Solving the $n_1 \times n_2 \times n_3$ Points Problem for $n_3 < 6$

Marco Ripà

sPIqr Society, World Intelligence Network
Rome, Italy
e-mail: marcokrt1984@yahoo.it

Abstract: In this paper, we show enhanced upper bounds of the nontrivial $n_1 \times n_2 \times n_3$ points problem for every $n_1 \leq n_2 \leq n_3 < 6$. We present new patterns that drastically improve the previously known algorithms for finding minimum-link covering paths, completely solving the fundamental case $n_1 = n_2 = n_3 = 3$.

Keywords: Graph theory, Topology, Three-dimensional, Creative thinking, Link-length, Connectivity, Outside the box, Upper bound, Point, Game, Covering path, Hamiltonian path.

2010 Mathematics Subject Classification: 91A43, 05C57.

1 Introduction

The $n_1 \times n_2 \times n_3$ points problem [11] is a three-dimensional extension of the classic *nine-dot problem* appeared in Samuel Loyd's *Cyclopedia of Puzzles* [1-8], and it is related to the well known NP-hard traveling salesman problem, minimizing the number of turns in the tour instead of the total distance traveled [1-13].

Given $n_1 \cdot n_2 \cdot n_3$ points in $\mathbb{R}^3$, our goal is to visit all of them (at least once) with a polygonal path that has the minimum number of line segments connected at their end-points (links or generically *lines*), the so called Minimum-link Covering Path [2-3-4-7]. In particular, we are interested in the best solutions to the nontrivial $n_1 \times n_2 \times n_3$ dots problem, where (by definition) $1 \leq n_1 \leq n_2 \leq n_3$ and $n_3 < 6$.

Let $h_l(n_1, n_2, n_3) \leq h(n_1, n_2, n_3) \leq h_u(n_1, n_2, n_3)$ be the length of the covering path with the minimum number of links for the $n_1 \times n_2 \times n_3$ points problem, we define the best known upper bound as $h_u(n_1, n_2, n_3) \geq h(n_1, n_2, n_3)$, and we denote as $h_l(n_1, n_2, n_3) \leq h(n_1, n_2, n_3)$ the proved lower bound. For the simplest cases, the same problem has already been solved [2].

Let $n_1 = 1$ and $n_2 < n_3$, we have that $h(n_1, n_2, n_3) = h(n_2) = 2 \cdot n_2 - 1$, while $h(n_1 = 1,\ n_2 = n_3 \geq 3) = 2 \cdot n_2 - 2$ [5].

Hence, for $n_1 = 2$, it can be easily proved that

$$h(2, n_2, n_3) = 2 \cdot h(1, n_2, n_3) + 1 = \begin{cases} 4 \cdot n_2 - 1 & \textbf{\textit{iff}} \quad n_2 < n_3 \\ 4 \cdot n_2 - 3 & \textbf{\textit{iff}} \quad n_2 = n_3 \end{cases}. \tag{1}$$

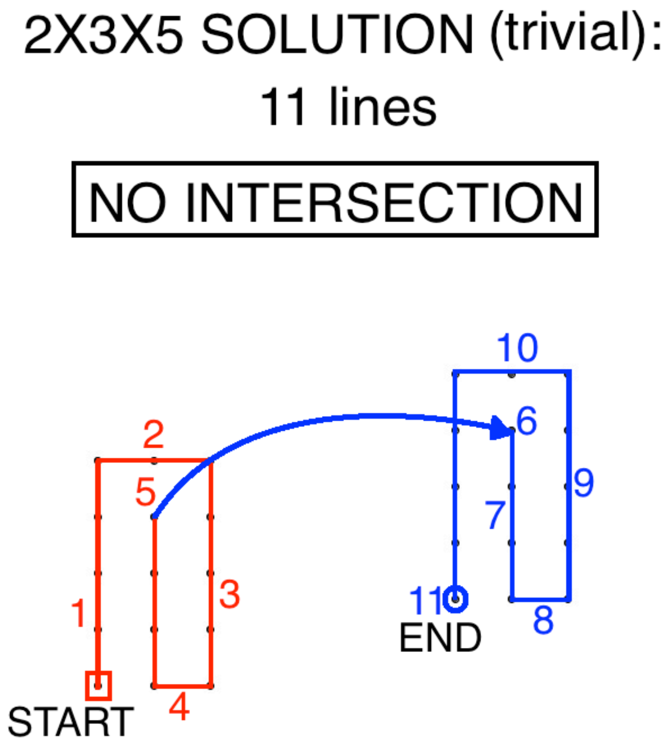

Figure 1. A trivial pattern that completely solves the 2 × 3 × 5 points puzzle (avoiding self-intersections).

