Exact Periodic Solutions of Truly Nonlinear Oscillator Equations and Quadratic Liénard-Type Equations

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The present research contribution is devoted to solving the integrability problem of Liénard type differential equations. It is shown that such a problem may be solved by nonlocal transformation for some classes of equations. By doing so, it is observed that the integrability of a class of restricted Duffing type equations with integral power or fractional power nonlinearity may be secured by that of a general class of quadratic Liénard type differential equation, and vice versa. Such a restricted Duffing type equation is also shown to be closely related to a quadratic Liénard type equation for which exact and explicit general solution may be computed. In this context it has been shown that exact and general periodic solutions may be computed for these two classes of restricted Duffing equations and quadratic Liénard type equations. The comparison of obtained solutions with some well-known results is carried out in some cases.

1 Introduction

The periodic solutions constitute an important class of solutions of nonlinear differential equations since many phenomena in physics and engineering applications are nonlinear and periodic. Such solutions are often computed by means of approximate methods as one can see in the literature due to the difficulty to solve exactly the nonlinear differential equations. Therefore the problem of finding exact and general periodic solutions to nonlinear differential equations is very less investigated in the literature. In this situation the integrability of nonlinear differential equations in terms of exact and general periodic solutions remains an interesting mathematical problem to be solved. For instance, analytical properties of the Liénard type differential equations

$$\ddot{x} + g(x)\dot{x}^2 + h(x) = 0 \tag{1.1}$$

and

$$\ddot{u} + f(u) = 0, \tag{1.2}$$

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which include some celebrated nonlinear differential equations like the Duffing and Bratu equations, and the Mathew-Lakshamanan equation, are not well understood exactly [1–3]. This is a shortcoming in the theory of nonlinear differential equations as no one can answer the following question: Can we secure the integrability of (1.1) from that of (1.2) in terms of exact and general periodic solutions, vice versa?

The current work assumes such a prediction. To demonstrate, it is first shown that equation (1.1) is mathematically equivalent to equation (1.2) for a certain definition of functions $f(x)$, $g(x)$, and $h(x)$ (section 2). Secondly it is shown that one may compute exact and general periodic solutions to equation (1.2) using a variable transformations (section 3). The work ends with illustrative examples (section 4) and a general conclusion.

2 Mathematical equivalence

With $g(x) = \frac{b}{x}$, and $h(x) = ax^s$, equations (1.1) becomes

$$\ddot{x} + b\frac{\dot{x}^2}{x} + ax^s = 0 \tag{2.1}$$

For $b = -\gamma$, $a = \omega^2$, and $s = 2\gamma + 1$, (2.1) takes the form [4]

$$\ddot{x} + -\gamma\frac{\dot{x}^2}{x} + \omega^2 x^{2\gamma+1} = 0 \tag{2.2}$$

On the other hand for $f(u) = pu^q$, (1.2) becomes

$$\ddot{u} + pu^q = 0 \tag{2.3}$$

The problem is now to show that equation (2.2) may be mapped under appropriate variable change to (2.3). In this way the following theorem may be considered

Theorem 1

Consider (2.2). Then by the application of variable change

$$u = x^{1-\gamma} \tag{2.4}$$

(2.2) reduces to (2.3) where

$$p = (1-\gamma)\omega^2, \quad q = \frac{1+\gamma}{1-\gamma} \tag{2.5}$$

Proof

From $u = x^{1-\gamma}$, one may compute

$$x = u^{\frac{1}{1-\gamma}}, \tag{2.6}$$

$$\dot{x} = \frac{1}{1-\gamma}\dot{u}u^{\frac{\gamma}{1-\gamma}} \tag{2.7}$$

and

$$\ddot{x} = \frac{1}{1-\gamma}\ddot{u}u^{\frac{\gamma}{1-\gamma}} + \frac{\gamma}{(1-\gamma)^2}\dot{u}^2 u^{\frac{2\gamma-1}{1-\gamma}} \tag{2.8}$$

Substituting equations (2.6), (2.7) and (2.8) into equation (2.2) leads immediately to [4]

$$\ddot{u} + (1-\gamma)\omega^2 u^{\frac{1+\gamma}{1-\gamma}} = 0 \tag{2.9}$$

which is nothing but equation (2.3) when (2.5) is taken into account. Thus the theorem 1 is proved. In such a situation equation (2.9) may be viewed as a Duffing type equation and the problem is then to prove that equation (2.9) may be solved in terms of exact and general periodic solutions.

