} + q\mathbf{A} = -\partial[\Delta S_{action}]$

$\mathbf{P} + q\mathbf{A} = -\partial[\hbar\Delta\Phi_{phase}]$

$\mathbf{P} + q\mathbf{A} = -\partial[\hbar\Delta\Phi_{phase,dyn} + (\hbar)\Delta\Phi_{phase,pot}]$

Inverse

SR Action Equation
$\Delta S_{action} = -\int_{path}\mathbf{P_T}\cdot\mathbf{dX}$

$\Delta S_{action} = -\int_{path}(\mathbf{P}+q\mathbf{A})\cdot\mathbf{dX}$

$\hbar\Delta\Phi_{phase} = -\int_{path}(\mathbf{P}+q\mathbf{A})\cdot\mathbf{dX}$

$\hbar\Delta\Phi_{phase,dyn} + \hbar\Delta\Phi_{phase,pot} = -\int_{path}(\mathbf{P}+q\mathbf{A})\cdot\mathbf{dX}$

+

4-TotMom Conservation
$\mathbf{P_T} = (\mathbf{P}+\mathbf{Q}) = (\mathbf{P}+q\mathbf{A})$
Minimal Coupling
$\mathbf{P} = (\mathbf{P_T}-q\mathbf{A}) = (\mathbf{P_T}-\mathbf{Q})$

Dynamic Part

4-Momentum
$\mathbf{P} = -\partial[\Delta S_{act,dync}]$
$-\partial[\hbar\Delta\Phi_{phase,dynamic}]$

Inverse

Dynamic Part

Action$_{\text{(free part)}}$
$\Delta S_{act,dyn} = \hbar\Delta\Phi_{phase,dynamic}$
$= -\int_{path}(\mathbf{P})\cdot\mathbf{dX}$

+

4-TotMom Conservation
$\mathbf{P_T} = (\mathbf{P}+\mathbf{Q}) = (\mathbf{P}+q\mathbf{A})$
Minimal Coupling
$\mathbf{P} = (\mathbf{P_T}-q\mathbf{A}) = (\mathbf{P_T}-\mathbf{Q})$

Potential Part

4-PotentialMomentum
$\mathbf{Q} = q\mathbf{A} = -\partial[\Delta S_{act,pot}]$
$-\partial[\hbar\Delta\Phi_{phase,potential}]$

Inverse

Potential Part

Action$_{\text{(potential part)}}$
$\Delta S_{act,pot} = \hbar\Delta\Phi_{phase,potential} =$
$-\int_{path}(q\mathbf{A})\cdot\mathbf{dX} = -\int_{path}(\mathbf{Q})\cdot\mathbf{dX}$

q

(ħ)

Technically, the standard Josephson Junction uses just the temporal part { $\mathbf{A} = (\varphi/c,\mathbf{a})$ } & Cooper-pair-electrons { $q = -2e$ }
giving $V(t) = \varphi = (\hbar/2e)\partial/\partial t[\Delta\Phi_{pot}]$.
There should be a spatial part as well.

Josephson Junction Relation
$\mathbf{A} = -(\hbar/q)\partial[\Delta\Phi_{potential}]$
$= -(1/q)\partial[\Delta S_{act,pot}]$
$=\mathbf{Q}/q$

Inverse

Aharonov-Bohm Relation
$\Delta\Phi_{potential} = -(q/\hbar)\int_{path}\mathbf{A}\cdot\mathbf{dX}$
$= -(1/\hbar)\int_{path}\mathbf{Q}\cdot\mathbf{dX}$
$=\Delta S_{act,pot}/\hbar$

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SR → QM

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SRQM 4-Vector Study: Einstein-de Broglie The (ħ) Connection

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The ħ Connection

P = ħK: Basic Einstein-de Broglie

$\mathbf{P}+\mathbf{Q} = \mathbf{P} + \mathbf{Q}$

$\mathbf{P}+\mathbf{Q} = \hbar\mathbf{K}_{dyn} + \hbar\mathbf{K}_{pot}$

$\mathbf{P}+\mathbf{Q} = \hbar(\mathbf{K}_{dyn} + \mathbf{K}_{pot})$

Sum over n particles: $\mathbf{P_T} = \Sigma_n(\mathbf{P}+\mathbf{Q})$, $\mathbf{K_T} = \Sigma_n(\mathbf{K}_{dyn}+\mathbf{K}_{pot})$

