Addendum

Let G be a group in which

(1) $$(ab)^n = a^n b^n$$

for some fixed integer $n > 1$ for all $a, b \in G$. For all $a, b \in G$, prove that:

(a) $(ab)^{n-1} = b^{n-1}a^{n-1}$.
(b) $a^n b^{n-1} = b^{n-1} a^n$.
(c) $(aba^{-1}b^{-1})^{n(n-1)} = e$.

Hint for Part (c): Note that

(2) $$(aba^{-1})^r = ab^r a^{-1}$$

for all integers r,

Proof.
Let $a, b \in G$. So

$$\begin{aligned} (bab^{-1})^n &= ba^n b^{-1} && \text{by (2)} \\ &= b(a^n b^{-1}). \end{aligned}$$

On the other hand,

$$\begin{aligned} (bab^{-1})^n &= (b(ab^{-1}))^n \\ &= b^n(ab^{-1})^n && \text{by (1)} \\ &= b^n(a^n(b^{-1})^n) && \text{by (1)} \\ &= b^n(a^n b^{-n}) \\ &= (bb^{n-1})(a^n b^{-n}) \\ &= b(b^{n-1}(a^n b^{-n})) \\ &= b((b^{n-1}a^n)b^{-n}). \end{aligned}$$

Combining the last two results,

$$b(a^n b^{-1}) = b((b^{n-1}a^n)b^{-n}).$$

It follows that

$$a^n b^{-1} = (b^{n-1}a^n)b^{-n}$$

by the cancellation laws in G.

Since $(a^n b^{n-1})b^{-n} = a^n(b^{n-1}b^{-n}) = a^n b^{-1}$, from the previous result,

$$(a^n b^{n-1})b^{-n} = (b^{n-1}a^n)b^{-n}.$$

Again by the cancellation laws in G,

(3) $$a^n b^{n-1} = b^{n-1}a^n.$$

Moreover,

$$
\begin{aligned}
(ab)^{n-1} &= (ab)^n (ab)^{-1} \\
&= (a^n b^n)(ab)^{-1} && \text{by (1)} \\
&= (a^n b^n)(b^{-1} a^{-1}) \\
&= ((a^n b^n) b^{-1}) a^{-1} \\
&= (a^n (b^n b^{-1})) a^{-1} \\
&= (a^n b^{n-1}) a^{-1} \\
&= (b^{n-1} a^n) a^{-1} && \text{by (3)} \\
&= b^{n-1} (a^n a^{-1}) \\
&= b^{n-1} a^{n-1}.
\end{aligned}
$$

To summarize,

$$
(ab)^{n-1} = b^{n-1} a^{n-1}. \tag{4}
$$

Finally,

$$
\begin{aligned}
(aba^{-1}b^{-1})^{n(n-1)} &= ((aba^{-1}) b^{-1})^{n(n-1)} \\
&= (((aba^{-1}) b^{-1})^n)^{n-1} \\
&= ((aba^{-1})^n (b^{-1})^n)^{n-1} && \text{by (1)} \\
&= ((ab^n a^{-1}) (b^{-1})^n)^{n-1} && \text{by (2)} \\
&= ((ab^n a^{-1}) b^{-n})^{n-1} \\
&= ((ab^n)(a^{-1} b^{-n}))^{n-1} \\
&= (a^{-1} b^{-n})^{n-1} (ab^n)^{n-1} && \text{by (4)} \\
&= ((b^{-n})^{n-1} (a^{-1})^{n-1})((b^n)^{n-1} a^{n-1}) && \text{by (4)} \\
&= (b^{-n(n-1)} a^{-(n-1)})((b^{n-1})^n a^{n-1}) \\
&= (b^{-n(n-1)} a^{-(n-1)})(a^{n-1} (b^{n-1})^n) && \text{by (3)} \\
&= (b^{-n(n-1)} a^{-(n-1)})(a^{n-1} b^{(n-1)n}) \\
&= (b^{-n(n-1)} a^{-(n-1)})(b^{-n(n-1)} a^{-(n-1)})^{-1} \\
&= e.
\end{aligned}
$$