3 Exact and general periodic solutions

The purpose of this section is to show that exact and general periodic solutions may be established for the Duffing type equation (2.9). To that end it is suitable to consider the nonlocal transformation

$$y(\tau) = u(t) \;\;,\;\; d\tau = u^{\frac{\gamma}{1-\gamma}}dt \tag{3.1}$$

Therefore the following theorem may be formulated.

Theorem 2

Consider the nonlocal transformation (3.1). Then by application of (3.1), equation (2.9) may be mapped into

$$y'' + \left(\frac{1}{1-\gamma} - 1\right)\frac{y'^2}{y} + (1-\gamma)\omega^2 y = 0 \tag{3.2}$$

where prime denotes the differentiation with respect to argument.

Proof

From (3.1) one may obtain

$$\frac{du}{dt} = y'(\tau)y^{\frac{\gamma}{1-\gamma}} \tag{3.3}$$

so that the second derivative $\frac{d^2u}{dt^2}$ may take the form

$$\frac{d^2u}{dt^2} = y''(\tau)y(\tau)^{\frac{2\gamma}{1-\gamma}} + \frac{\gamma}{1-\gamma}y'^2(\tau)y(\tau)^{\frac{3\gamma-1}{1-\gamma}} \tag{3.4}$$

Also from equation (3.1), one may notice that

$$u^{\frac{1+\gamma}{1-\gamma}} = y^{\frac{1+\gamma}{1-\gamma}} \tag{3.5}$$

Then the substitution of equations (3.4) and (3.5) into (2.9) yields precisely equation (3.2). Therefore theorem 2 is demonstrated. According to [4] the exact and explicit general solution to (3.2) may be written as

$$y(\tau) = [A_0 \sin(\omega\tau + \alpha]^{1-\gamma} \tag{3.6}$$

Therefore, using (3.5) one may obtain the exact and general solution to (2.9) in the form

$$u(t) = (A_0 \sin \phi(t))^{1-\gamma} \tag{3.7}$$

where

$$\phi(t) = \omega\tau + \alpha \tag{3.8}$$

is given by quadrature

$$\int \frac{d\phi}{sin^{\gamma}(\phi)} = \omega A_0^{\gamma}(t + C) \tag{3.9}$$

The parameter C is a constant of integration. In this context the exact and general solution to (2.2) may take the form

$$x(t) = A_0 \sin \phi(t) \tag{3.10}$$

So with that some examples may be given to illustrate the usefulness of the developed theory.

4 Examples

In this section the exact and general periodic solutions to some generalized equations of well known equations and to other equations are computed as illustration. Exact solutions under specific initial conditions as well as approximate solutions are deduced in view of comparison.

4.1 Generalized inverse cube-root oscillator equation

For $\gamma = -2$, equation (2.9) reduces to the generalized truly nonlinear oscillator equation

$$\ddot{u}(t) + 3\omega^2 u^{-1/3}(t) = 0 \tag{4.1}$$

The equation (4.1) is investigated in [6] for the specific value $3\omega^2 = 1$. Using equation (3.7), the exact and general periodic solution to equation (4.1) may take the form

$$u(t) = A_0^3 \sin^3 \phi(t) \tag{4.2}$$

where the function ϕ satisfies

$$\frac{1}{2}\phi - \frac{1}{4}\sin 2\phi = \omega A_0^{-2}(t + C)$$

that is

$$\phi - \frac{1}{2}\sin 2\phi = 2\omega A_0^{-2}(t + C) \tag{4.3}$$

Now, an approximate closed form solution to (4.1) may be deduced by neglecting the term $\frac{1}{2}\sin 2\phi$ in equation (4.3) so that one may obtain

$$\phi(t) = 2\omega A_0^{-2}(t + C) \tag{4.4}$$

In this context the approximate closed-form solution takes the form

$$u(t) = A_0^3 \sin^3[2\omega A_0^{-2}(t + C)] \tag{4.5}$$

The solution (4.5) can be applied to a variety of initial conditions. Under the initial conditions [6]

$$u(t = 0) = u_0 \ , \ \ \dot{u}(t = 0) = 0 \tag{4.6}$$

the approximate solution (4.5) becomes

$$u(t) = \frac{3}{4}u_0 \cos\left(\frac{2\omega}{u_0^{\frac{2}{3}}}t\right) + \frac{1}{4}u_0 \cos\left(3\left(\frac{2\omega}{u_0^{\frac{2}{3}}}\right)t\right) \tag{4.7}$$