$\mathbf{P_T} = \hbar\mathbf{K_T}$

$\mathbf{P_T}\cdot\mathbf{X} = \hbar\mathbf{K_T}\cdot\mathbf{X}$

$(\mathbf{P_T}\cdot\mathbf{X}) = \hbar(\mathbf{K_T}\cdot\mathbf{X})$

$-S_{action} = -\hbar\Phi_{phase}$

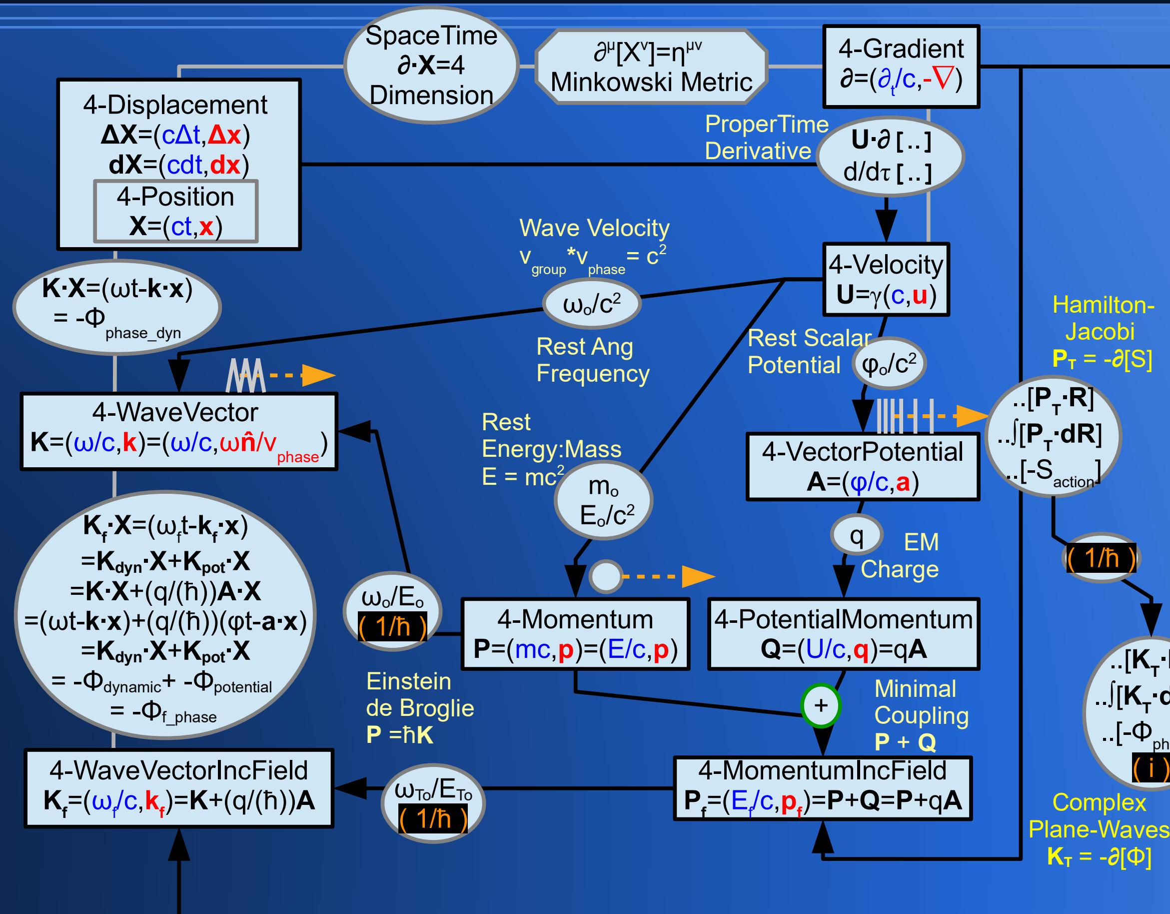
$S_{action} = \hbar\Phi_{phase}$

$-\partial[S_{action}] = -\hbar\partial[\Phi_{phase}]$

$\mathbf{P_T} = \hbar\mathbf{K_T}$

{SR Hamilton-Jacobi} = ħ{QM Complex Plane-Waves}

The SR Hamilton-Jacobi Equation, and the QM idea of Complex Plane-Waves, are related by a simple constant (ħ) relation.



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SRQM 4-Vector Study:
Dimensionless Physical Objects

A Tensor Study of Physical 4-Vectors

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Dimensionless Physical Objects

There are a number of dimensionless physical objects in SR that can be constructed from Physical 4-Vectors.
Most are 4-Scalars, but there are few 4-Vector and 4-Tensors.

$\partial\cdot\mathbf{X}=4$: SpaceTime Dimension
$\partial^\mu[X^\nu]=\eta^{\mu\nu}$: The SR Minkowski Metric

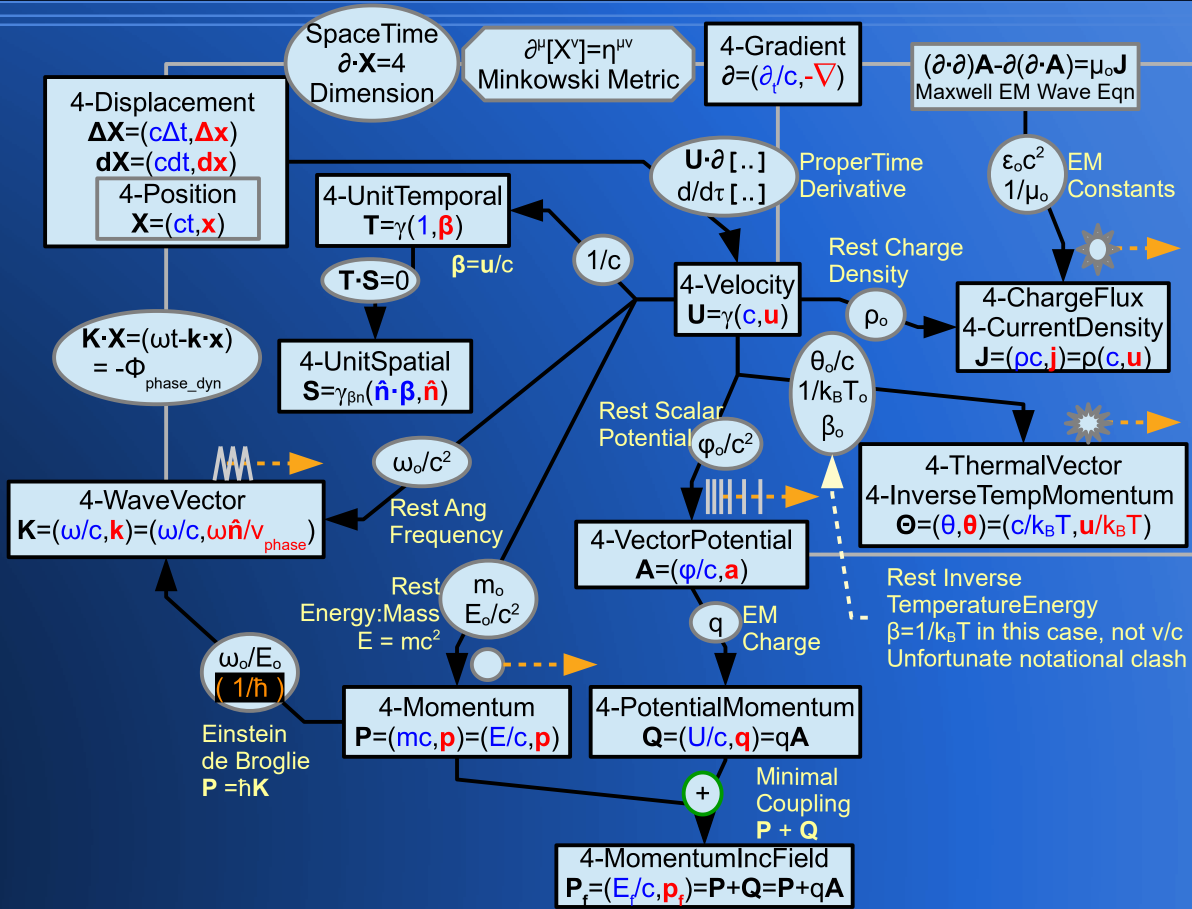
$\mathbf{T}\cdot\mathbf{T}=\ \ 1$: Lorentz Scalar "Magnitude" of the 4-UnitTemporal
$\mathbf{T}\cdot\mathbf{S}=\ \ 0$: Lorentz Scalar of 4-UnitTemporal with 4-UnitSpatial
$\mathbf{S}\cdot\mathbf{S}=-1$: Lorentz Scalar "Magnitude" of the 4-UnitSpatial

