Quick Disproof of the Riemann Hypothesis

Jonathan W. Tooker

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Abstract

In this brief note, we propose a set of operations for the affinely extended real number called infinity. Under the terms of the proposition, we show that the Riemann zeta function has infinitely many non-trivial zeros on the complex plane and off the critical line.

§1 Definitions

Definition 1.1 The number infinity, which like the imaginary number is not a real number, is defined as

$$\lim_{x \to 0^{\pm}} \frac{1}{x} = \pm\infty \quad .$$

Definition 1.2 The real number line is a 1D space extending infinitely far in both directions. It is represented in set and interval notations respectively as

$$\mathbb{R} = \{x \mid -\infty < x < \infty\} \quad , \qquad \text{and} \qquad \mathbb{R} \equiv (-\infty, \infty) \quad .$$

Definition 1.3 A number x is a real number if and only if it is a cut in the real number line

$$(-\infty, \infty) = (-\infty, x) \cup [x, \infty) \quad .$$

Definition 1.4 The affinely extended real numbers are constructed as $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. They are represented in set and interval notations respectively as

$$\overline{\mathbb{R}} = \{x \mid -\infty \leq x \leq \infty\} \quad , \qquad \text{and} \qquad \overline{\mathbb{R}} \equiv [-\infty, \infty] \quad .$$

$\overline{\mathbb{R}}$ is called the affinely extended real number line.

Definition 1.5 A number x is an affinely extended real number $x \in \overline{\mathbb{R}}$ if and only if $x = \pm\infty$ or it is a cut in the affinely extended real number line

$$[\infty, \infty] = [-\infty, x) \cup [x, \infty] \quad .$$

Theorem 1.6 *If $x \in \overline{\mathbb{R}}$ and $x \neq \pm\infty$, then $x \in \mathbb{R}$.*

Proof. Proof follows from Definition 1.4.

Definition 1.7 Infinity has the properties of additive and multiplicative absorption:

$$x \in \mathbb{R} \quad , \quad x > 0 \qquad \Longrightarrow \qquad \begin{cases} \pm x + \infty = \infty \\ \pm x \times \infty = \pm\infty \end{cases} \quad .$$

Proposition 1.8 Suppose the additive absorptive property of $\pm\infty$ is taken away when it appears as $\pm\widehat{\infty}$. Further suppose that $\|\widehat{\infty}\| = \infty$ and that the ordering is such that

$$\begin{gathered} n < \widehat{\infty} - b < \widehat{\infty} - a < \infty \\ -\infty < -\widehat{\infty} + a < -\widehat{\infty} + b < -n \quad , \end{gathered}$$

for any positive $a, b \in \mathbb{R}$, $a < b < n$, and any natural number $n \in \mathbb{N}$.

Theorem 1.9 *$\widehat{\infty}$ is*

$$\pm\widehat{\infty} = \lim_{x \to 0^\pm} \frac{1}{x} \quad .$$

Proof. Proof follows from the $\|\widehat{\infty}\| = \infty$ condition given in Propositon 1.8.

Theorem 1.10 *If $x = \pm(\widehat{\infty} - b)$ and $0 < b < n$ for some $n \in \mathbb{N}$, then $x \in \mathbb{R}$.*

Proof. By the ordering given in Proposition 1.8, we have

$$[\infty, \infty] = [-\infty, x) \cup [x, \infty] \quad .$$

It follows from Definition 1.5 that $x \in \overline{\mathbb{R}}$. Since $\widehat{\infty}$ does not have additive absorption and the theorem states that $b > 0$, it follows from the ordering that

$$x \neq \pm\widehat{\infty} \quad , \qquad \text{and} \qquad x \neq \pm\infty \quad .$$

It follows from Theorem 1.6 that $x \in \mathbb{R}$.