The angular frequency

$$\Omega = \frac{2\omega}{u_0^{\frac{2}{3}}} \tag{4.8}$$

for $3\omega^2 = 1$, that is $\omega = \frac{1}{\sqrt{3}}$, becomes

$$\Omega = \frac{2}{\sqrt{3}}\frac{1}{u_0^{2/3}} \tag{4.9}$$

which is equal to the exact angular frequency given in [6].

Figure 1 shows the comparison of the exact and general periodic solution (4.2) with the approximate analytical solution (4.7) for $u_0 = 1$, and $\omega = 0.45$.

On the other hand, for $\gamma = -2$, equation (2.2) becomes

$$\ddot{x} + 2\frac{\dot{x}^2}{x} + \omega^2 x^{-3} = 0 \tag{4.10}$$

The exact and general periodic solution of (4.10) is secured by that of the truly nonlinear equation (4.1) so that one may obtain

$$x(t) = A_0 sin\phi(t) \tag{4.11}$$

where (4.3) holds. Using the approximate solution (4.5) and the initial conditions $x(t = 0) = x_0$ and $\dot{x}(t = 0) = 0$, the approximate closed-form solution to equation (4.10) may be written as

$$x(t) = x_0 \cos\left(\frac{2\omega}{x_0^2}t\right) \tag{4.12}$$

Figure 2 shows the comparison of the exact and general periodic solution (4.11) with the approximate solution (4.12) for $x_0 = 1$ and $\omega = 0.45$.

4.2 Generalized quadratic oscillator equation

According to [6] the quadratic truly nonlinear oscillator equation reads

$$\ddot{u} + |u|u = 0 \tag{4.13}$$

which is equivalent to the set of equations

$$\ddot{u} + u^2 = 0 \tag{4.14}$$

and

$$\ddot{u} - u^2 = 0 \tag{4.15}$$

when $u > 0$, and $u < 0$, respectively. Substituting $\gamma = \frac{1}{3}$, into (2.9) yields the generalized quadratic oscillator equation

$$\ddot{u}(t) + \frac{2}{3}\omega^2 u^2(t) = 0 \tag{4.16}$$

which includes equations (4.14) and (4.15) as special cases when $\frac{2}{3}\omega^2 = 1$, and $\frac{2}{3}\omega^2 = -1$, respectively. Now the problem is to compute the exact and general periodic solution to equation (4.16). In this way the exact and general solution to (4.16) may, using (3.7), become

$$u(t) = A_0^{\frac{2}{3}} \sin^{\frac{2}{3}} \phi(t) \tag{4.17}$$

where $\phi(t)$ is given by quadrature

$$\int \frac{d\phi}{sin^{\frac{1}{3}}(\phi)} = \omega A_0^{\frac{1}{3}} t + C_1 \tag{4.18}$$

C_1 is a constant of integration. By a suitable change of variable the indefinite integral

$$\int \frac{d\phi}{sin^{\frac{1}{3}}(\phi)} \tag{4.19}$$

reduces to

$$\frac{3}{2}\varepsilon \int \frac{dv}{\sqrt{1-v^3}} \tag{4.20}$$

where

$$v = sin^{\frac{2}{3}}\phi \tag{4.21}$$

and $\varepsilon = \pm 1$.

