Proof of Twin Prime Conjecture

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Author’s Biography

The author of this research paper is K.H.K. Geerasee Wijesuriya . And this proof of twin prime conjecture is completely K.H.K. Geerasee Wijesuriya's proof.

Geerasee she studied before at Faculty of Science, University of Colombo Sri Lanka. And she graduated with BSc (Hons) in Physics and Mathematics from the University of Colombo, Sri Lanka in 2014. And in March 2018, she completed her first Doctorate Degree in Physics with first class recognition. Now she is following her second PhD in Astrophysics with Belarusian National Technical University.

Geerasee has been invited by several Astronomy/Physics institutions and organizations world-wide, asking to get involve with them. Also, She has received several invitations from some private researchers around the world asking to contribute to their researches. She worked as Mathematics tutor/Instructor at Mathematics department, Faculty of Engineering, University of Moratuwa, Sri Lanka. Now she is a research scientist in Physics as her career. Furthermore she has achieved several other scientific achievements already.

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Abstract

Twin prime numbers are two prime numbers which have the difference of 2 exactly. In other words, twin primes is a pair of prime that has a prime gap of two. Sometimes the term twin prime is used for a pair of twin primes; an alternative name for this is prime twin or prime pair. Up to date there is no any valid proof/disproof for twin prime conjecture. Through this research paper, my attempt is to provide a valid proof for twin prime conjecture.

Literature Review

The question of whether there exist infinitely many twin primes has been one of the great open questions in number theory for many years. This is the content of the twin prime conjecture, which states that there are infinitely many primes p such that p + 2 is also prime. In 1849, de Polignac made the more general conjecture that for every natural number k, there are infinitely many primes p such that p + 2k is also prime. The case k = 1 of de Polignac's conjecture is the twin prime conjecture.

A stronger form of the twin prime conjecture, the Hardy–Littlewood conjecture, postulates a distribution law for twin primes akin to the prime number theorem. On April 17, 2013, Yitang Zhang announced a proof that for some integer N that is less than 70 million, there are infinitely many pairs of primes that differ by N. Zhang's paper was accepted by Annals of Mathematics in early May 2013. Terence Tao subsequently proposed a Polymath Project collaborative effort to optimize Zhang's bound. As of April 14, 2014, one year after Zhang's announcement, the bound has been reduced to 246. Further, assuming the Elliott–Halberstam conjecture and its generalized form, the Polymath project wiki states that the bound has been reduced to 12 and 6, respectively. These improved bounds were discovered using a different approach that was simpler than Zhang's and was discovered independently by James Maynard and Terence Tao.

Assumption

Let's assume that there are finitely many twin prime numbers…………….(1.0)

Therefore we proceed by considering that there are finitely many twin prime numbers. Then let the highest twin prime numbers are P_{n-1} and $(P_{n-1} + 2)$. Then for all prime numbers P_N greater than $(P_{n-1} + 2)$, $(P_N + 2)$ is not a prime number.

Methodology

With this mathematical proof, I use the contradiction method to prove that there are infinitely many twin prime numbers.

Let P_n is an odd number equals to 3. But let P_n such that $P_n \mid (P_3 . x_3)$. To see the meaning of P_3 and x_3 , please refer the below content.

Let P_N is an arbitrary prime number greater than $(P_{n-1} + 2)$. Because there are infinitely many prime numbers. And here $(P_N – 2) > (P_{n-1} + 2)$. And that arbitrary P_N should obey $[3 \mid (P_N – 2)]$ (since for all $(P_N – 2) > (P_{n-1} + 2)$, $(P_N – 2)$ is not a prime number).

Then according to our assumption, $(P_N + 2)$ is not a prime number. Here P_N is a prime number such that $(P_N + 2)$ is dividing by prime number P_2. …………………(1)

Thus $(P_N + 2) = P_2 * x_2$ for some x_2 natural number. Since P_N is a prime number, for some r_2 (rational number which is not a natural number): $P_N / r_2 = P_2$.

