The nature of Yukawa's nucleon charge

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Abstract

This paper builds on our previous paper and further explores the math and the physics of Yukawa's potential function for the nucleus. It calculates forces and provides a formula for the squared nucleon charge. This is the equivalent of the squared electron charge for the nuclear force. We find it is equal to the product of Euler's number, the fine-structure constant, Planck's constant and the speed of light. The interpretation of this formula is not easy, but it yields sensible results: the calculated forces and the equilibrium between the electromagnetic repulsion and the nuclear attraction make sense.

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Introduction

In our previous paper[1], we mentioned Yukawa's potential as some kind of mandatory exercise to help one think through what might or might not be going on inside the nucleus. However, we got off on a tangent and started thinking about the size and mass of a nucleon. We're still on that tangent, but we will now think through about what one might usefully say about its charge—we might call it the Yukawa charge. What *is* it, really?

Let's remind ourselves of the basics. The Yukawa potential is written as follows[2]:

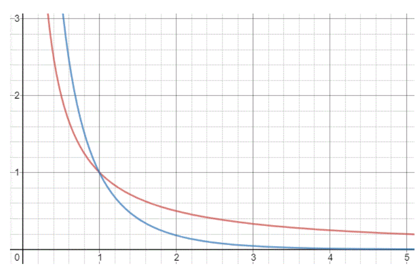
$$\mathrm{U}(r) = -\frac{\mathrm{g_N^2}}{4\pi}\frac{e^{-r/a}}{r}$$

It is just the same as the electrostatic potential V(r), , except for the $e^{-r/a}$ function and the fact that we have the luxury of defining the unit for this new *nucleon* charge g_N. To make sure you see the similarity, we'll remind you of the formula for the electrostatic (Coulomb) potential:

$$\mathrm{V}(r) = -\frac{\mathrm{q_e^2}}{4\pi\varepsilon_0}\frac{1}{r}$$

I found it helpful to play with a graphing tool[3] to get a quick grasp of what might be going on here. We can simply things by forgetting about the 4π factor. This factor is common to both and, in any case, it is just the 4π factor in the formulas for the surface *area* ($4\pi r^2$) and the *volume* ($4\pi r^3$) of a sphere[4]. We may also want to think of the radius of the nucleon as a natural distance unit and, therefore, equate a to 1. So that's what we do in the plot below (Figure 1).

Figure 1: The Yukawa versus the Coulomb potential



[1] *An Oscillator Model for Nuclear Mass*, 15 June 2019 (http://vixra.org/abs/1906.0250).

[2] The Wikipedia article uses a mass factor – and we will come back to that – but we prefer the formula given in Aitchison and Hey's *Gauge Theories in Particle Physics* (2013). It is a widely used textbook in advanced courses and, hence, we will use it as a reference point.

[3] There are a few but I find the free online desmos.com graphing tool very intuitive. The easy parametrization of a function through the addition of a slider, for example, helps to get a quick understanding of the basic properties of some complicated function.

[4] Gauss' Law can be expressed in integral or differential form and these spherical surface area and volume formulas pop up when you go from one to the other. Hence, you shouldn't think of this 4π factor as something weird: it just shows that circles and spheres are more natural shapes to work with in physics.

How can we plot the Yukawa potential if we have no idea whatsoever of what that nucleon charge actually is? You are right. The plot we get in Figure 1 assumes these two functions are equal to unity for $r = a = 1$. It's easy to show that's the case if $g_N^2 = (e/\varepsilon_0)\cdot q_e^2$:

$$\mathrm{U}(1) = \mathrm{V}(1) = 1 \Leftrightarrow -\frac{g_N^2}{4\pi}\frac{e^{-1}}{1} = -\frac{q_e^2}{4\pi\varepsilon_0}\frac{1}{1} \Leftrightarrow g_N^2 = \frac{e}{\varepsilon_0}q_e^2$$

What is this? Some kind of coupling constant showing the relative strength of both forces? Maybe. Maybe not. Our assumption that the two functions are equal to 1 for $r = a = 1$ is quite random. At the same time, the two functions have to cross somewhere if we want that Yukawa potential to serve the purpose it serves, and that is to show the nuclear force is stronger than the Coulomb force *inside* of the nucleus and, vice versa, that the electrostatic force is stronger outside.

