SOLUTIONS TO ERDŐS-STRAUS CONJECTURE

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ABSTRACT. Erdős-Straus conjecture states that for all integers $n \geq 2$, the rational number 4/n can be expressed as the sum of three positive unit fractions. This conjecture was formulated in 1948, by Paul Erdős and Ernst G.Straus. Here we demonstrate solutions with some ways and this offers evidence that many large numbers can be shown by this method.

1.Introduction

Many years ago, Egyptians searched expansions of rational numbers as unit fractions in mathematics. Egyptian fraction expansions of this type were used as a notation for recording fractional quantities. The Egyptians produced tables as the Rhind Mathematical Papyrus 2/n table of expansion of fraction and it created curiosity for mathematicians for some time. Our interest in this problems initialized with finding some techniques with unit fractions.

It is known that positive rational numbers can be written as sum of unit fractions. The algorithm for these fractions was firstly described by Fibonacci in his book, *Liber Abaci*; he found expansions for 1/n=1/(n+1)+1/n(n+1). It is clearly seen that 4/n can be written as sum of two unit fractions, but when we look at three unit fractions, prime numbers should be observed. Computer searches have verified the truth of the conjecture up to $n \leq 10^{17}$, but proving it for all n remains an open problem.

As an interest, 269 can be written in this form:

$$\frac{4}{269}=\frac{1}{68}+\frac{1}{9146}+\frac{1}{18292}$$

THEOREM 1.1 For all positive integers x, y, z then

$$\frac{4}{xy+z}=\frac{1}{\frac{(x+1)y+(z+1)}{4}}+\frac{1}{(xy+z)(\frac{x+1}{4})}+\frac{1}{\frac{((x+1)y+(z+1))(xy+z)}{2}}$$

PROPOSITION 1.2. *The equation has infinitely many solutions for those primes of the form n≡1 (mod 4) by this method:*

(1) $$\frac{4}{7n+3}=\frac{1}{\frac{8n+4}{4}}+\frac{1}{2(7n+3)}+\frac{1}{\frac{(8n+4)(7n+3)}{2}}$$

(2) $$\frac{4}{15n+7}=\frac{1}{\frac{16n+8}{4}}+\frac{1}{4(15n+7)}+\frac{1}{\frac{(16n+8)(15n+7)}{2}}$$

(3) $$\frac{4}{23n+11}=\frac{1}{\frac{24n+12}{4}}+\frac{1}{6(23n+11)}+\frac{1}{\frac{(24n+12)(23n+11)}{2}}$$

Proof. From Theorem 1.1, x is in the form of (8k+7) and z is in the form of 4k+3.

By this technique, we show that every number in the form of x=8y+7 multiplied by n plus 4y+3 can create these equations.

COROLLARY 1.3*. By using this method, finding solutions becomes easy.*

In each solution of this case we prove many prime numbers which can be demonstrated as expansion of unit fractions. However, it is not only solution for this expansion; there are at least four ways to find prime numbers for solving this equation.

COROLLARY 1.4 All next equations are made changing (x+1) by (x+n), n depends on equation and also (z+1) is changed by (z+k) in theorem above. In denominator, 4 is changed by all positive numbers greater than 2.

PROPOSITION 1.5. *There are infinitely many solutions for this* *equation by this method:*

$$(4)\quad \frac{4}{15n+11}=\frac{1}{\frac{16n+12}{4}}+\frac{1}{4(15n+11)}+\frac{1}{(16n+12)(15n+11)}$$

$$(5)\quad \frac{4}{31n+23}=\frac{1}{\frac{32n+24}{4}}+\frac{1}{8(31n+23)}+\frac{1}{(32n+24)(31n+23)}$$

$$(6)\quad \frac{4}{47n+35}=\frac{1}{\frac{48n+36}{4}}+\frac{1}{12(47n+35)}+\frac{1}{(48n+36)(47n+35)}$$

$$(7)\quad \frac{4}{63n+47}=\frac{1}{\frac{64n+48}{4}}+\frac{1}{16(63n+47)}+\frac{1}{(64n+48)(63n+47)}$$

By using this method, we show that every number in the form of x=16y+15 multiplied by n plus 12y+1 can create these equations.