Thus $(P_N + 2) = P_2 * x_2$ ………………(02) and $P_N = r_2 * P_2$ ……………….(03)

x_2 is a natural number and P_2 is a prime number.

Since P_N is a prime number , $(P_N – 2)$ is also not a prime number (Since $P_N – 2 > P_{n-1} + 2$)

Then for some prime P_3 , $(P_N – 2) / P_3 = x_3$; where x_3 is an integer.

$(P_N – 2) = P_3 * x_3$ ……………………..(04)

But $(P_N + 2)$, P_n both are odd numbers. Thus $(P_N + 2) = P_n + 2.l$ for some l integer number…………….(05)

Then $(P_N - 2) = P_n + 2.l - 4 = P_n + 2 . (l - 2)$ ....(6.1)'

And we know that $(P_N + 2) = P_n + 2.l \rightarrow P_N = P_n + 2.l - 2$ ……………………(*)

Thus by (*): $P_n + 2.l - 2 = P_N$. Thus by (04) and (*): $P_3 * x_3 + 2 = P_n + 2.l - 2$

Thus $P_3 * x_3 - 2.l + 4 = P_n$ ....(6.1.0)

Thus $P_3 * x_3 + 2. (l - 2) = P_n + 4. (l - 2) = P_n + 2.P_N - 4 - 2.P_n = 2.P_N – 4 – P_n$ (by (6.1)')

Thus $P_3 * x_3 + 2. (l - 2) = 2.P_N – 4 – P_n = P_{n''}$

Thus $P_3 * x_3 + 2. (l - 2) = P_{n''} = 2. P_3 * x_3 - P_n$ ....(7)

Thus $P_3 * x_3 +2.l = 4 + 2. P_3 * x_3 - P_n$

$P_3 * x_3 + (2.l + M) = (4 + M - P_n) + 2. P_3 * x_3$; Where M is an integer

$(2.l + M) = (4 + M - P_n) + P_3 * x_3$; Where M is an integer ………(8)

But we chose M such that (M + 4) is divisible by P_n. And we know that $(P_3 * x_3)$ is divisible by P_n ....(8.1) .

Thus by (8), $P_n \mid (2.l + M)$ …………….(i)

But P_N is an arbitrary prime greater than $(P_{n-1} + 2)$. Then let $(P_N + A_1)$ and P_N are two arbitrary consecutive primes greater than $(P_{n-1} + 2)$.

Here since $P_N > (P_{n-1} + 2)$ and since $P_N – 2 > (P_{n-1} + 2)$, $A_1 \neq (+/-) 2$. Because for any two arbitrary consecutive primes greater than $(P_{n-1} + 2)$, the difference between those consecutive primes is greater than 2 (since the greatest twin primes are P_{n-1} and $[P_{n-1} + 2]$). But that arbitrary prime number P_N is obeying $3 \mid (P_N – 2)$.

But $A_1 \neq 2.(x_3 - 1)$. But here $[P_n \mid (A_1 – 2)]$. **Since $A_1 \neq - 2$, there exists an odd number P_n greater than 1 such that $[P_n \mid (A_1 – 2)]$.**

But we know that $(P_N + A_1) > (P_{n-1} + 2)$. Here $A_1 \neq (+/-) 2$, since there are finite number of twin primes according to our assumption. BUT REMEMBER THAT P_N AND $(P_N + A_1)$ ARE CONSECUTIVE PRIMES.

{ Here $(P_N - 2) = P_3.x_3$ and $(P_N + A_1) = P$ = Prime. That means $P_3.x_3 + (A_1 + 2) = P$

But $(A_1 - 2)$ is divisible by P_n . Thus $(A_1 + 2)$ is not divisible by P_n. Because P_n does not divide 4. But since $P_3 * x_3$ is divisible by P_n , P is not divisible by P_n.