This line of reasoning yields a hypothesis which might be smarter. The *Compton* radius is a natural distance unit, right? Hence, the order of magnitude of the *range* parameter *a* in Yukawa's formula is equal to the order of magnitude of the Compton radius of the nucleon, which we write as a_N and which is equal to:

$$a_N = \frac{\hbar}{m_N \cdot c} = \frac{\hbar}{E_N/c} = \frac{(6.582\times 10^{-16}\ \mathrm{eV}\cdot s)\cdot(3\times 10^8\ m/s)}{938\times 10^6\ \mathrm{eV}} \approx 0.21\times 10^{-15}\ \mathrm{m}$$

Let us, for the time being, assume that *a* is equal to a_N. If that would be the case, then we may want to impose the condition that the U(*r*) and V(*r*) potentials should be the same for $r = a = a_N$. This equality implies the following:

$$\mathrm{U}(a_N) = \mathrm{V}(a_N) \Leftrightarrow -\frac{g_N^2}{4\pi}\frac{e^{-a_N/a_N}}{a_N} = -\frac{q_e^2}{4\pi\varepsilon_0}\frac{1}{a_N} \Leftrightarrow g_N^2 = \frac{e}{\varepsilon_0}q_e^2$$

We get the same condition![5] This is quite interesting. Why? Because we can relate ε_0 to the fine-structure constant.

The electric, magnetic and fine-structure constants

As a result of the recent (2019) redefinition of SI units, the electric constant has now been *defined* as:

$$\varepsilon_0 = \frac{1}{\mu_0 c^2} = \frac{q_e^2}{2\alpha hc}$$

You may not have seen this formula before so let me say a few words about it. It comes straight out of the redefinition of SI units which was, effectively, adopted this year only so, yes, it all feels somewhat new. The current theoretical framework for SI units thinks of the electron charge as some given number. You'll say: sure. So what? It means that we accept its definition and, importantly, that *we will measure other things as a function of this and other given numbers*. What other things? The *magnetic* constant μ_0. How do we measure that? By measuring the magnetic moment of an electron. We have a theoretical value for that magnetic moment:

[5] Just to make sure: *e* is Euler's number in this formula. Don't think of the e we use for the electron or – when writing classical equations – the electron charge.

$$\mu_e = \frac{q_e}{2m_e}\hbar$$

It may be good to remind ourselves of where this comes from. In the *Zitterbewegung* model of an electron, we will think of the electron as a circular current – not unlike a current in superconducting material – and the area of this loop of current is defined by the Compton radius of the electron. The current is the charge times the frequency f = E/h and we could, therefore, write the following[6]:

$$\mu = I \cdot \pi a^2 = q_e \cdot f \cdot \pi a^2 = I \cdot \pi a^2 = q_e \frac{mc^2}{h} \cdot \pi a^2 = q_e c \frac{\pi a^2}{2\pi a} = \frac{q_e c}{2}\frac{\hbar}{mc} = \frac{q_e}{2m}\hbar$$

This is the magnetic moment for an electron in free space – a spin-only electron as we called it. For an electron in an electron orbital, we got the following formula:

$$\mu_n = I \cdot \pi {r_n}^2 = \frac{q_e}{2m} n\hbar$$

If n = 1, which is the case for the first atomic orbital or when thinking of an electron in a Penning trap[7], Now, we also know the *experimental* value is slightly off, and the anomaly is related to the fine-structure constant. Of course, theory also explains the difference. To be precise, quantum field theory yields Schwinger's $\alpha/2\pi$ factor, which explains about 99.85% of the anomaly. Schwinger's analysis involves the calculation of a "one loop electron vertex function in an external magnetic field", which is probably at least as complicated as it sounds.[8] We offer an easier geometric explanation[9] based on the interpretation of α as the (relative) radius of the *Zitterbewegung* charge.