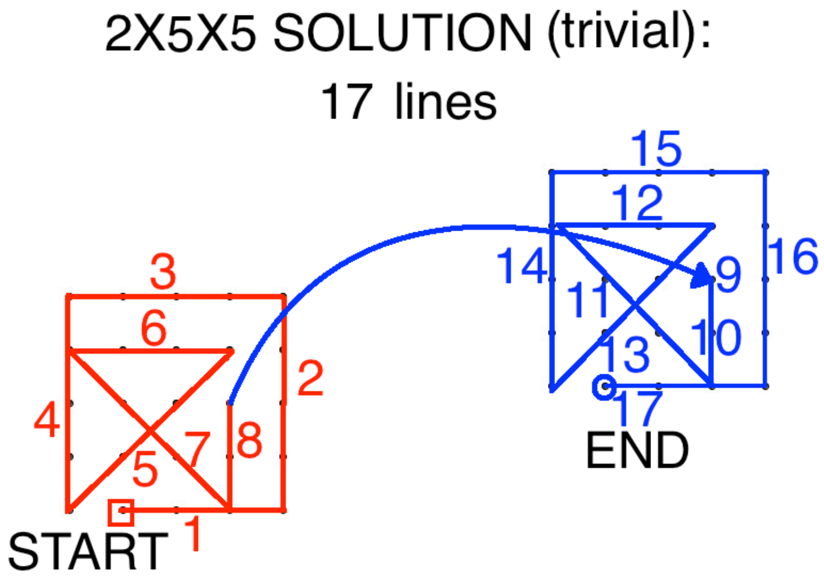

Figure 2. Another example of a trivial case: the 2 × 5 × 5 points puzzle.

Therefore, the aim of the present paper is to solve the ten aforementioned nontrivial cases where the current upper bound does not match the proved lower bound.

2 Improving the solution of the $n_1 \times n_2 \times n_3$ points problem for $n_3 < 6$

In this complex brain challenge we need to stretch our pattern recognition [6-9] in order to find a plastic strategy that improves the known upper bounds [2-12] for the most interesting cases (and the $3 \times 3 \times 3$ puzzle, which is the three-dimensional extension of the immortal nine-dot problem, is by far the most valuable one), avoiding those standardized methods which are based on fixed patterns that lead to suboptimal covering paths, as the approaches presented in [7-10].

Theorem 1

If $3 \leq n_1 \leq n_2 \leq n_3$, then a lower bound of the general $n_1 \times n_2 \times n_3$ problem is given by

$$h_l(n_1, n_2, n_3) = \left\lceil \frac{3 \cdot (n_3 \cdot n_2 \cdot n_1 - n_1)}{2 \cdot n_3 + n_2 - 3} \right\rceil + 1. \tag{2}$$

Proof Let $n_1 \times n_2 \times \ldots \times n_k$ be a set of $\prod_{i=1}^{k} n_i$ points in $\mathbb{R}^k$ such that $n_1 \leq n_2 \leq \ldots \leq n_k$, it is not possible to intersect more than $(n_k - 1) + (n_{k-1} - 1) + (n_k - 1) = 2 \cdot n_k + n_{k-1} - 3$ points using three straight lines connected at their endpoints; however, there is one exception (which, for simplicity, we may assume as in the case of the first line drawn). In this circumstance, it is possible to fit n_k points with the first line, $n_{k-1} - 1$ points using the second line, $n_k - 1$ points with the next one, and so forth. In general, the third and the last line of the aforementioned group will join (at most) $n_k - 1$ points each.

In order to complete the covering path, reaching every edge of our hyper-parallelepiped, we need at least one more link for any of the remaining n_i, and this implies that $k - 2$ lines cannot join a total of more than $n_{k-2} - 1 + n_{k-3} - 1 + \ldots + n_1 - 1 = \sum_{i=1}^{k-2} n_i - k + 2$ unvisited points.

Thus, the considered lower bound $h_l(n_1, n_2, \ldots, n_k)$ satisfies the relation

$$\prod_{i=1}^{k} n_i - \sum_{i=1}^{k-2} n_i + k - 2 - 1 \leq (2 \cdot n_k + n_{k-1} - 3) \cdot \left(\frac{h_l(n_1, n_2, \ldots, n_k)}{3} - k + 2 \right). \tag{3}$$

Hence,

$$h_l(n_1, n_2, \ldots, n_k) = \left\lceil 3 \cdot \frac{\prod_{i=1}^{k} n_i - \sum_{i=1}^{k-2} n_i + k - 3}{2 \cdot n_k + n_{k-1} - 3} \right\rceil + k - 2. \tag{4}$$

Substituting $k = 3$ into equation (4), we get the statement of Theorem 1. □

The current best results are listed in Table 1, and a direct proof follows for each nontrivial upper bound shown below.

n_1	n_2	n_3	**Best Lower Bound (h_l)**	**Best Upper Bound (h_u)**	**Discovered by**	**Gap (h_u-h_l)**
2	2	3	7	**7**	trivial	0
2	3	3	9	**9**	trivial	0
3	3	3	13	**13**	Marco Ripà (proved on Jun. 19, 2020 [v6])	0
2	2	4	7	**7**	trivial	0
2	3	4	11	**11**	trivial	0
2	4	4	13	**13**	trivial	0
3	3	4	14	15	Marco Ripà (proved on Jun. 27, 2019 [v1])	1
3	4	4	16	19	Marco Ripà (ibid.)	3
4	4	4	21	23	Marco Ripà (NNTDM [12])	2
2	2	5	7	**7**	trivial	0
2	3	5	11	**11**	trivial	0
2	4	5	15	**15**	trivial	0

n_1	n_2	n_3	Best Lower Bound (h_l)	Best Upper Bound (h_u)	Discovered by	Gap ($h_u - h_l$)
2	5	5	17	**<u>17</u>**	trivial	0
3	3	5	14	16	Marco Ripà (proved on Jun. 27, 2019 [v1])	2
3	4	5	17	20	Marco Ripà (ibid.)	3
3	5	5	19	24	Marco Ripà (ibid.)	5
4	4	5	22	26	Marco Ripà (ibid.)	4
4	5	5	25	31	Marco Ripà (ibid.)	6
5	5	5	31	36	Marco Ripà (proved on Jul. 9, 2019 [v4])	5

Table 1: Current solutions to the $n_1 \times n_2 \times n_3$ points problem, where $n_1 \leq n_2 \leq n_3 \leq 5$.