$\mathbf{K}\cdot\mathbf{X}=(\omega t-\mathbf{k}\cdot\mathbf{x}) = -\Phi_{phase\_dyn}$: Phase of an SR Wave
used in SRQM wave functions $\psi=a{*}e$^$-(\mathbf{K}\cdot\mathbf{X})$

$(\mathbf{P}\cdot\mathbf{\Theta}) = (E_o/k_BT_o)$: 4-Momentum with 4-InvThermalMomentum
used in statistical mechanics particle distributions
F(state) ~ e^$-(\mathbf{P}\cdot\mathbf{\Theta})$ = e^$-(E_o/k_BT_o)$

$\alpha = (1/4\pi\varepsilon_o)(e^2/\hbar c) = (\mu_o/4\pi)(ce^2/\hbar)$: Fine Structure Constant
constructed from Lorentz 4-Scalars, which are themselves
constructed from 4-Vectors via the Lorentz Scalar Product.
ex. $\hbar=(\mathbf{P}\cdot\mathbf{X})/(\mathbf{K}\cdot\mathbf{X})$; $q=(\mathbf{Q}\cdot\mathbf{X})/(\mathbf{A}\cdot\mathbf{X})$ →e for electron; $c=(\mathbf{T}\cdot\mathbf{U})$
$\mu_o=\{(\partial\cdot\partial)[\mathbf{A}]\cdot\mathbf{X}\}/(\mathbf{J}\cdot\mathbf{X})$ when $(\partial\cdot\mathbf{A})=0$

$\{\gamma^\mu\}$: Dirac Gamma Matrix ("4-Vector")
$\{\sigma^\mu\}$: Pauli Spin Matrix ("4-Vector")
Components are matrices of numbers, not just numbers



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SRQM: QM Axioms Unnecessary
QM Principles emerge from SR

QM is derivable from SR plus a few empirical facts – the "QM Axioms" aren't necessary
These properties are either empirically measured or are emergent from SR properties...

3 "QM Axioms" are really just empirical constant relations between purely SR 4-Vectors:
- Particle-Wave Duality [(**P**) = ħ(**K**)]
- Unitary Evolution [∂ = (-i)**K**]
- Operator Formalism [(∂) = -i**K**]

2 "QM Axioms" are just the result of the Klein-Gordon Equation being a linear wave PDE:
- Hilbert Space Representation (<bra|,|ket>, wavefunctions, etc.) & The Principle of Superposition

3 "QM Axioms" are a property of the Minkowski Metric and the empirical fact of Operator Formalism
- The Canonical Commutation Relation
- The Heisenberg Uncertainty Principle (time-like-separated measurement exchange)
- The Pauli Exclusion Principle (space-like-separated particle exchange)

1 "QM Axiom" only holds in the NRQM case
- The Born QM Probability Interpretation – Not applicable to RQM, use Conservation of Worldlines instead

1 "QM Axiom" is really just another level of limiting cases, just like SR → CM in limit of low velocity
- The QM Correspondence Principle (QM → CM in limit of $\{\nabla^2[\phi] << (\nabla[\phi])^2\}$)

SRQM Interpretation: Relational QM & EPR

The SRQM interpretation fits fairly well with Carlo Rovelli's Relational QM interpretation:

Relational QM treats the state of a quantum system as being observer-dependent, that is, the QM State is the relation between the observer and the system. This is inspired by the key idea behind Special Relativity, that the details of an observation depend on the reference frame of the observer.

All systems are quantum systems: no artificial Copenhagen dichotomy between classical/macroscopic/conscious objects and quantum objects.

The QM States reflect the observers' information about a quantum system.
Wave function “collapse” is informational – not physical. A particle always knows it’s complete properties. An observer has at best only partial information about the particle’s properties.

No Spooky Action at a Distance. When a measurement is done locally on an entangled system, it is only the partial information about the distant entangled state that “changes/becomes-available-instantaneously”. There is no superluminal signal. Measuring/physically-changing the local particle does not physically change the distant particle.

ex. Place two identical-except-for-color marbles into a box, close lid, and shake. Without looking, pick one marble at random and place it into another box. Send that box very far away. After receiving signal of the far box arrival at a distant point, open the near box and look at the marble. You now instantaneously know the far marble’s color as well. The information did not come by signal. You already had the possibilities (partial knowledge). Looking at the near marble color simply reduced the partial knowledge of both marble’s color to complete knowledge of both marbles’ color. No signal was required, superluminal or otherwise.

ex. The quantum version of the same experiment uses the spin of entangled particles. When measured on the same axis, one will always be spin-up, the other will be spin-down. It is conceptually analogous. Entanglement is only about correlations of system that interacted in the past and are determined by conservation laws.