The mathematical idea is quite simple: we do think of the *naked charge* q_e as a pointlike but, at the same time, we don't think pointlike necessarily means it has no dimension whatsoever. We think the charge itself as some tiny spherical object – with zero rest mass – and a radius that's equal to the classical electron radius (aka Thomson or Lorentz radius) $r_e = \alpha \cdot a_e \approx a_e/137 \approx 2.818 \times 10^{-15}$ m. We think this is consistent with elastic scattering experiments: low-energy photons do seem to just bounce off some core: there is no interference—as opposed to Compton scattering. We, therefore, think this core might be the pointlike charge which – in itself – has zero rest mass but gives the electron as a whole a moment of inertia because of its rotational motion. We can't dwell on this here – we do so in our other papers[10] - and we shouldn't. The point here is that there is, effectively, some *physical* explanation for the formula that – unlike our formula for ε_0 – you probably did see many times:

$$\alpha = \frac{q_e^2}{4\pi\varepsilon_0 \hbar c} = \frac{q_e^2}{2\varepsilon_0 hc} \Leftrightarrow \varepsilon_0 = \frac{q_e^2}{2\alpha hc}$$

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[6] See: Jean Louis Van Belle, *The Electron as a Harmonic Electromagnetic Oscillator*, 31 May 2019 (http://vixra.org/abs/1905.0521).

[7] Real-life experiments measuring the magnetic moment of an electron use a device which, through a clever combination of the electric and magnetic fields of a cyclotron and a magnetron, is effectively able to capture one electron and keep it in a circular orbit.

[8] The quote is taken from Ivan Todorov's excellent 2018 paper on the history of this thing (https://arxiv.org/abs/1804.09553).

[9] Jean Louis Van Belle, *The Anomalous Magnetic Moment: Classical Calculations*, 6 June 2019 (http://vixra.org/abs/1906.0007).

[10] For a full list of our papers, see: http://vixra.org/author/jean_louis_van_belle.

We can quickly show the various formulas are consistent by calculating the magnetic constant using the formulas above:

$$\mu_0 = \frac{1}{\varepsilon_0 c^2} = \frac{2\alpha hc}{q_e^2 c^2} = \frac{2h}{q_e^2 c} \cdot \frac{q_e^2}{2\varepsilon_0 hc} = \frac{1}{\varepsilon_0 c^2}$$

You may wonder why we inserted this digression: what's the point? We needed this discussion to think about the *physics* in that equation we jotted down:

$$U(a_N) = V(a_N) \Leftrightarrow -\frac{g_N^2}{4\pi} \frac{e^{-a_N/a_N}}{a_N} = -\frac{q_e^2}{4\pi\varepsilon_0} \frac{1}{a_N} \Leftrightarrow g_N^2 = \frac{e}{\varepsilon_0} q_e^2$$

This equation suggests we can calculate the *physical* dimension of Yukawa's nucleon charge. Let us try to think that through.

The nature of the nucleon charge

We started off by saying that the idea of a nucleon charge is something new: we associate some potential with it but we shouldn't think of it as an electrostatic charge. We have no positive or negative charge, for example: all nucleons – positive or negative – share the same charge and should attract each other by the same (strong) force. So, *a priori*, we should just define some new *unit* for it. The *Einstein* unit, perhaps, but I checked: this unit exists already so we need some other term.[11] Jokes apart, we might think of using the equation above to try to derive a unit for the nucleon charge:

$$g_N^2 = \frac{e}{\varepsilon_0} q_e^2 \Leftrightarrow [g_N] = \left[\frac{q_e}{\sqrt{\varepsilon_0}}\right] = \frac{C}{\sqrt{\frac{C^2}{N \cdot m^2}}} = \sqrt{N} \cdot m$$