Figures 3 to 12 show the patterns used to solve the $n_1 \times n_2 \times n_3$ puzzle (case by case). In particular, combining equation (2) with the original results shown in figures 3-4, we obtain a formal proof for the major $3 \times 3 \times 3$ points problem, plus very tight bounds for the $3 \times 3 \times 4$ case.

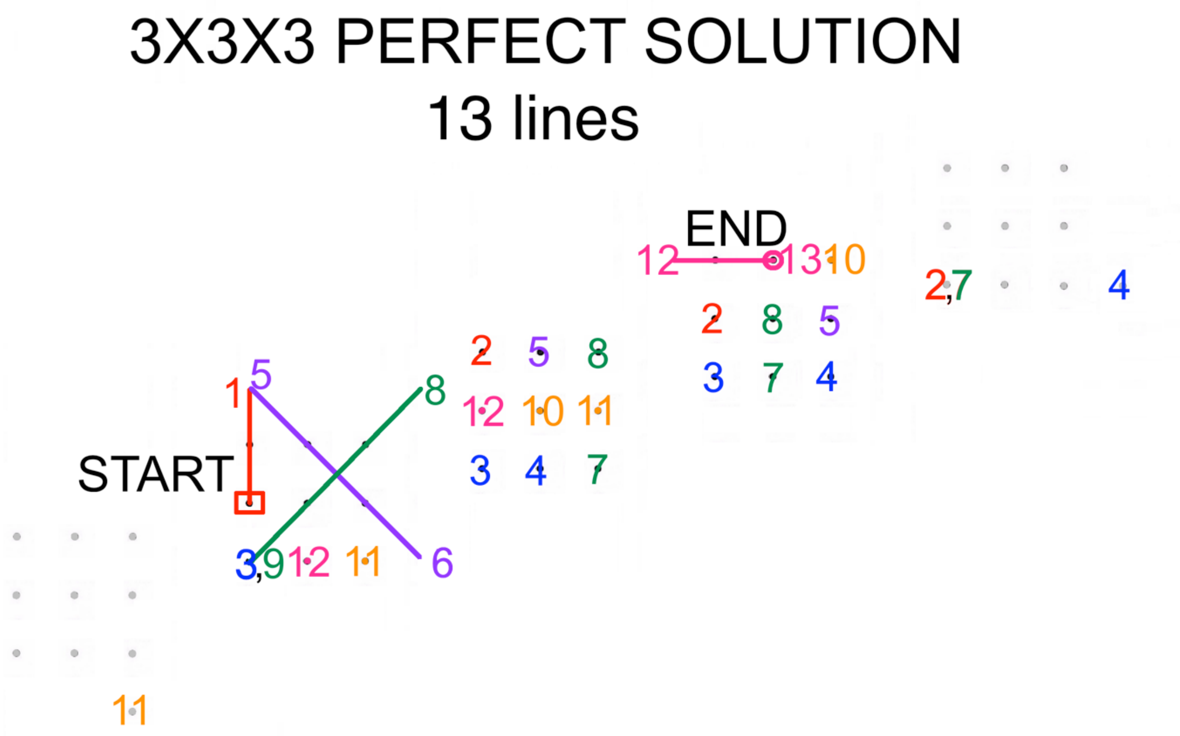

Figure 3. The $3 \times 3 \times 3$ puzzle has finally been solved: $h_u(3,3,3) = h_l(3,3,3) = 13$. This solution can trivially be proved to be optimal.

Corollary 1

$$h_l(3,3,3) = h_u(3,3,3) = h(3,3,3) = 13. \quad (5)$$

Proof The covering path of the $3 \times 3 \times 3$ case shown in Figure 3 consists of 13 straight lines connected at their end-points, and equation (2) gives $h_l(3,3,3) = \lceil 12 \rceil + 1 = 13$. □