Using an appropriate choice of initial conditions

$$u(t=0) = u_0 = A_0^{2/3} \;\; , \;\; \dot{u}(t=0) = 0 \tag{4.22}$$

one may compute t as

$$\omega A^{1/3} t = \frac{3}{2}\varepsilon \int_v^1 \frac{dv}{\sqrt{1-v^3}} \tag{4.23}$$

that is [7]

$$\omega A_0^{1/3} t = \frac{3}{2}\varepsilon \left[\frac{1}{3^{1/4}} F(\beta, m) \right] \tag{4.24}$$

where $F(\beta, m)$ is the elliptic integral of first kind, $m = k^2 = \frac{2+\sqrt{3}}{4}$, and $\beta = arccos\frac{\sqrt{3}-1+v}{\sqrt{3}+1-v}$. From (4.24) one may find

$$cos\beta = \frac{\sqrt{3}-1+v}{\sqrt{3}+1-v} = cn\left(\frac{2}{3}(3)^{1/4}\omega A_0^{1/3}t, m\right) \tag{4.25}$$

from which

$$v = \frac{1 + cn(\frac{2}{3}(3)^{1/4}\omega A_0^{1/3}t, m) - \sqrt{3}(1 - cn(\frac{2}{3}(3)^{1/4}\omega A_0^{1/3}t, m))}{1 + cn(\frac{2}{3}(3)^{1/4}\omega A_0^{1/3}t, m)} \tag{4.26}$$

Knowing v, one may compute $sin\phi$ using the relation (4.21) so that the solution (4.17) takes the form

$$u(t) = A_0^{2/3} - A_0^{2/3}\sqrt{3}\frac{1 - cn(\frac{2}{3}(3)^{1/4}\omega A_0^{1/3}t, m)}{1 + cn(\frac{2}{3}(3)^{1/4}\omega A_0^{1/3}t, m)} \tag{4.27}$$

or

$$u(t) = u_0\left[\frac{(1-\sqrt{3}) + (1+\sqrt{3})cn(\frac{2}{3}(3)^{1/4}\omega u_0^{1/2}t, m)}{1 + cn(\frac{2}{3}(3)^{1/4}\omega u_0^{1/2}t, m)}\right] \tag{4.28}$$

In such a situation the oscillation period T may be computed as

$$\omega A_0^{1/3}\frac{T}{4} = \frac{3}{2}\int_0^1 \frac{dv}{\sqrt{1-v^3}} \tag{4.29}$$

so that one may obtain

$$T = \frac{3(\Gamma(\frac{1}{3}))^3}{2^{1/3}\pi\sqrt{3}\omega A_0^{1/3}} \tag{4.30}$$

that is

$$T = \frac{3(\Gamma(\frac{1}{3}))^3}{\pi\sqrt{3u_0}2^{1/3}\omega} \tag{4.31}$$

The period (4.31) is the exact period found by [6] when $\frac{2}{3}\omega^2 = 1$. Under the initial condition

$$u(t=0) = 0 \tag{4.32}$$

the time t may read

$$\omega A_0^{1/3}t = \frac{3}{2}\varepsilon\int_0^v \frac{dv}{\sqrt{1-v^3}} \tag{4.33}$$

The evaluation of (4.33) leads to obtain

$$\omega A_0^{1/3}t = \varepsilon\left\{\omega A_0^{1/3}\frac{T}{4} - \frac{3}{2}\frac{1}{3^{1/4}}F(\beta, m)\right\} \tag{4.34}$$

so that

$cos\beta = \frac{\sqrt{3}-1+v}{\sqrt{3}+1-v} = cn[\frac{2}{3}(3)^{1/4}\omega A_0^{1/3}(t - \frac{T}{4}), m]$ which may give

$$v = 1 - \sqrt{3}\frac{1 - cn[\frac{2}{3}(3)^{1/4}\omega A_0^{1/3}(t - \frac{T}{4}), m]}{1 + cn[\frac{2}{3}(3)^{1/4}\omega A_0^{1/3}(t - \frac{T}{4}), m]} \tag{4.35}$$

Using (4.21), one may find

$$u(t) = A_0^{2/3} - A_0^{2/3}\sqrt{3}\frac{1 - cn[\frac{2}{3}(3)^{1/4}\omega A_0^{1/3}(t - \frac{T}{4}), m]}{1 + cn[\frac{2}{3}(3)^{1/4}\omega A_0^{1/3}(t - \frac{T}{4}), m]} \tag{4.36}$$

or

$$u(t) = A_0^{2/3}\left[\frac{(1-\sqrt{3}) + (1+\sqrt{3})cn(\frac{2}{3}(3)^{1/4}\omega A_0^{1/3}(t - \frac{T}{4}), m)}{1 + cn(\frac{2}{3}(3)^{1/4}\omega A_0^{1/3}(t - \frac{T}{4}), m)}\right] \tag{4.37}$$