This can't work, can it? What's that $N^{1/2}\cdot m$ dimension for the nucleon charge? We have no idea, but the logic is sound. Of course, we cut some corners. Yukawa left a constant out of his equation because he had the luxury of defining some new unit: the nucleon charge. However, it is obvious that the Yukawa potential would also need a factor like ε_0 to fix the physical dimensions. We need to think in terms of force units. Why? Because a force is a force: we should *not* be thinking in terms of equating potential but in terms of equating forces. Let us, therefore, start all over again and see what we get when we use this force formula:

$$F = -\frac{dU}{dr} = -\frac{dV}{dr}$$

Let us think about the minus signs here. The forces should be opposite, right? Right, but the formula should take care of that. We should keep our wits with us here, so let us remind ourselves of whatever is that we are trying to do here. We are thinking of two protons here, and these two protons carry an electric charge (q_e) as well as what we vaguely referred to as a nucleon charge (g_N). The electric charge

[11] Believe it or not, but the *Einstein* is defined as a one *mole* (6.022×10^{23}) of photons. It is used, for example, when discussing photosynthesis: we can then define the flux of light – or the flux of photons, to be precise – in terms of *x* micro-einsteins per second per square meter. For more information, see the Wikipedia article on the Einstein as a unit: https://en.wikipedia.org/wiki/Einstein_(unit). If we would truly want to honor Einstein, I would suggest we re-define the Einstein as the unit of charge of the nucleon.

pushes them away from each other, but the nucleon charge pulls them together. At some in-between point, the two forces are equal but opposite. So we should find some value for a force – expressed in *newton*. So it's independent of charge – even if we know it *acts* on a charge. A unit charge, to be precise. So… Well… We have two *different* unit charges here: q_e versus g_N. What does that mean? Let us go through the calculations and see where we get. The Coulomb force is easy to calculate:

$$F_C = -\frac{dV}{dr} = -\frac{d\left(-\frac{q_e^2}{4\pi\varepsilon_0}\frac{1}{r}\right)}{dr} = \frac{q_e^2}{4\pi\varepsilon_0}\frac{d\left(\frac{1}{r}\right)}{dr} = -\frac{q_e^2}{4\pi\varepsilon_0}\frac{1}{r^2}$$

This is just Coulomb's Law, of course! The calculation of the nucleon force – should we say: nucle*ar*? – is somewhat more complicated because of the $e^{-r/a}$ factor[12]:

$$F_N = -\frac{dU}{dr} = -\frac{d\left(-\frac{g_N^2}{4\pi}\frac{e^{-\frac{r}{a}}}{r}\right)}{dr} = \frac{g_N^2}{4\pi\varepsilon_0}\frac{d\left(\frac{e^{-\frac{r}{a}}}{r}\right)}{dr}$$

$$= \frac{g_N^2}{4\pi}\cdot\frac{\frac{d\left(e^{-\frac{r}{a}}\right)}{dr}\cdot r - e^{-\frac{r}{a}}\cdot\frac{dr}{dr}}{r^2} = \frac{g_N^2}{4\pi}\cdot\frac{-\frac{r}{a}\cdot e^{-\frac{r}{a}} - e^{-\frac{r}{a}}}{r^2} = -\frac{g_N^2}{4\pi}\cdot\frac{(\frac{r}{a}+1)\cdot e^{-\frac{r}{a}}}{r^2}$$

The condition for these forces to be equal is:

$$\frac{q_e^2}{4\pi\varepsilon_0}\frac{1}{r^2} = \frac{g_N^2}{4\pi}\cdot\frac{\left(\frac{r}{a}+1\right)\cdot e^{-\frac{r}{a}}}{r^2} \Leftrightarrow \frac{q_e^2}{g_N^2} = \varepsilon_0\cdot\left(\frac{r}{a}+1\right)\cdot e^{-\frac{r}{a}}$$

This condition is not very restrictive. Let us analyze this:

1. We know the $e^{-r/a}$ function already: it decreases from 1 for $r = 0$ to zero as r increases. The range parameter a determines the *shape* of this function. Indeed, an $N_0\cdot e^{-\lambda\cdot t}$ function describes exponential decay, and the $\lambda = 1/a$ parameter gives us the decay rate. It is interesting to note that the *inverse* of the decay rate ($\tau = 1/\lambda$) would give you the mean lifetime, so that's a natural scaling constant. This is compatible with our interpretation of a as some natural distance unit.