SRQM Interpretation:
Interpretation of EPR-Bell Experiment

Einstein and Bohr can both be “right” about EPR:
Per Einstein: The QM State measured is not a “complete” description, just one observer's point-of-view.
Per Bohr: The QM State measured is a “complete” description, it's all that a single observer can get.

The point is that many observers can all see the “same” system, but see different facets of it. But a single measurement is the maximal information that a single observer can get without re-interacting with the system, which of course changes the system in general. Remember, the Heisenberg Uncertainty comes from non-zero commutation properties which *require separate measurement arrangements*. The properties of a particle are always there. Properties define particles. We as observers simply have only partial information about them.

Relativistic QM, being derived from SR, should be local – The low-velocity limit to QM may give unexpected anomalous results if taken out of context, or out of the applicable validity range, such as with velocity addition $v_{12} = v_1+v_2$, where the correct formula should be the relativistic velocity composition $v_{12} = (v_1+v_2)/[1+v_1 v_2/c^2]$

These ideas lead to the conclusion that the wavefunction is just one observer’s state of information about a physical system, not the state of the physical system itself. The “collapse” of the wavefunction is simply the change in an observer’s information about a system brought about by a measurement or, in the case of EPR, an inference about the physical state.

EPR doesn’t break Heisenberg because measurements are made on different particles. The happy fact is that those particles interacted and became correlated in the causal past. The EPR-Bell experiments prove that it is possible to maintain those correlations over long distances. It does not prove superluminal signaling

SRQM Interpretation:
Range-of-Validity Facts & Fallacies

We should not be surprised by the "quantum" probabilities being correct instead of "classical" in the EPR and Bell Inequalities experiments.
Classical thinking (in both CM and QM) has a number of fallacies when it is mistakenly applied outside of its range-of-validity.

Examples

*The limit of $\hbar \to 0$ {Fallacy}:
ħ is a Lorentz Scalar Invariant and Fundamental Physical Constant. It never becomes 0. {Fact}

*The classical commutator being zero $[p^k,x^j] = 0$ {Fallacy}:
$[P^\mu,X^\nu] = i\hbar\eta^{\mu\nu}$; $[p^k,x^j] = -i\hbar\delta^{kj}$; $[p^0,x^0] = [E/c,ct] = [E,t] = i\hbar$; Again, it never becomes 0 {Fact}

*Using Maxwell-Boltzmann (distinguishable) statistics for counting probabilities of (indistinguishable) quantum states {Fallacy}:
Must use Fermi-Dirac statistics for Fermions:Spin=(n+1/2); Bose-Einstein statistics for Bosons:Spin=(n) {Fact}

*Using sums of classical probabilities on quantum states {Fallacy}:
Must use sums of quantum probability-amplitudes {Fact}

*Ignoring phase cross-terms and interference effects in calculations {Fallacy}:
Quantum systems and entanglement require phase cross-terms {Fact}

*Assuming that one can simultaneously "measure" non-commuting properties at a single spacetime event {Fallacy}:
Particle properties always exist. However, non-commuting ones require <u>separate</u> measurement arrangements to get information about the properties.
The required measurement arrangements on a single particle/worldline are at best sequential events, where the temporal order plays a role; {Fact}
However, EPR allows one to "infer (not measure)" the other property of a particle by the <u>separate</u> measurement of an entangled partner. {Fact}
This does not break Heisenberg Uncertainty, which is about the order of operations (measurement events) on a single worldline. {Fact}
In the entangled case, both/all of the entangled partners share common past-causal entanglement events, typically due to a conservation law. {Fact}
Information is not transmitted at FTL. The particles simply carried their normal respective "correlated" properties (no hidden variables) with them. {Fact}

*Assuming that QM is a generalization of CM, or that classical probabilities apply to QM {Fallacy}:
CM is a limiting-case of QM for when changes in a system by a few quanta have a negligible effect on the whole/overall system. {Fact}

SRQM Interpretation: Quantum Information

We should not be surprised by the “quantum” probabilities being correct instead of “classical” in the EPR and Bell Inequalities experiments. Classical thinking (in both CM and QM) has a number of fallacies when it is mistakenly applied outside of its range-of-validity.

{from Wikipedia}
No-Communication Theorem/No-Signaling:
A no-go theorem from quantum information theory which states that, during measurement of an entangled quantum state, it is not possible for one observer, by making a measurement of a subsystem of the total state, to communicate information to another observer. The theorem shows that quantum correlations do not lead to what could be referred to as "spooky communication at a distance". SRQM: There is no FTL signaling.

No-Teleportation Theorem:
The no-teleportation theorem stems from the Heisenberg uncertainty principle and the EPR paradox: although a qubit |ψ> can be imagined to be a specific direction on the Bloch sphere, that direction cannot be measured precisely, for the general case |ψ>. The no-teleportation theorem is implied by the no-cloning theorem.
SRQM: Ket states are informational, not physical.

No-Cloning Theorem:
In physics, the no-cloning theorem states that it is impossible to create an identical copy of an arbitrary unknown quantum state. This no-go theorem of quantum mechanics proves the impossibility of a simple perfect non-disturbing measurement scheme. The no-cloning theorem is normally stated and proven for pure states; the no-broadcast theorem generalizes this result to mixed states. SRQM: Measurements are arrangements of particles that interact with a subject particle.

No-Broadcast Theorem:
Since quantum states cannot be copied in general, they cannot be broadcast. Here, the word "broadcast" is used in the sense of conveying the state to two or more recipients. For multiple recipients to each receive the state, there must be, in some sense, a way of duplicating the state. The no-broadcast theorem generalizes the no-cloning theorem for mixed states. The no-cloning theorem says that it is impossible to create two copies of an unknown state given a single copy of the state.
SRQM: Conservation of worldlines.