Theorem 1.11 *If a, b are positive numbers less than some natural number $n \in \mathbb{N}$, then*

$$(\widehat{\infty} - a) - (\widehat{\infty} - b) = b - a \quad .$$

Proof. Observe that

$$(\widehat{\infty} - a) - (\widehat{\infty} - b) = \lim_{x \to 0} \left(\frac{1}{x} - a - \frac{1}{x} + b \right) = b - a \quad .$$

Theorem 1.12 *If $a, b \in \mathbb{R}$ are positive numbers less than some natural number $n \in \mathbb{N}$, then the quotient $(\widehat{\infty} - b)/(\widehat{\infty} - a)$ is identically one.*

Proof. Observe that

$$\frac{\widehat{\infty} - b}{\widehat{\infty} - a} = \lim_{x \to 0} \left(\frac{\frac{1}{x} - b}{\frac{1}{x} - a} \right) = \lim_{x \to 0} \left(\frac{\frac{1}{x} - b}{\frac{1}{x} - a} \cdot \frac{x}{x} \right) = \lim_{x \to 0} \frac{1 - bx}{1 - ax} = 1 \quad .$$

Definition 1.13 A number is a complex number $z \in \mathbb{C}$ if and only if

$$z = x + iy \quad , \qquad \text{and} \qquad x, y \in \mathbb{R} \quad .$$

§2 Disproof of the Riemann Hypothesis

Theorem 2.1 *If $b, y_0 \in \mathbb{R}$, if $0 < b < n$ for some $n \in \mathbb{N}$, if $z_0 = (\widehat{\infty} - b) + iy_0$, and if $\zeta(z)$ is the Riemann ζ function, then $\zeta(z_0) = 1$.*

Proof. Observe that the Dirichlet sum form of ζ [1] takes z_0 as

$$\begin{aligned} \zeta(z_0) &= \sum_{n=1} \frac{1}{n^{(\widehat{\infty} - b) + iy_0}} \\ &= \sum_{n=1} \frac{n^b}{n^{\widehat{\infty}}} \Big(\cos(y_0 \ln n) - i \sin(y_0 \ln n) \Big) \\ &= 1 + \sum_{n=2} 0 \Big(\cos(y_0 \ln n) - i \sin(y_0 \ln n) \Big) = 1 \quad . \end{aligned}$$

Theorem 2.2 *The Riemann ζ function has non-trivial zeros at certain $z \in \mathbb{C}$ outside of the critical strip.*

Proof. Riemann's functional form of ζ [1] is

$$\zeta(z) = \frac{(2\pi)^z}{\pi} \sin\left(\frac{\pi z}{2} \right) \Gamma(1 - z) \zeta(1 - z) \quad .$$

Theorem 2.1 gives $\zeta(\widehat{\infty} - b)$ when we set $y_0 = 0$ so we will use Riemann's equation to prove this theorem by solving for $z = -(\widehat{\infty} - b) + 1$. (This value for z follows from $1 - z = \widehat{\infty} - b$.) We have

$$\begin{aligned} \zeta\big[-(\widehat{\infty} - b) + 1 \big] &= \lim_{z \to -(\widehat{\infty} - b) + 1} \left(\frac{(2\pi)^z}{\pi} \sin\left(\frac{\pi z}{2} \right) \right) \lim_{z \to (\widehat{\infty} - b)} \Big(\Gamma(z) \zeta(z) \Big) \\ &= \lim_{z \to -(\widehat{\infty} - b) + 1} \Big(2 \sin\left(\pi z / 2 \right) \Big) \lim_{z \to (\widehat{\infty} - b)} \Big((2\pi)^{-z} \Gamma(z) \zeta(z) \Big) \quad . \end{aligned}$$

For the limit involving Γ, we will compute the limit as a product of two limits. We separate terms as

$$\lim_{z\to(\widehat{\infty}-b)}\Big((2\pi)^{-z}\Gamma(z)\zeta(z)\Big)=\lim_{z\to(\widehat{\infty}-b)}\Big((2\pi)^{-z}\Gamma(z)\Big)\lim_{z\to(\widehat{\infty}-b)}\zeta(z) \quad .$$

From Theorem 2.1, we know the limit involving ζ is equal to one. For the remaining limit, we will insert the identity and again compute it as the product of two limits. If z approaches $(\widehat{\infty}-b)$ along the real axis, it follows from Theorem 1.12 that

$$1=\frac{z-(\widehat{\infty}-b)}{z-(\widehat{\infty}-b)} \quad .$$

Inserting the identity yields

$$\lim_{z\to(\widehat{\infty}-b)}\Big((2\pi)^{-z}\Gamma(z)\Big)=\lim_{z\to(\widehat{\infty}-b)}\left((2\pi)^{-z}\Gamma(z)\frac{z-(\widehat{\infty}-b)}{z-(\widehat{\infty}-b)}\right) \quad .$$