On the other hand, for $\gamma = \frac{1}{3}$, equation (2.5) leads to the quadratic Liénard type equation

$$\ddot{x} - \frac{1}{3}\frac{\dot{x}^2}{x} + \omega^2 x^{5/3} = 0 \tag{4.38}$$

for which the exact and general solution, according to (3.9) and (4.28) becomes

$$x(t) = u_0\left[1 - \sqrt{3}\frac{1 - cn(\frac{2}{3}(3)^{1/4}\omega A_0^{1/3}t, m)}{1 + cn(\frac{2}{3}(3)^{1/4}\omega A_0^{1/3}t, m)}\right]^{3/2} \tag{4.39}$$

which may read

$$x(t) = x_0^{2/3}\left[\frac{(1-\sqrt{3}) + (1+\sqrt{3})cn(\frac{2}{3}(3)^{1/4}\omega x_0^{3/4}t, m)}{1 + cn(\frac{2}{3}(3)^{1/4}\omega x_0^{3/4}t, m)}\right]^{3/2} \tag{4.40}$$

under the initial conditions (4.22) that is

$$x(t=0) = x_0 = u_0^{3/2} \ \ , \ \ \dot{x}(t=0) = 0 \tag{4.41}$$

According to (3.10) and (4.37), one may write also the exact solution of (4.38) as

$$x(t) = x_0^{2/3}\left[\frac{(1-\sqrt{3}) + (1+\sqrt{3})cn(\frac{2}{3}(3)^{1/4}\omega x_0^{3/4}(t - \frac{T}{4}), m)}{1 + cn(\frac{2}{3}(3)^{1/4}\omega x_0^{3/4}(t - \frac{T}{4}), m)}\right]^{3/2} \tag{4.42}$$

4.3 Restricted quintic Duffing equation

Letting $\gamma = \frac{2}{3}$, into (2.9), yields the restricted quintic Duffing oscillateur equation

$$\ddot{u}(t) + \frac{\omega^2}{3}u^5(t) = 0 \tag{4.43}$$

The exact and general solution to (4.43) may, according to the above, read

$$u(t) = (A_0 sin\phi(t))^{1/3} \tag{4.44}$$

where $\phi(t)$ is given by

$$\int \frac{d\phi}{sin^{\frac{2}{3}}(\phi)} = \omega A_0^{\frac{2}{3}}t + C_2 \tag{4.45}$$

C_2 is an integration constant. The change of variable

$$v^{3/2} = sin\phi \tag{4.46}$$

leads to obtain

$$I = \int \frac{d\phi}{sin^{\frac{2}{3}}(\phi)} = \frac{3}{2}\varepsilon \int \frac{dv}{\sqrt{v(1-v^3)}} \tag{4.47}$$

For an appropriate choice of the initial condition

$$u(t=0) = 0 \tag{4.48}$$

equation (4.45) may be rearranged, using (4.47), in the form

$$\omega A_0^{2/3} t = \frac{3}{2}\varepsilon \int_0^v \frac{dv}{\sqrt{v(1-v^3)}} \tag{4.49}$$

which leads to obtain

$$\omega A_0^{1/3} t = \frac{3}{2}\varepsilon \left(\frac{1}{(3)^{1/4}} F(\beta, m) \right) \tag{4.50}$$

where $m = k^2 = \frac{2-\sqrt{3}}{4}$, and

$$cos\beta = \frac{1-(\sqrt{3}+1)v}{1+(\sqrt{3}-1)v} = cn\left(\frac{2}{3}(3)^{1/4}(A_0)^{2/3}\omega t, m\right) \tag{4.51}$$

so that one may find

$$v = \frac{1 - cn\left(\frac{2}{3}(3)^{1/4}(A_0)^{2/3}\omega t, m\right)}{\sqrt{3}+1+(\sqrt{3}-1)cn\left(\frac{2}{3}(3)^{1/4}(A_0)^{2/3}\omega t, m\right)} \tag{4.52}$$