No-Deleting Theorem:
In physics, the no-deleting theorem of quantum information theory is a no-go theorem which states that, in general, given two copies of some arbitrary quantum state, it is impossible to delete one of the copies. It is a time-reversed dual to the no-cloning theorem, which states that arbitrary states cannot be copied.
SRQM: Conservation of worldlines.

No-Hiding Theorem:
the no-hiding theorem is the ultimate proof of the conservation of quantum information. The importance of the no-hiding theorem is that it proves the conservation of wave function in quantum theory.
SRQM: Conservation of worldlines. RQM wavefunctions are Lorentz Scalars (spin=0), Spinors (spin=1/2), 4-Vectors (spin=1), all of which are Lorentz Invariant.

SRQM Interpretation: Quantum Information

We should not be surprised by the “quantum” probabilities being correct instead of “classical” probabilities in the EPR/Bell-Inequalities experiments.
Classical thinking (in both CM and QM) has a number of fallacies when it is mistakenly applied outside of its range-of-validity.
{from Wikipedia}
Quantum information (qubits) differs strongly from classical information, epitomized by the bit, in many striking and unfamiliar ways. Among these are the following:

A unit of quantum information is the qubit. Unlike classical digital states (which are discrete), a qubit is continuous-valued, describable by a direction on the Bloch sphere. Despite being continuously valued in this way, a qubit is the smallest possible unit of quantum information, as despite the qubit state being continuously-valued, it is impossible to measure the value precisely.

A qubit cannot be (wholly) converted into classical bits; that is, it cannot be "read". This is the no-teleportation theorem.

Despite the awkwardly-named no-teleportation theorem, qubits can be moved from one physical particle to another, by means of quantum teleportation. That is, qubits can be transported, independently of the underlying physical particle. SRQM: Ket states are informational, not physical.

An arbitrary qubit can neither be copied, nor destroyed. This is the content of the no cloning theorem and the no-deleting theorem. SRQM: Conservation of worldlines.

Although a single qubit can be transported from place to place (e.g. via quantum teleportation), it cannot be delivered to multiple recipients; this is the no-broadcast theorem, and is essentially implied by the no-cloning theorem. SRQM: Conservation of worldlines.

Qubits can be changed, by applying linear transformations or quantum gates to them, to alter their state. While classical gates correspond to the familiar operations of Boolean logic, quantum gates are physical unitary operators that in the case of qubits correspond to rotations of the Bloch sphere.

Due to the volatility of quantum systems and the impossibility of copying states, the storing of quantum information is much more difficult than storing classical information. Nevertheless, with the use of quantum error correction quantum information can still be reliably stored in principle. The existence of quantum error correcting codes has also led to the possibility of fault tolerant quantum computation.

Classical bits can be encoded into and subsequently retrieved from configurations of qubits, through the use of quantum gates. By itself, a single qubit can convey no more than one bit of accessible classical information about its preparation. This is Holevo's theorem. However, in superdense coding a sender, by acting on one of two entangled qubits, can convey two bits of accessible information about their joint state to a receiver.

Quantum information can be moved about, in a quantum channel, analogous to the concept of a classical communications channel. Quantum messages have a finite size, measured in qubits; quantum channels have a finite channel capacity, measured in qubits per second.

Minkowski still applies in local GR
QM is a local phenomenon

The QM Schrodinger Equation is not fundamental. It is just the low-energy limiting-case of the RQM Klein-Gordon Equation. All of the standard QM Axioms are shown to be empirically measured constants or emergent properties of SR. It is a bad approach to start with NRQM as an axiomatic starting point and try to generalize it to RQM, in the same way that one cannot start with CM and derive SR. Since QM *can* be derived from SR, this partially explains the difficulty of uniting QM with GR:
QM is not a “separate formalism” outside of SR that can be used to “quantize” just anything...

Strictly speaking, the use of the Minkowski space to describe physical systems over finite distances applies only in the SR limit of systems without significant gravitation. In the case of significant gravitation, SpaceTime becomes curved and one must abandon SR in favor of the full theory of GR.

Nevertheless, even in such cases, based on the GR Equivalence Principle, Minkowski space is still a good description in a local region surrounding any point (barring gravitational singularities). More abstractly, we say that in the presence of gravity, SpaceTime is described by a curved 4-dimensional manifold for which the tangent space to any point is a 4-dimensional Minkowski Space. Thus, the structure of Minkowski Space is still essential in the description of GR.

So, even in GR, at the local level things are considered to be Minkowskian:
i.e. SR → QM “lives inside the surface” of this local SpaceTime, GR curves the surface.

SRQM Interpretation: Main Result
QM is derivable from SR!

Hopefully, this interpretation will shed light on why Quantum Gravity has been so elusive. Basically, QM rules of “quantization” don’t apply to GR. They are a manifestation-of/derivation-from SR. Relativity *is* the “Theory of Measurement” that QM has been looking for.

This would explain why no one has been able to produce a successful theory of Quantum Gravity,
and why there have been no violations of Lorentz Invariance nor of the Equivalence Principle.

If quantum effects “live” in Minkowski SpaceTime with SR,
then GR curvature effects are at a level above the RQM description, and two levels above standard QM.
SR+QM are “in” SpaceTime, GR is the “shape” of SpaceTime…

<u>Thus, this treatise explains the following:</u>

- Why GR works so well in it's realm of applicability {large scale systems}.
- Why QM works so well in it's realm of applicability {micro scale systems and certain macroscopic systems}.
  i.e. The tangent space to any point in GR curvature is locally Minkowskian, and thus QM is typically found in small local volumes...
- Why RQM explains more stuff than QM without SR {because QM is just the low-velocity limiting-case of RQM}.
- Why all attempts to "quantize gravity" have failed {essentially, everyone has been trying to put the cart (QM) before the horse (GR)}.
- Why all attempts to modify GR keep conflicting with experimental data {because GR is apparently fundamental}.
- Why QM works perfectly well with SR as RQM but not with GR {because QM is derivable from SR, hence a manifestation of SR rules}.
- How Minkowski Space, 4-Vectors, and Lorentz Invariants play vital roles in RQM, and give the SRQM Interpretation of Quantum Mechanics.