REMARK 1.6. *It is the second way to find solutions for this equation.*

PROPOSITION 1.7. *There are infinitely many solutions of this form, satisfying this equation:*

$$(8)\quad \frac{4}{15n+13}=\frac{1}{\frac{16n+16}{4}}+\frac{1}{4(15n+13)}+\frac{1}{\frac{(16n+16)(15n+13)}{8}}$$

$$(9)\quad \frac{4}{23n+21}=\frac{1}{\frac{24n+24}{4}}+\frac{1}{6(23n+21)}+\frac{1}{\frac{(24n+24)(23n+21)}{8}}$$

$$(10)\quad \frac{4}{31n+29}=\frac{1}{\frac{32n+32}{4}}+\frac{1}{8(31n+29)}+\frac{1}{\frac{(32n+32)(31n+29)}{8}}$$

$$(11)\quad \frac{4}{39n+37}=\frac{1}{\frac{40n+40}{4}}+\frac{1}{10(39n+37)}+\frac{1}{\frac{(40n+40)(39n+37)}{8}}$$

By using this technique, we demonstrate that every number in the form of x=8y+7 multiplied by n plus 8y+5 can reveal these equations. This method can be key for solving conjecture, because it contains sufficiently large numbers. In the next method, we prove different structure but same logic.

PROPOSITION 1.8. *There are infinitely many solutions of this form:*

(12) $\frac{4}{15n+29} = \frac{1}{\frac{16n+32}{4}} + \frac{1}{4(15n+29)} + \frac{1}{\frac{(16n+32)(15n+29)}{4}}$

(13) $\frac{4}{23n+45} = \frac{1}{\frac{24n+48}{4}} + \frac{1}{6(23n+45)} + \frac{1}{\frac{(24n+48)(23n+45)}{4}}$

(14) $\frac{4}{31n+61} = \frac{1}{\frac{32n+64}{4}} + \frac{1}{8(31n+61)} + \frac{1}{\frac{(32n+64)(31n+61)}{4}}$

By this technique, we show that every number in the form of x=8y+7 multiplied by n plus 16y+3 can create these equations.

This logic reveal many solutions of this form and it consists of many large numbers in the form of n≡1(mod 4).

2. New method for solutions

PROPOSITION 1.9. *By using this new method, we observe that there are infinitely many solutions:*

(15) $\frac{4}{8n+7} = \frac{1}{\frac{6n+6}{3}} + \frac{1}{\frac{(6n+6)(8n+7)}{2}} + \frac{1}{(6n+6)(8n+7)}$

(16) $\frac{4}{12n+11} = \frac{1}{\frac{18n+18}{3}} + \frac{1}{\frac{18n+18}{3}} + \frac{1}{\frac{(9n+9)(12n+11)}{3}}$

(17) $\frac{4}{16n+15} = \frac{1}{\frac{40n+40}{5}} + \frac{1}{\frac{40n+40}{5}} + \frac{1}{\frac{(20n+20)(16n+15)}{5}}$

(18) $\frac{4}{24n+23} = \frac{1}{\frac{84n+84}{7}} + \frac{1}{\frac{84n+84}{7}} + \frac{1}{\frac{(42n+42)(24n+23)}{7}}$

It is just one method to find solutions with many large numbers.

By this technique we illustrate these numbers with their divisors.

Next method is expansion of first.

PROPOSITION 2. *There are infinitely many solutions in this form:*

(19) $\frac{4}{23n+22} = \frac{1}{\frac{18n+18}{3}} + \frac{1}{6(23n+22)} + \frac{1}{(6n+6)(23n+22)}$

(20) $\frac{4}{47n+46} = \frac{1}{\frac{60n+60}{5}} + \frac{1}{12(47n+46)} + \frac{1}{(12n+12)(47n+46)}$

(21) $\frac{4}{46n+45} = \frac{1}{\frac{60n+60}{5}} + \frac{1}{6(46n+45)} + \frac{1}{(12n+12)(46n+45)}$

For example, 60 has five divisors which have prime factor – 5. As a result, 47n+46, 46n+45, 45n+44, 44n+43, 42n+41 can be written as the expansion of unit fractions, because (60*4-5*44) can be divided by 60. It is the last method of our solutions, except one more technique can be used.

3. Last method for finding solutions

Besides these methods above, there is another technique which can be used for many sufficiently large numbers. We propose that in 4/n, if n divides 4, closer even positive integer to this quotient can be written in denominator of first fraction. For instance, we can write 997 by this method:

$$\frac{4}{997} = \frac{1}{250} + \frac{1}{124625} + \frac{1}{249250}$$

If we look at small integers same process will be repeated:

$$\frac{4}{17} = \frac{1}{5} + \frac{1}{34} + \frac{1}{170}$$

It is conclusion of our new search for Erdős-Straus conjecture.

References

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