In this context, the solution (4.44) becomes

$$u(t) = A_0^{1/3}\sqrt{\frac{1 - cn\left(\frac{2}{3}(3)^{1/4}(A_0)^{2/3}\omega t, m\right)}{\sqrt{3}+1+(\sqrt{3}-1)cn\left(\frac{2}{3}(3)^{1/4}(A_0)^{2/3}\omega t, m\right)}} \tag{4.53}$$

In such a situation the period T may be found as

$$\omega A_0^{2/3}\frac{T}{4} = \frac{1}{2}\int_0^1 r^{1/6-1}(1-r)^{1/2-1}dr \tag{4.54}$$

that is *[7]*

$$T = \frac{2\Gamma(1/6)\Gamma(1/2)}{\omega A_0^{2/3}\Gamma(2/3)} \tag{4.55}$$

with the change of variable

$$r^{1/2} = sin\phi \tag{4.56}$$

Using $u_0 = A_0^{1/3}$, the period T may take the form

$$T = \frac{2\Gamma(1/6)\Gamma(1/2)}{\omega u_0^2\Gamma(2/3)} \tag{4.57}$$

For $\frac{\omega^2}{3} = 1$, the period (4.57) reduces to the exact period found in [8]. However in [8] the exact closed-form periodic solution to the quintic truly nonlinear oscillator equation (4.43) is not given when $\frac{\omega^2}{3} = 1$. Figure 3 shows the periodic behavior of the solution (4.53) for $A_0 = 2$, $\omega = 0.5$. Knowing the solution (4.53) one may easily compute the solution to the quadratic Liénard type equation

$$\ddot{x} - \frac{2}{3}\frac{\dot{x}^2}{x} + \omega^2 x^{7/3} = 0 \tag{4.58}$$

obtained from (2.2) for $\gamma = \frac{2}{3}$. Thus using (4.53) the solution of (4.58) may be expressed in the form

$$x(t) = A_0 \left[\sqrt{\frac{1 - cn\left(\frac{2}{3}(3)^{1/4}(A_0)^{2/3}\omega t, m\right)}{\sqrt{3} + 1 + (\sqrt{3} - 1)cn\left(\frac{2}{3}(3)^{1/4}(A_0)^{2/3}\omega t, m\right)}} \right]^3 \tag{4.59}$$

Figure 4 shows the periodic behavior of (4.59) obtained for $A = 2$, $\omega = 0.5$

4.4 Generalized restricted cubic Duffing equation

The generalized restricted cubic Duffing equation

$$\ddot{u} + \frac{\omega^2}{2} u^3 = 0 \tag{4.60}$$

is obtained from (2.9) using $\gamma = 1/2$. For $\frac{\omega^2}{2} = 1$, equation (4.60) becomes the restricted cubic Duffing equation investigated in [6]. Using (3.7) the solution of (4.60) may take immediately the form

$$u(t) = A_0^{1/2} \sin^{1/2}\phi(t) \tag{4.61}$$

where

$$\omega A_0^{1/2}(t + C_3) = \int \frac{d\phi}{\sin^{\frac{1}{2}}\phi} \tag{4.62}$$

Therefore, (4.62) may reduce to [7]

$$\omega A_0^{1/2}(t + C_3) = -\sqrt{2}F(\beta, m) \tag{4.63}$$

where $m = k^2 = 1/2$, and $\beta = \sqrt{1 - sin\phi} = -sn\left(\frac{\omega A_0^{1/2}}{\sqrt{2}}(t + C_3), m\right)$. In this way, one may find $sin\phi = 1 - sn^2\left(\frac{\omega A_0^{1/2}}{\sqrt{2}}(t + C_3), m\right)$ so that the exact and explicit general solution (4.61) becomes

$$u(t) = A_0^{1/2} cn\left(\omega\sqrt{\frac{A_0}{2}}(t + C_3), 1/2\right) \tag{4.64}$$