SRQM:
SR→QM Interpretation Simplified

A Tensor Study of Physical 4-Vectors

SciRealm.org
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http://scirealm.org/SRQM.pdf

SRQM: The [SR→QM] Interpretation of Quantum Mechanics

SR Axioms: Invariant Interval + (c) as Physical Constant lead to SR,
although technically SR is itself the low-curvature limiting-case of GR

$\{c,\tau,m_o,\hbar,i\}$: All Empirically Measured SR Lorentz Invariants and/or Mathematical Constants

Standard SR 4-Vectors: Related by these SR Lorentz Invariants

4-Position	$\mathbf{R} = (ct,\mathbf{r})$	= <Event>	$(\mathbf{R}\cdot\mathbf{R}) = (c\tau)^2$
4-Velocity	$\mathbf{U} = \gamma(c,\mathbf{u})$	$= (\mathbf{U}\cdot\partial)\mathbf{R} = ({}^{d}/_{d\tau})\mathbf{R} = d\mathbf{R}/d\tau$	$(\mathbf{U}\cdot\mathbf{U}) = (c)^2$
4-Momentum	$\mathbf{P} = (E/c,\mathbf{p})$	$= m_o\mathbf{U}$	$(\mathbf{P}\cdot\mathbf{P}) = (m_oc)^2$
4-WaveVector	$\mathbf{K} = (\omega/c,\mathbf{k})$	$= \mathbf{P}/\hbar$	$(\mathbf{K}\cdot\mathbf{K}) = (m_oc/\hbar)^2$
4-Gradient	$\partial = (\partial_t/c,-\nabla)$	$= -i\mathbf{K}$	$(\partial\cdot\partial) = -(m_oc/\hbar)^2$ = KG Eqn→RQM $\xrightarrow{\lvert\mathbf{v}\rvert<<c}$ QM

SR + Emipirically Measured Physical Constants lead to RQM via the Klein-Gordon Eqn,
and thence to QM via the low-velocity limit { |**v**| << c }, giving the Schrödinger Eqn.
The relation also leads to the Dirac, Maxwell, Pauli, Proca, Weyl, & Scalar Wave QM Eqns.

SRQM: A treatise of SR→QM by John B. Wilson (SciRealm@aol.com)

SR → QM

4-Vector SRQM Interpretation of QM

SRQM Diagram:
Special Relativity → Quantum Mechanics
RoadMap of SR→QM

A Tensor Study of Physical 4-Vectors

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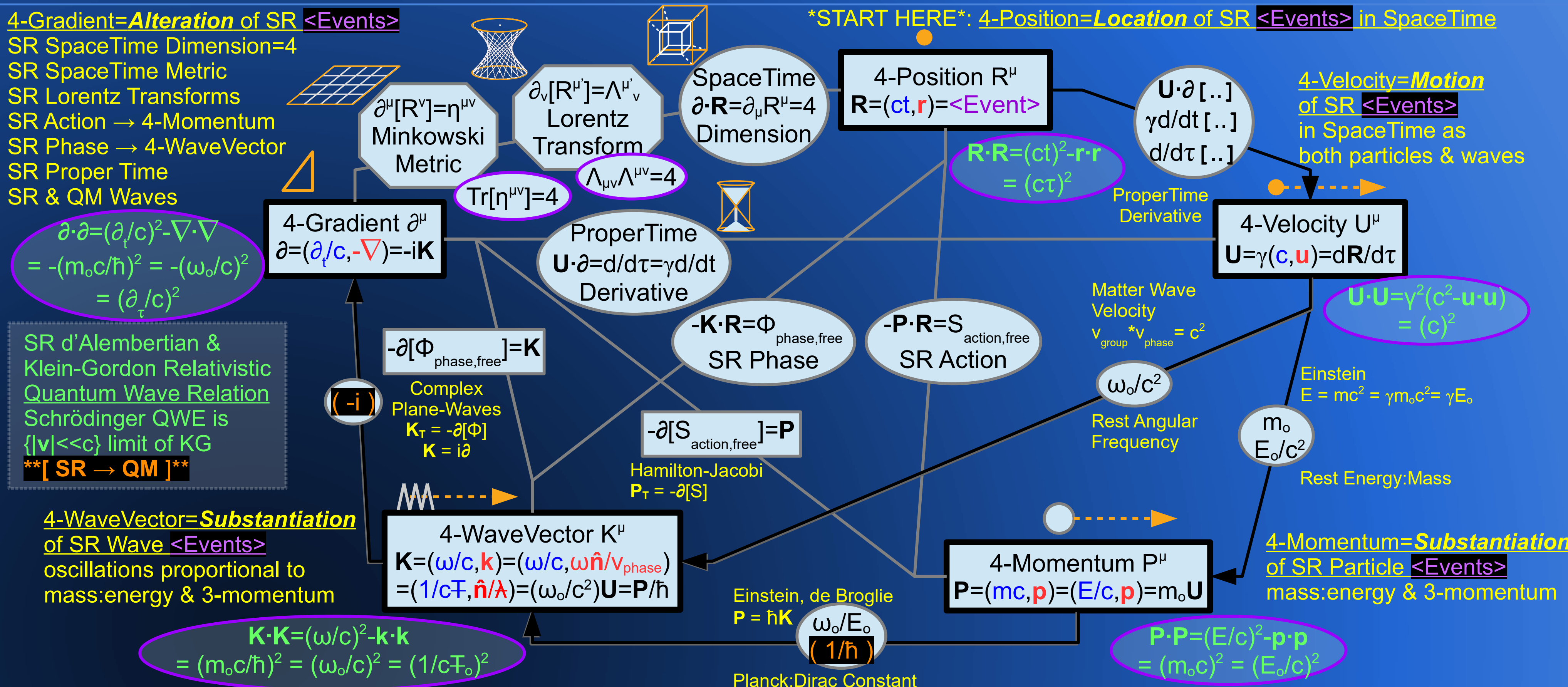