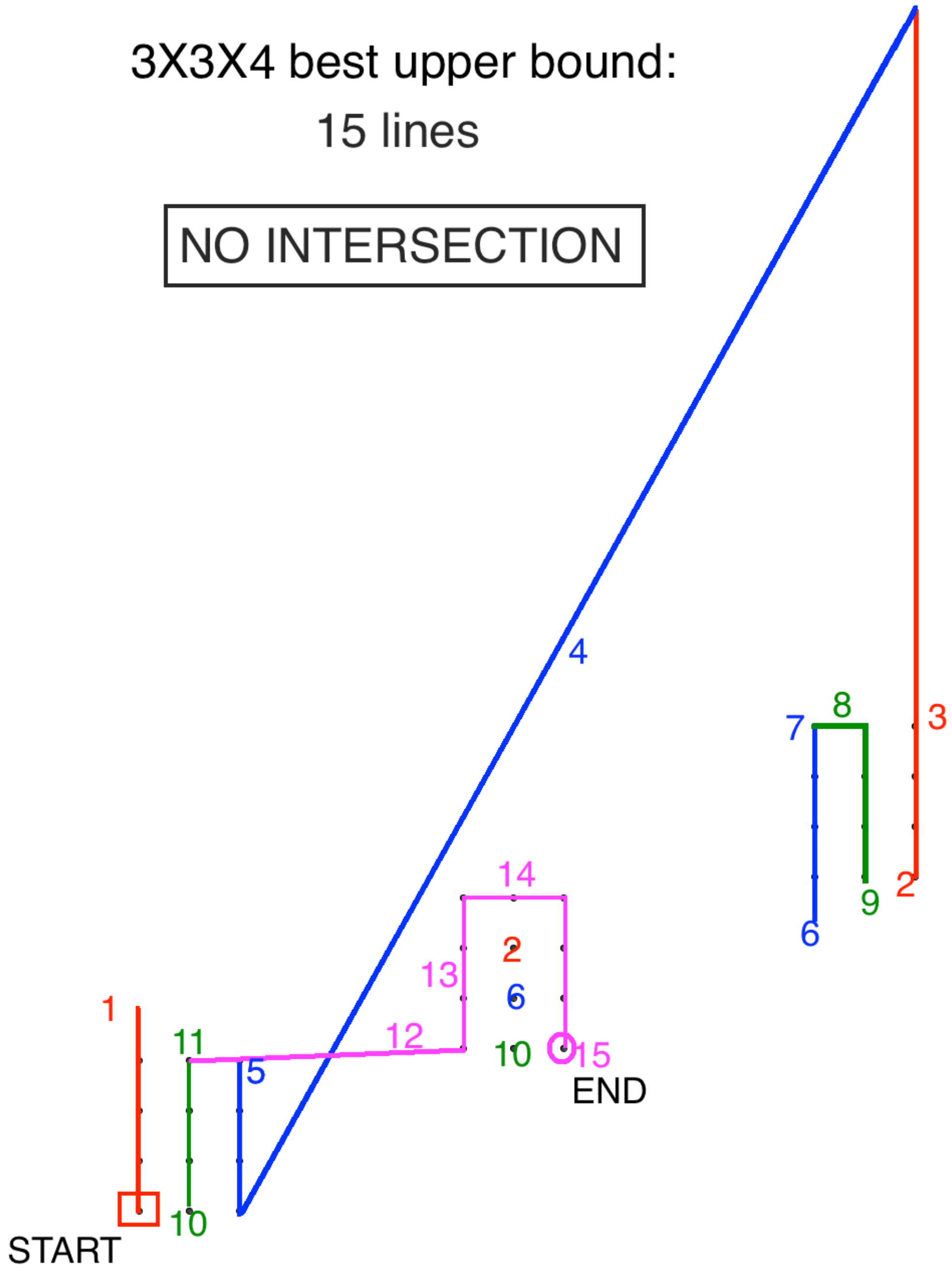

Figure 4. Best known (non-crossing) Hamiltonian path for the $3 \times 3 \times 4$ puzzle.
$15 = h_u = h_l + 1$.

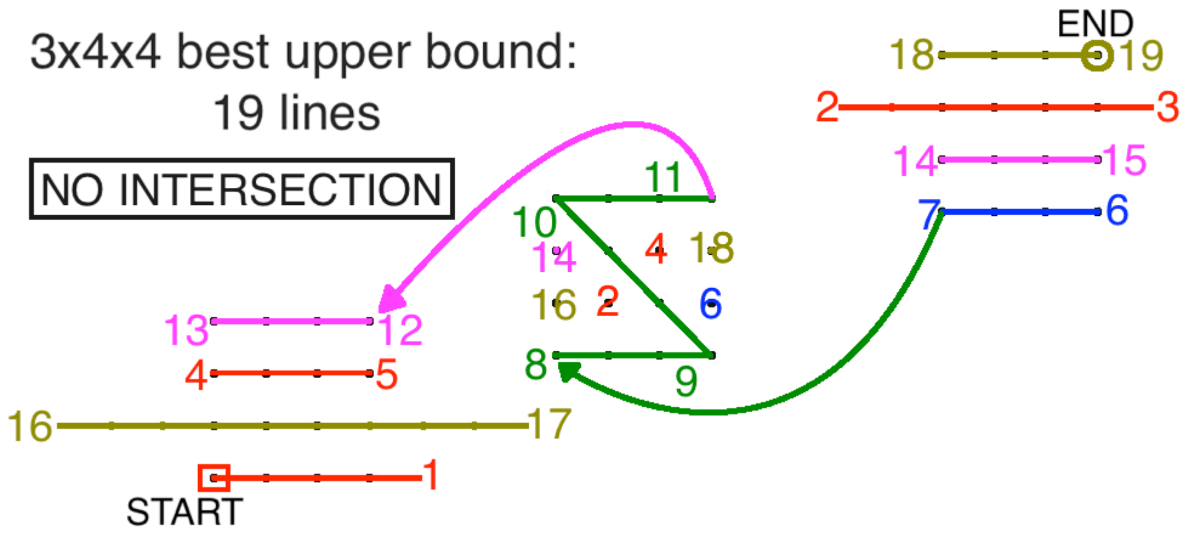

Figure 5. Best known (non-crossing) Hamiltonian path for the $3 \times 4 \times 4$ puzzle. $19 = h_u = h_l + 3$.