Under the initial conditions [6]

$$u(t = 0) = u_0 = A_0^{1/2} \;\; , \;\; \dot{u}(t = 0) = 0 \tag{4.65}$$

one may find $C_3 = 0$, such that (4.64) may read

$$u(t) = u_0 cn\left(\omega \frac{u_0}{\sqrt{2}} t, 1/2\right) \tag{4.66}$$

It is interesting to notice that the solution (4.66) becomes the solution obtained by [6] when $\frac{\omega^2}{2} = 1$. Figure 5 shows the periodic behavior of the solution (4.66) for $\omega = 1$, $u_0 = 0.5$ On the other hand, the exact and explicit general solution to the quadratic Liénard type equation

$$\ddot{x} - \frac{1}{2}\frac{\dot{x}^2}{x} + \omega^2 x^2 = 0 \tag{4.67}$$

obtained from (2.2) by letting $\gamma = 1/2$, may be immediately, using (2.6), computed as

$$x(t) = A_0 cn^2\left(\omega\sqrt{\frac{A_0}{2}}(t + C_3), \frac{1}{\sqrt{2}}\right) \tag{4.68}$$

Under the initial conditions $x(t = 0) = A_0$, $\dot{x}(t = 0) = 0$, the constant $C_3 = 0$, so that (4.68) gives

$$x(t) = A_0 cn^2\left(\omega\sqrt{\frac{A_0}{2}}t, \frac{1}{\sqrt{2}}\right) \tag{4.69}$$

Setting $x_0 = A_0$, equation (4.69) may take the form

$$x(t) = x_0 cn^2\left(\omega\sqrt{\frac{x_0}{2}}t, \frac{1}{\sqrt{2}}\right) \tag{4.70}$$

The periodic behavior of (4.70) is shown in Figure 6 under the conditions that $x_0 = 2$, $\omega = 0.5$

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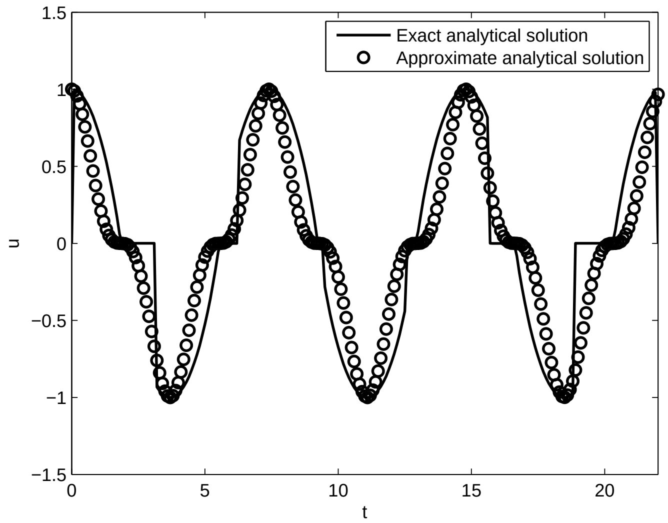

Figure 1: Comparison of the solution (4.2) with (4.7)

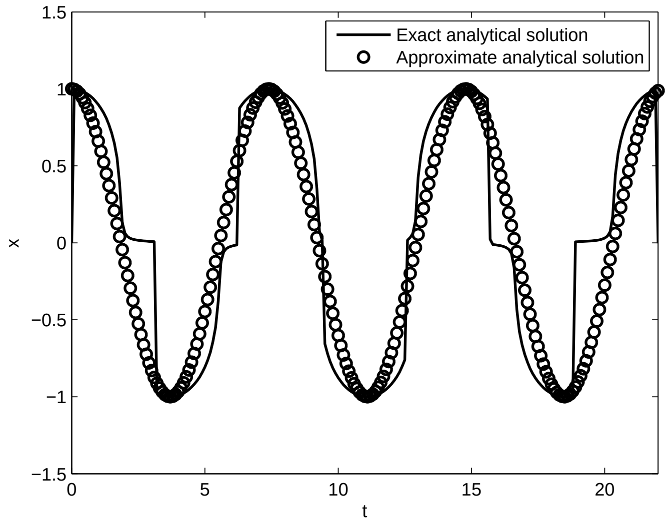

Figure 2: Comparison of the solution (4.11) with (4.12)