SR 4-Tensor (2,0)-Tensor $T^{\mu\nu}$ (1,1)-Tensor $T^\mu{}_\nu$ or $T_\mu{}^\nu$ (0,2)-Tensor $T_{\mu\nu}$	**SR 4-Vector** (1,0)-Tensor V^μ = **V** = (v^0,**v**) **SR 4-CoVector** (0,1)-Tensor V_μ = (v_0,-**v**)	**SR 4-Scalar** (0,0)-Tensor S Lorentz Scalar	Existing SR Rules Quantum Principles

$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu{}_\mu = T$

$\mathbf{V}\cdot\mathbf{V} = V^\mu\eta_{\mu\nu}V^\nu = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0{}_o)^2$ = Lorentz Scalar

SR → QM

4-Vector SRQM Interpretation of QM

SRQM Diagram:

Special Relativity → Quantum Mechanics

RoadMap of SR→QM (EM Potential)

A Tensor Study of Physical 4-Vectors

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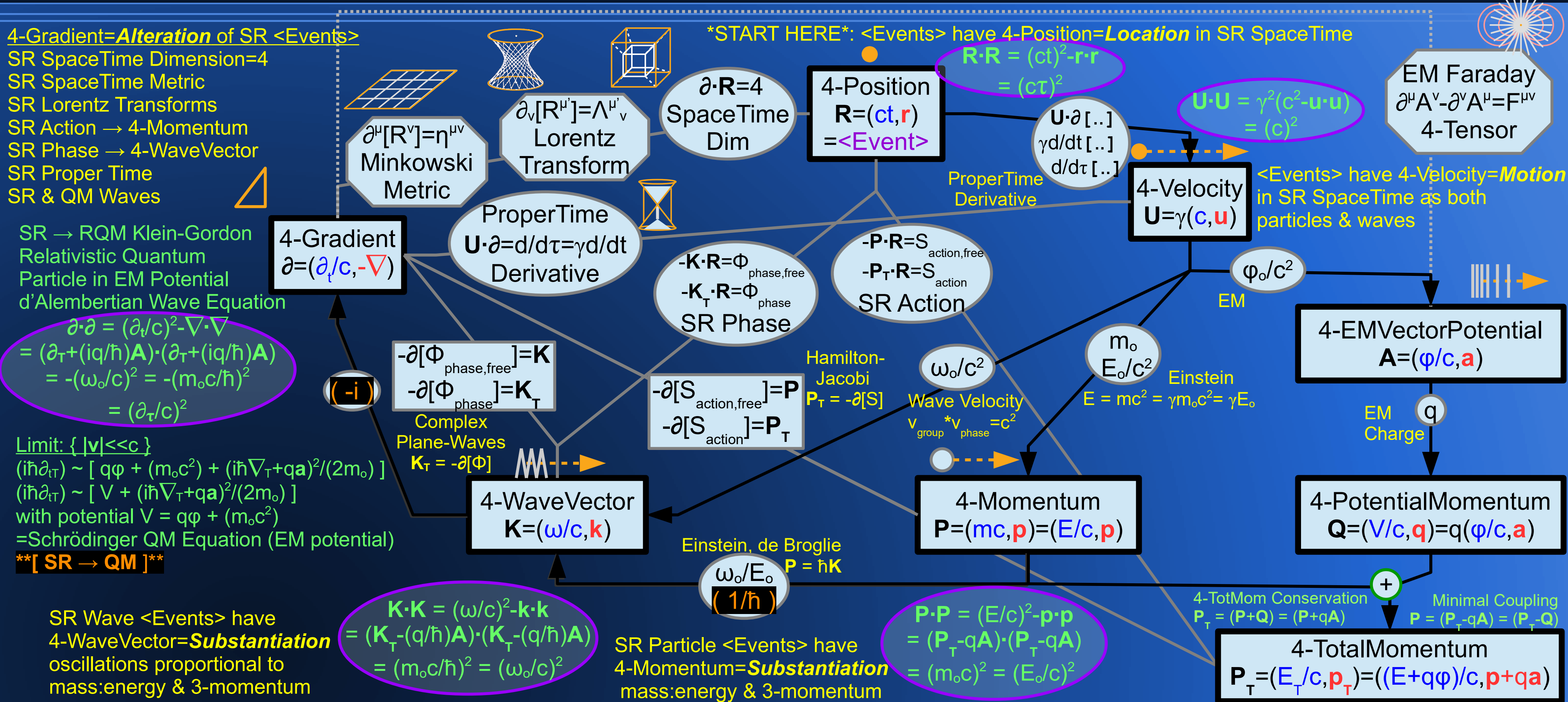

SR 4-Tensor
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(1,1)-Tensor $T^\mu{}_\nu$ or $T_\mu{}^\nu$
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(0,1)-Tensor $V_\mu = (v_0,-\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Existing SR Rules
Quantum Principles

$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu{}_\mu = T$
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SR → QM