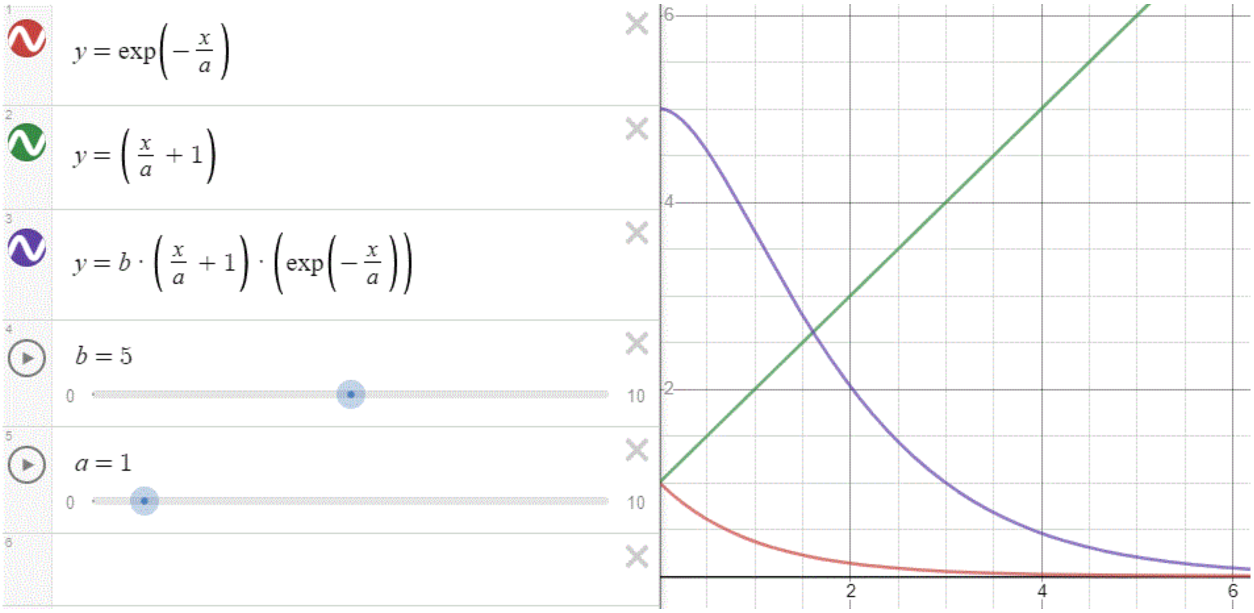
2. The electric constant ε_0 causes the $e^{-r/a}$ to decrease from ε_0 to 0 over the domain (as opposed to decreasing from 1 to 0). Hence, it determines the maximum value for our $\varepsilon_0\cdot(r/a + 1)\cdot e^{-r/a}$ function.

3. Finally, the (r/a + 1) factor is just a linear function which also alters the shape of our function: it makes it look like (half) of a (normal) distribution function but you shouldn't think of our condition as a distribution because a distribution function will have a *squared* exponent. we don't have unction is just a linear

[12] We need to take the derivative of a quotient of two functions here, so you will want to check that rule.