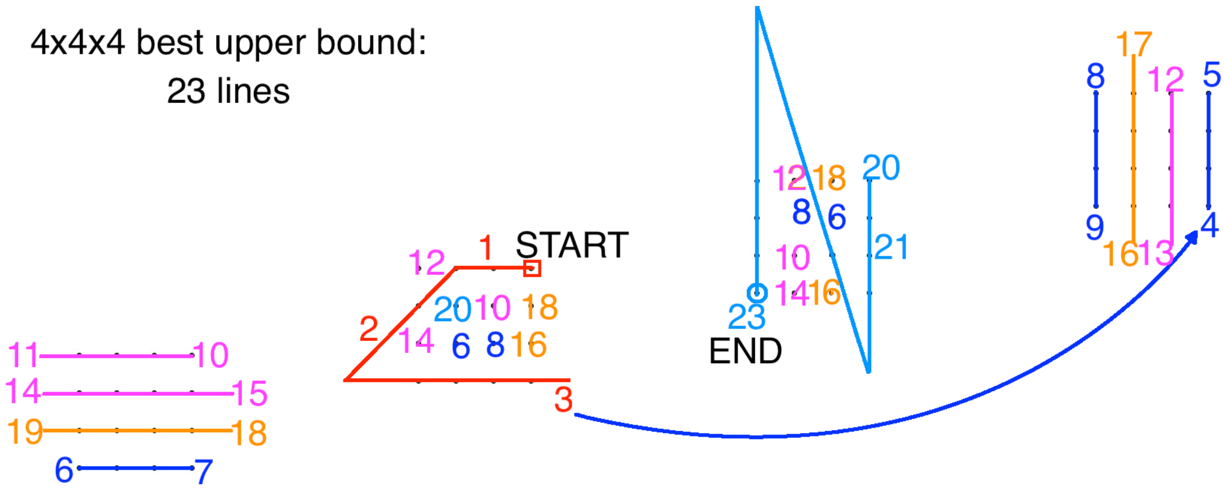

Figure 6. An original Hamiltonian path for the $4 \times 4 \times 4$ puzzle. $23 = h_u = h_l + 2$ [12].

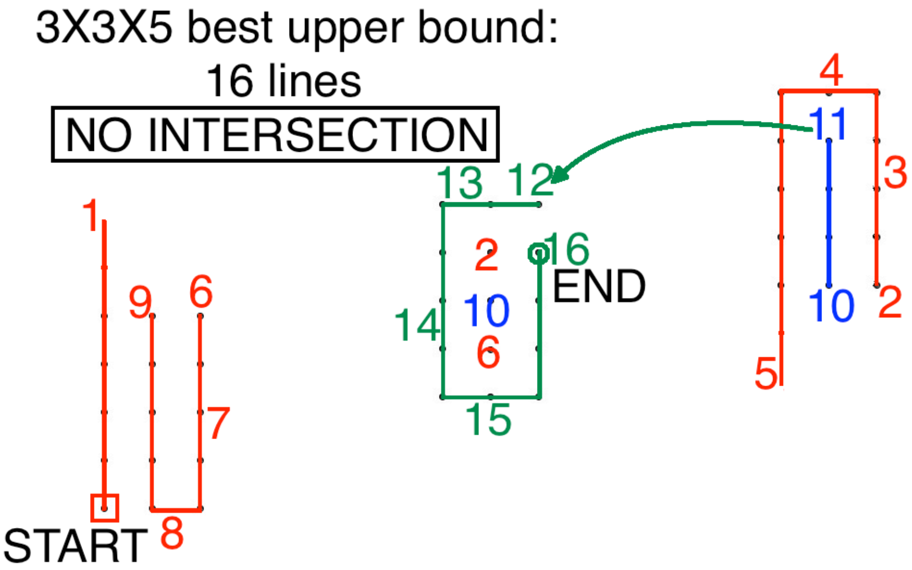

Figure 7. Best known (non-crossing) Hamiltonian path for the $3 \times 3 \times 5$ puzzle.
$16 = h_u = h_l + 2$.