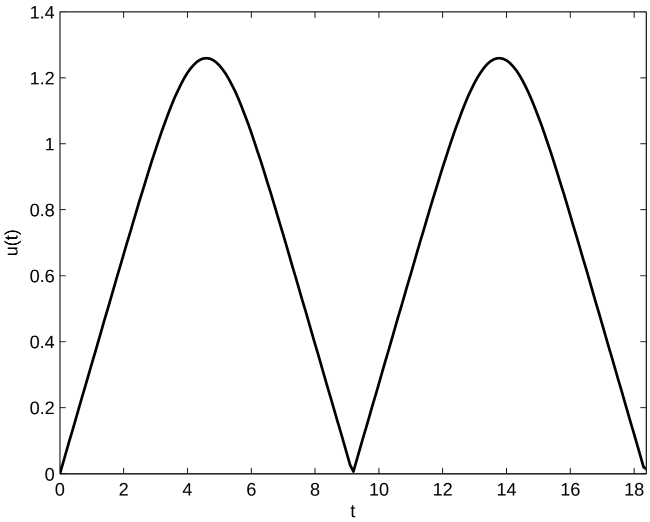

Figure 3: Typical behavior of the solution (4.53)

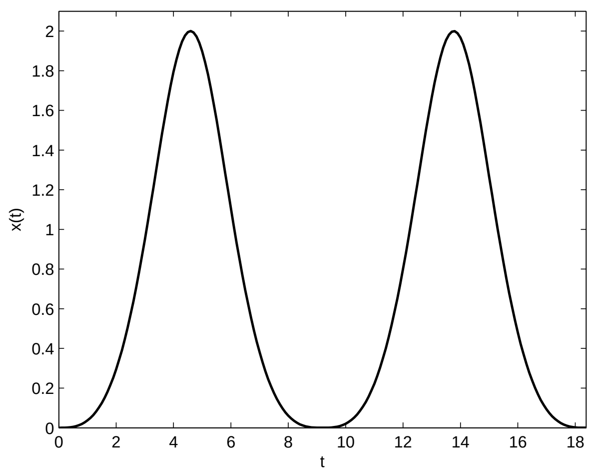

Figure 4: Typical behavior of the solution (4.59)

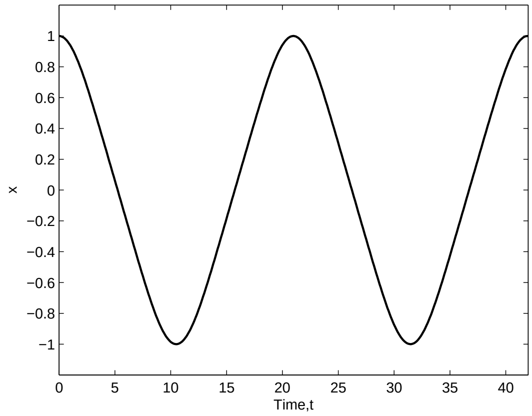

Figure 5: Typical behavior of the solution (4.66)

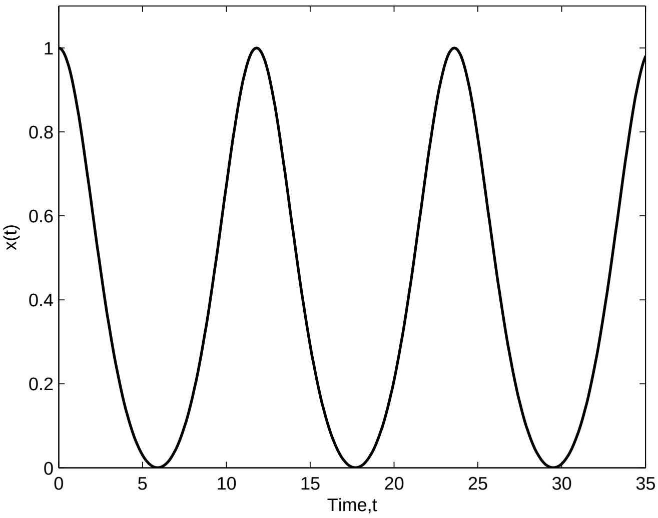

Figure 6: Typical behavior of the solution (4.70)