Figure 2 shows how this thing looks like for a = 1 and ε_0 = 5.[13]

Figure 2: The shape of the q_e^2/g_N^2 ratio function



What can we do with this? Plenty of things. We can think of some wild assumption again: didn't we assume the two forces would be equal if *r* was equal to *a*? To be precise, we should say: if *r* is about the same order of magnitude of *a*. But let us just equate the two. If *r* = *a*, then our condition becomes:

$$\frac{q_e^2}{g_N^2} = \varepsilon_0 \cdot \left(\frac{a}{a} + 1\right) \cdot e^{-\frac{a}{a}} = \frac{2}{e} \cdot \varepsilon_0 \approx 3.26 \times 10^{-12} \frac{C^2}{N \cdot m^2}$$

You may think this is a sensible value but we can't say much about it because we have these weird physical dimension: it's the dimension of the electric constant. Let us re-write this thing using that expression for ε_0 in terms of the fine-structure constant: $\varepsilon_0 = q_e^2/2\alpha hc$:

$$\frac{q_e^2}{g_N^2} = \frac{2}{e} \cdot \varepsilon_0 \Leftrightarrow g_N^2 = \frac{e}{2\varepsilon_0} \cdot q_e^2 = \frac{e \cdot 2\alpha hc}{2 \cdot q_e^2} \cdot q_e^2$$

$$\Leftrightarrow g_N^2 = e \cdot \alpha \cdot h \cdot c$$

Interpretation

The $g_N^2 = e\alpha hc$ is is a weird formula: we have the product of two pure numbers (Euler's number and the fine-structure constant) and two physical constants (Planck's constant and the speed of light). In fact, although it has no physical dimension, we should probably think of the fine-structure constant as a physical constant too, so we have one mathematical constant (e) and three physical constants (α, h and

[13] The order of magnitude of a will be 10^{-15} m, while the order of magnitude of ε_0 – when using SI units – is 10^{-12}. Hence, one should not attach any importance to the values we use here. They just serve to illustrate the shape of this function.

c). The *physical* dimension of this product is that of action times velocity, which gives us the N·m^2 dimension:

$$(\text{N·m·s})\cdot(\text{m/s}) = \text{N·m}^2$$

This dimension is consistent with the result we found when doing a dimensional analysis after equating potentials, but we've found the missing ½ factor. Indeed, if g_N^2 is equal to $e\alpha hc$, then the Yukawa and Coulomb potentials at $r = a = 1$ can be calculated as:

$$\mathrm{U}(1) = -\frac{\mathrm{g}_\mathrm{N}^2}{4\pi}e^{-1} = -\frac{e\alpha hc}{4\pi e} = -\frac{\alpha hc}{4\pi}$$

$$\mathrm{V}(1) = -\frac{\mathrm{q}_\mathrm{e}^2}{4\pi\varepsilon_0} = -\frac{\mathrm{q}_\mathrm{e}^2\cdot 2\alpha hc}{4\pi\cdot \mathrm{q}_\mathrm{e}^2} = -\frac{\alpha hc}{2\pi} = 2\cdot\mathrm{U}(1)$$

The Coulomb potential is *twice* the Yukawa potential at the distance where the two forces are equal but opposite.

But let us say a few words about the N·m^2 = J·m dimension. It is weird. We can, of course, re-write it using the mass unit (and Newton's Law): 1 N·m^2 = 1 kg·(s^2/m)·m^2 = 1 kg·m·s^2. However, that doesn't make us much wiser. The joule·second (J·s) is the unit of (physical) action but what is one joule·meter? Energy times a distance? We have not been able to find an explanation.[14]

The last question we need to answer is: what *is* that distance? Let us try to calculate it:

$$\mathrm{U}(r) = -\frac{\mathrm{g}_\mathrm{N}^2}{4\pi}\frac{e^{-\frac{r}{a}}}{r} = -\frac{\alpha hc}{4\pi} \Leftrightarrow \frac{e\alpha hc}{4\pi}\frac{e^{-\frac{r}{a}}}{r} = \frac{\alpha hc}{4\pi} \Leftrightarrow e^{1-\frac{r}{a}} = r$$