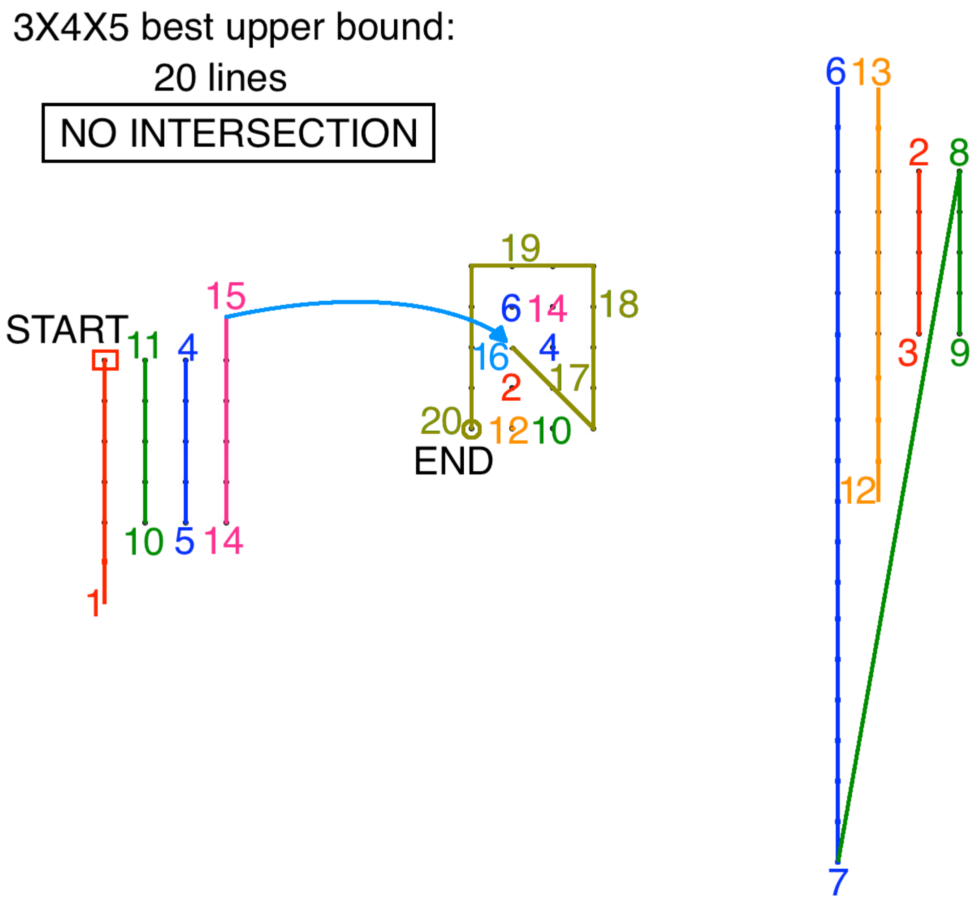

Figure 8. Best known (non-crossing) Hamiltonian path for the $3 \times 4 \times 5$ puzzle, consisting of
$20 = h_u = h_l + 3$ lines.

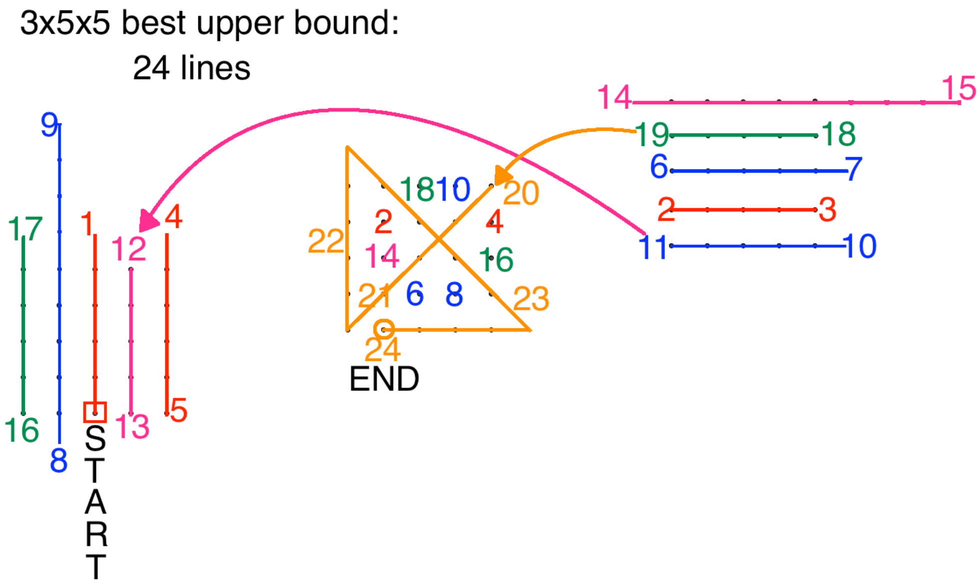

Figure 9. Best known Hamiltonian path for the $3 \times 5 \times 5$ puzzle. $24 = h_u = h_l + 5$.