4-Vector SRQM Interpretation of QM

SRQM Diagram: SRQM 4-Vectors and Lorentz Scalars / Physical Constants

A Tensor Study of Physical 4-Vectors

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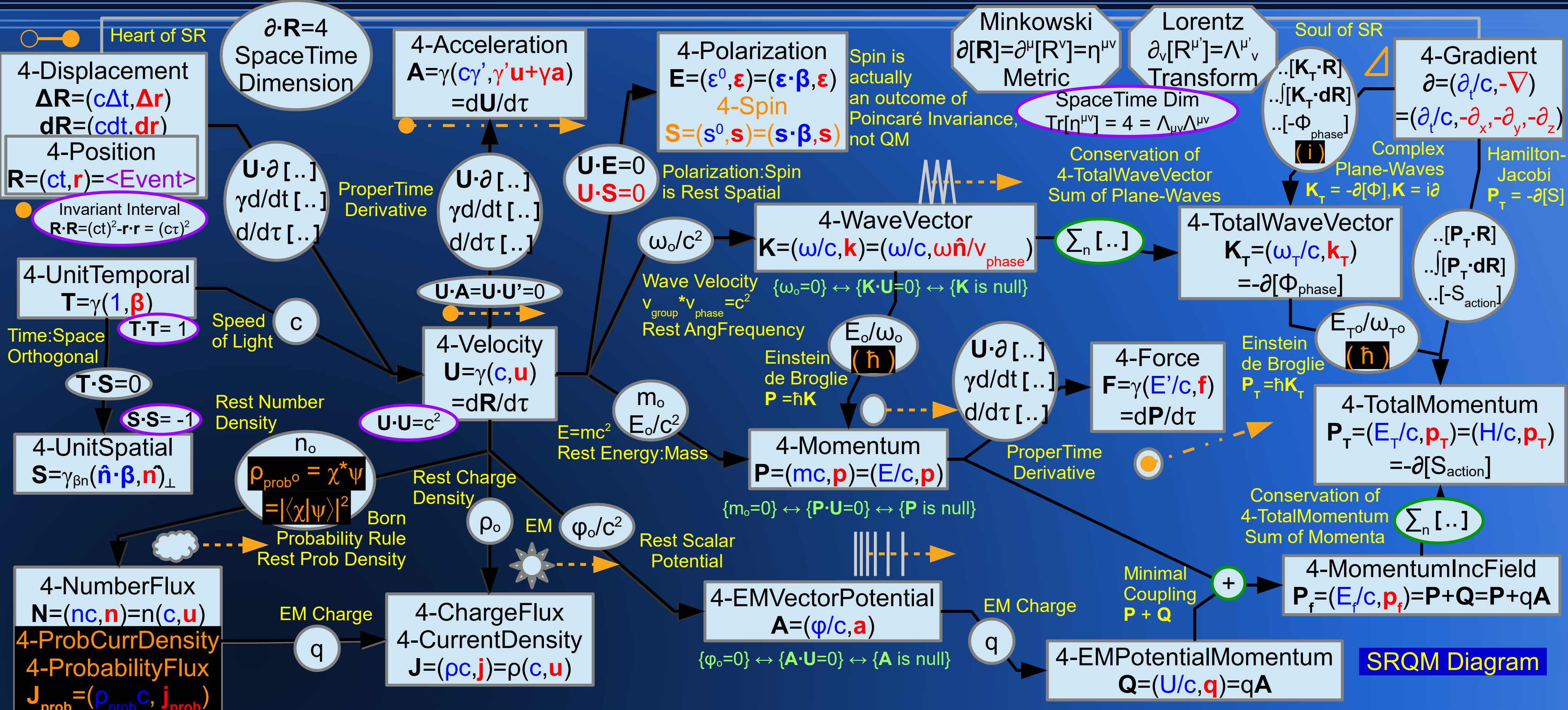

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^{\mu}{}_{\nu}$ or $T_{\mu}{}^{\nu}$
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SR 4-Vector
(1,0)-Tensor $V^{\mu} = \mathbf{V} = (v^0,\mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_{\mu} = (v_0,-\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Existing SR Rules
Quantum Principles

$\mathrm{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^{\mu}{}_{\mu} = T$
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= Lorentz Scalar

Special Relativity → Quantum Mechanics
The SRQM Interpretation: Links

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See also:
http://scirealm.org/SRQM.html (alt discussion)
http://scirealm.org/SRQM-RoadMap.html (main SRQM website)
http://scirealm.org/4Vectors.html (4-Vector study)
http://scirealm.org/SRQM-Tensors.html (Tensor & 4-Vector Calculator)
http://scirealm.org/SciCalculator.html (Complex-capable RPN Calculator)

or Google "SRQM"

http://scirealm.org/SRQM.pdf (this document)

The 4-Vector SRQM Interpretation

QM is derivable from SR!

A Tensor Study of Physical 4-Vectors

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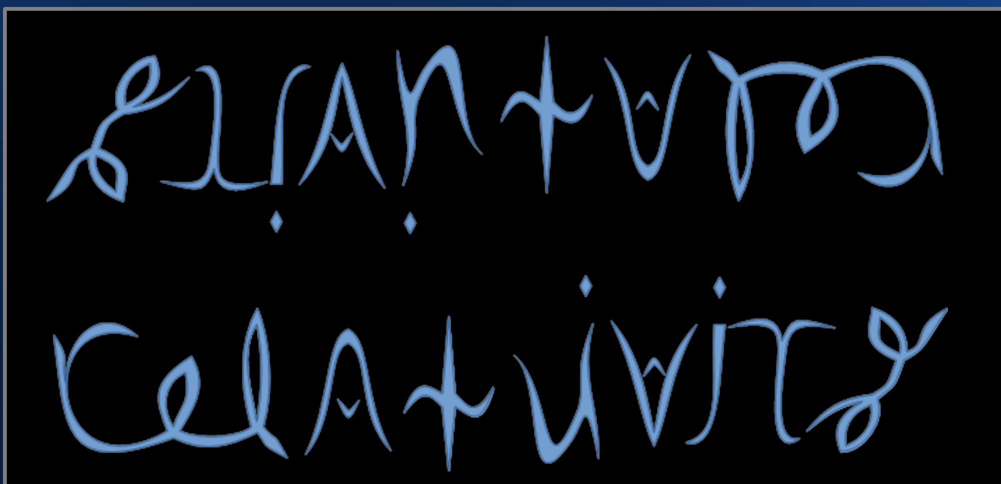
The SRQM or [SR→QM] Interpretation of Quantum Mechanics
A Tensor Study of Physical 4-Vectors

quantum
relativity



SRQM = SciRealm QM? A happy coincidence… :) Ambigrams