This formula only makes sense if $r = a$. However, that's a condition that does not allow us to write a as $a = a_N$. We can only do that when assuming that the *naked* nucleon charge has zero rest mas. In other words, we can only do that if we think our oscillator model – which is nothing but an extension of the *Zitterbewegung* model of our electron – makes sense. If so, then the grand result is what we would like it to be:

$$r = a = a_\mathrm{N} = \frac{\hbar}{\mathrm{m}_\mathrm{N}\cdot c} = \frac{\hbar}{\mathrm{E}_\mathrm{N}/c} = \frac{(6.582\times 10^{-16}\ \mathrm{eV}\cdot s)\cdot(3\times 10^8\ m/s)}{938\times 10^6\ \mathrm{eV}} \approx 0.21\times 10^{-15}\ \mathrm{m}$$

How can we know? We can calculate the forces. For the Coulomb force, we get:

$$\mathrm{F_C} = -\frac{\mathrm{q}_\mathrm{e}^2}{4\pi\varepsilon_0}\frac{1}{r^2} = -\frac{4\pi\mathrm{q}_\mathrm{e}^2\alpha\hbar c}{4\pi\mathrm{q}_\mathrm{e}^2}\frac{\mathrm{m}_\mathrm{N}^2c^2}{\hbar^2} = -\frac{\alpha\mathrm{m}_\mathrm{N}^2c^3}{\hbar} = -\frac{\alpha\cdot\mathrm{m}_\mathrm{N}c\cdot\mathrm{m}_\mathrm{N}c^2}{\hbar} = -\frac{\alpha\mathrm{E}_\mathrm{N}}{a_\mathrm{N}}$$

For the nucleon force, we find the same result, so we're fine:

$$\mathrm{F_N} = -\frac{\mathrm{g}_\mathrm{N}^2}{4\pi}\cdot\frac{\left(\frac{r}{a}+1\right)\cdot e^{-\frac{r}{a}}}{r^2} = -\frac{e\alpha hc}{4\pi}\cdot\frac{\left(\frac{a_\mathrm{N}}{a_\mathrm{N}}+1\right)\cdot e^{-\frac{a_\mathrm{N}}{a_\mathrm{N}}}}{r^2} = \frac{4\pi\alpha\hbar c}{4\pi}\cdot\frac{\mathrm{m}_\mathrm{N}^2c^2}{\hbar^2} = -\frac{\alpha\mathrm{m}_\mathrm{N}^2c^3}{\hbar} = -\frac{\alpha\mathrm{E}_\mathrm{N}}{a_\mathrm{N}}$$

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[14] This site offers an excellent overview of physical units: http://www.ebyte.it/library/educards/sidimensions/SiDimensionsByCategory.html.

Too good to be true? What is the *numerical* value of that force?

$$\mathrm{F_N} = \mathrm{F_C} = \frac{\alpha \mathrm{E_N}}{a_\mathrm{N}} \approx \frac{1.5 \times 10^{-10}\ \mathrm{J}}{137 \cdot 0.21 \times 10^{-15}\ \mathrm{m}} \approx 5{,}212\ \mathrm{N}$$

This force is equivalent to a force that gives a mass of 5.2 metric ton (1 g = 10^{-3} kg) an acceleration of 1 m/s per second. That's huge, but it's quite reasonable as compared to the force inside the nucleon itself, which we calculated to be equal to about 358,000 N.[15] Now that we are here, we can compare the two. We calculated that force using our oscillator model, which yields the F = $(m_\gamma/m)\cdot(E/a)$ = E/2*a* formula:

$$\mathrm{F} = \frac{\mathrm{E_N}}{2a_\mathrm{N}} \approx \frac{1.5 \times 10^{-10}\ \mathrm{J}}{2 \cdot 0.21\ \times 10^{-15}\ \mathrm{m}} \approx 358{,}000\ \mathrm{N}$$

It is easy to see that the two forces differ by a factor that is two times the fine-structure constant (2α). These results are all quite remarkable.

Jean Louis Van Belle, 18 June 2019

[15] See: Jean Louis Van Belle, *An Oscillator Model for Nuclear Mass*, 15 June 2019 (http://vixra.org/abs/1906.0250)