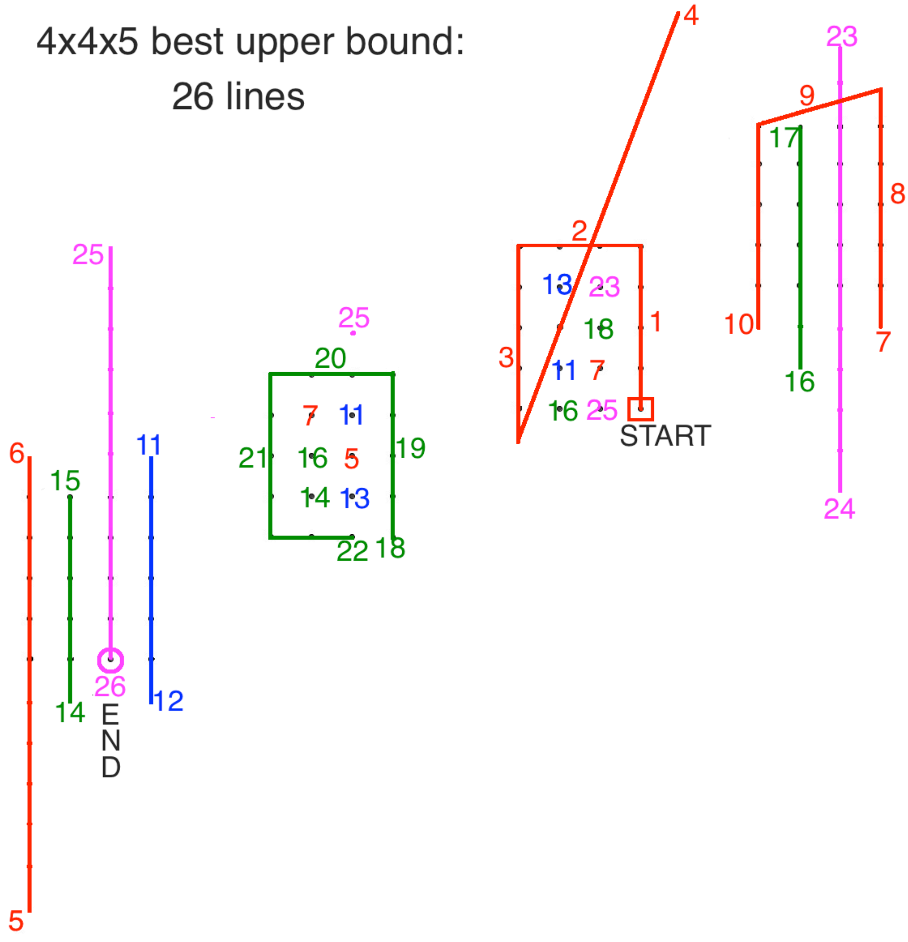

Figure 10. Best known Hamiltonian path for the $4 \times 4 \times 5$ puzzle. $26 = h_u = h_l + 4$.

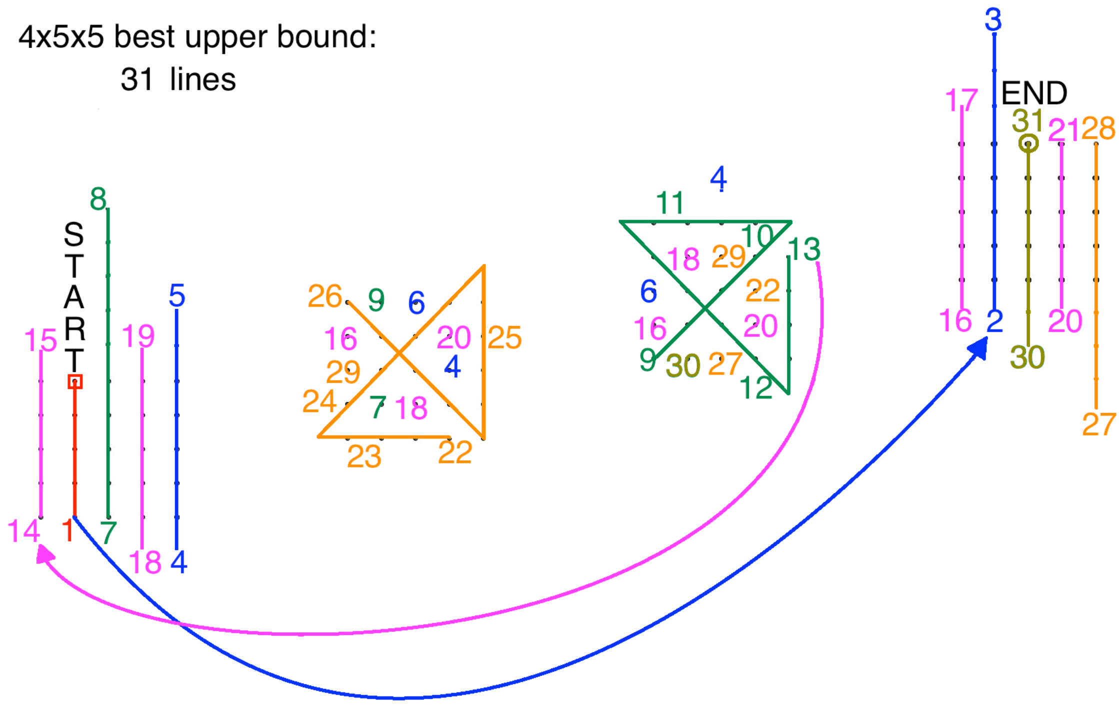

Figure 11. Best known Hamiltonian path for the $4 \times 5 \times 5$ puzzle. $31 = h_u = h_l + 6$.