Let

$$A=\Gamma(z)\Big(z-(\widehat{\infty}-b)\Big) \quad , \qquad \text{and} \qquad B=\frac{(2\pi)^{-z}}{z-(\widehat{\infty}-b)} \quad .$$

To get the limit of A into workable form we will use the property $\Gamma(z)=z^{-1}\Gamma(z+1)$ to derive an expression for $\Gamma[z-(\widehat{\infty}-b)+1]$. If we can write $\Gamma(z)$ in terms of $\Gamma[z-(\widehat{\infty}-b)+1]$, then the limit as z approaches $(\widehat{\infty}-b)$ will be very easy to compute. Observe that

$$\Gamma\big[z-(\widehat{\infty}-b)+1\big]=\Gamma\big[z-(\widehat{\infty}-b)+2\big]\Big(z-(\widehat{\infty}-b)+1\Big)^{-1} \quad .$$

By recursion we obtain

$$\Gamma\big[z-(\widehat{\infty}-b)+1\big]=\Gamma(z)\lim_{n\to(\widehat{\infty}-b)}\prod_{k=1}^{n}\Big(z-(\widehat{\infty}-b)+k\Big)^{-1} \quad .$$

Rearrangement yields

$$\Gamma(z)=\Gamma\big[z-(\widehat{\infty}-b)+1\big]\lim_{n\to(\widehat{\infty}-b)}\prod_{k=1}^{n}\Big(z-(\widehat{\infty}-b)+k\Big) \quad .$$

It follows that

$$A=\Gamma\big[z-(\widehat{\infty}-b)+1\big]\lim_{n\to(\widehat{\infty}-b)}\prod_{k=0}^{n}\Big(z-(\widehat{\infty}-b)+k\Big) \quad .$$

The limit of A is

$$\lim_{z\to(\widehat{\infty}-b)}A=\Gamma\big[(\widehat{\infty}-b)-(\widehat{\infty}-b)+1\big]\lim_{n\to(\widehat{\infty}-b)}\prod_{k=0}^{n}\Big((\widehat{\infty}-b)-(\widehat{\infty}-b)+k\Big) \,.$$

Theorem 1.11 gives $(\widehat{\infty}-b)-(\widehat{\infty}-b)=0$ so

$$\lim_{z\to(\widehat{\infty}-b)} A=\Gamma(1)\lim_{n\to(\widehat{\infty}-b)}\prod_{k=0}^{n}k=0 \quad .$$

Direct evaluation of the limit of B gives $0/0$ so we need to use L'Hôpital's rule which gives

$$\begin{aligned}\lim_{z\to(\widehat{\infty}-b)} B &\overset{*}{=} \lim_{z\to(\widehat{\infty}-b)}\left(\frac{\dfrac{d}{dz}(2\pi)^{-z}}{\dfrac{d}{dz}\Big(z-(\widehat{\infty}-b)\Big)}\right)\\ &= \lim_{z\to(\widehat{\infty}-b)}\frac{d}{dz}e^{-z\ln(2\pi)}\\ &= -\ln(2\pi)\ e^{-(\widehat{\infty}-b)\ln(2\pi)}=\frac{-1}{e^{\widehat{\infty}}}\ln(2\pi)\ e^{b\ln(2\pi)}=0\end{aligned}$$

Therefore, we find that the limit of AB is 0. It follows that

$$\zeta\big[-(\widehat{\infty}-b)+1\big]=\lim_{z\to-(\widehat{\infty}-b)+1}2\sin\left(\frac{\pi z}{2}\right)\times 0=0 \quad .$$

Definition 2.3 The Riemann hypothesis as defined by the Clay Mathematics Institute [2] is

> **The non-trivial zeros of the Riemann ζ function have real parts equal to one half.**

Definition 2.4 According to the Clay Mathematics Institute [2], the trivial zeros of ζ are the even negative integers.

Remark 2.5 The zeros demonstrated in Theorem 2.2 are neither on the critical line $\text{Re}(z)=1/2$ nor are they the negative even integers. Theorem 2.2, therefore, is the negation of the Riemann hypothesis.

References

[1] Bernhard Riemann. On the Number of Primes Less than a Given Quantity. *Monatsberichte der Berliner Akademie*, (1859).

[2] Enrico Bombieri. Problems of the Millennium : The Riemann Hypothesis. 2000.