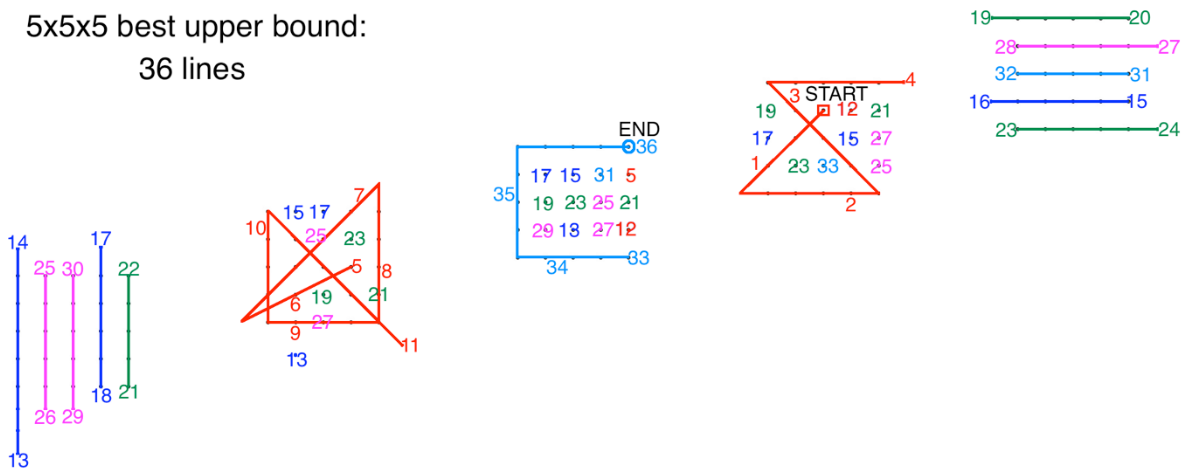

Figure 12. Best known upper bound of the $5 \times 5 \times 5$ puzzle. $36 = h_u = h_l + 5$.

Finally, it is interesting to note that the improved $h_u(n_1, n_2, n_3)$ can lower down the upper bound of the generalized k-dimensional puzzle too. As an example, we can apply the aforementioned 3D patterns to the generalized $n_1 \times n_2 \times \ldots \times n_k$ points problem using the simple method described in [11].

Let $k \geq 4$, given $n_k \leq n_{k-1} \leq \ldots \leq n_4 \leq n_1 \leq n_2 \leq n_3$, we can conclude that

$$h_u(n_1, n_2, n_3, \ldots, n_k) = (h_u(n_1, n_2, n_3) + 1) \cdot \prod_{j=4}^{k} n_j - 1. \qquad (6)$$

3 Conclusion

In the present paper, we have drastically reduced the gap $h_u(n_1, n_2, n_3) - h_l(n_1, n_2, n_3)$ for every previously unsolved puzzle such that $n_3 < 6$.

Moreover, by equation (6), $h(3,3,3) = 13$ naturally provides a covering path with link-length $h_u(3,3,3,3) = 41$ for the $3 \cdot 3 \cdot 3 \cdot 3$ points in $\mathbb{R}^4$.

We do not know if any of the patterns shown in figures 4 to 12 represent optimal solutions, since (by definition) $h_l(n_1, n_2, n_3) \leq h(n_1, n_2, n_3)$. Therefore, some open questions about the NP-complete [2] $n_1 \times n_2 \times n_3$ points problem remain to be answered, and the research in order to cancel the gap $h_u(n_1, n_2, n_3) - h_l(n_1, n_2, n_3)$, at least for every $n_3 \leq 5$, is not over yet.

References

[1] Aggarwal, A., Coppersmith, D., Khanna, S., Motwani, R., Schieber, B. (1999). The angular-metric traveling salesman problem. *SIAM Journal on Computing* **29**, 697–711.

[2] Bereg, S., Bose, P., Dumitrescu, A., Hurtado, F., Valtr, P. (2009). Traversing a set of points with a minimum number of turns. *Discrete & Computational Geometry* **41(4)**, 513–532.

[3] Collins, M. J. (2004). Covering a set of points with a minimum number of turns. *International Journal* of *Computational Geometry* & Applications **14(1-2)**, 105–114.

[4] Collins, M.J., Moret, M.E. (1998). Improved lower bounds for the link length of rectilinear spanning paths in grids. *Information Processing Letters* **68(6)**, 317–319.

[5] Keszegh, B. (2013). Covering Paths and Trees for Planar Grids. *arXiv*, 3 Nov. 2013, `https://arxiv.org/abs/1311.0452`

[6] Kihn, M. (1995). Outside the Box: The Inside Story. *FastCompany*.

[7] Kranakis, E., Krizanc, D., Meertens, L. (1994). Link length of rectilinear Hamiltonian tours in grids. *Ars Combinatoria* **38**, 177–192.

[8] Loyd, S. (1914). Cyclopedia of Puzzles. *The Lamb Publishing Company*, p. 301.

[9] Lung, C. T., Dominowski, R. L. (1985). Effects of strategy instructions and practice on nine-dot problem solving. *Journal of Experimental Psychology: Learning, Memory, and Cognition* **11(4)**, 804–811.

[10] Ripà, M., Bencini, V. (2018). n × n × n Dots Puzzle: An Improved “Outside The Box” Upper Bound. *viXra*, 25 Jul. 2018, `http://vixra.org/pdf/1807.0384v2.pdf`

[11] Ripà, M. (2014). The Rectangular Spiral or the $n_1 \times n_2 \times \ldots \times n_k$ Points Problem. *Notes on Number Theory and Discrete Mathematics* **20(1)**, 59-71.

[12] Ripà, M. (2019). The 3 × 3 × … × 3 Points Problem solution. *Notes on Number Theory and Discrete Mathematics* **25(2)**, 68-75.

[13] Stein, C., Wagner, D.P. (2001). Approximation algorithms for the minimum bends traveling salesman problem. In: Aardal K., Gerards B. (eds) *Integer Programming and Combinatorial Optimization*. IPCO 2001. LNCS, vol 2081, 406–421. Springer, Berlin, Heidelberg.