But $P = P_3.x_3 + A_1 + 2 \neq P_3.x_3 + 2.(x_3 - 1) + 2 = P_3.x_3 + 2.x_3 = x_3.(P_3 + 2)$. Thus $P \neq x_3.(P_3 + 2)$.

Therefore according to above steps, we can write $P_3.x_3 + (A_1 + 2) = P$ as a prime }

Here $A_1 \neq -2$, since there are finite number of twin primes and since $(P_N + A_1) > (P_{n-1} + 2)$, since $(P_N - 2)$ is not a prime and since $(P_N - 2) > (P_{n-1} + 2)$. Therefore there exists odd number P_n such that $[P_n | (A_1 - 2)]$.

But $(2.l + M) = P_N - P_n + 2 + M = (P_N + A_1) + (M + 2 - A_1 - P_n)$…………..(9)

By (8.1): $P_n | (M + 4)$. Since $P_n | (P_3.x_3)$, $[P_n | (P_N - 2)]$. But $[P_n | (A_1 - 2)]$. …….(10)

Since $[P_n | (A_1 - 2)]$, P_n does not divide $(A_1 + 2)$.

But since $[P_n | (P_N - 2)]$, $\{ (A_1 + 2) + (P_N - 2) \}$ does not divide by P_n . i.e. P (= $(P_N + A_1)$) does not divide by P_n. Thus our choice of A_1 such that $[P_n | (A_1 - 2)]$ is okay.

But $[P_n | (P_N - 2)]$ and $[P_n | (A_1 - 2)]$. Thus $P_n | (P_N + A_1 - 4)$.

i.e $P_n | (P - 4)$………………(11)

Let's choose M integer such that $M = a.P - C$; for some integer 'a' and for some integer C ………..……….(12).

But $P_n | (M + 4)$ and $P_n | (P - 4)$ by (8.1) and (11).

By (12): $P = (M + C) / a$. Thus $[(M + C) / a] - 4 = P_n . P_L$ …………(13)

Where $P_L = [(P - 4) / P_n]$ = integer. But $[(M + 4) / P_n] = P_Q$ = integer.

Thus by (13): $[(P_n .P_Q - 4 + C) / a] - 4 = P_n. P_L$

Thus $a = [(P_n .P_Q + C - 4) / (P_n. P_L + 4)]$ ……………………….(14)

$(a + 1) = [(P_n .P_Q + C - 4) + (P_n. P_L + 4)] / (P_n. P_L + 4)$

Thus $(a + 1) = [P_n .(P_Q + P_L) + C] / (P_n. P_L + 4)$ …………………..(15)

Let C = ($\alpha - A_1 + 1$). Thus - C = ($A_1 - \alpha - 1$).

But 'α' is an integer such that (α - 1) = θ.P and $P_n \mid (\alpha - 5)$………………(16)

Where θ is an integer such that $P_n \mid (4.\theta + 2)$.

Refer the 'Proof' below to see the verification of the possibility of $P_n \mid (\alpha - 5)$ whenever $P_n \mid (4.\theta + 2)$ and $(\alpha - 1) = [\theta. P]$.

But C = ($\alpha - A_1 + 1$). Thus C – 4 = ($\alpha - A_1 - 3$) = ($\alpha - 5 - A_1 + 2$). By (16): $P_n \mid (C - 4)$. Thus by (14): ($P_n \mid a$)……………….(16.1)

By (09): ($2.l + M$) = ($P_N + A_1$) + ($M + 2 - A_1 - P_n$) = P + ($M + 2 - A_1 - P_n$)

= P + a.P + ($A_1 - \alpha - 1$) + 2 – $A_1 - P_n$ = (a + 1).P - (α – 1) – P_n = (a + 1 - θ).P – P_n ……..(17)

(Because (α – 1) = θ.P).

But, [$P_n \mid a$] . But $P_n \mid (4.\theta + 2)$. But P_n does not divide (θ – 1). i.e. P_n does not divide (1 – θ).

See the verification of the existence of θ such that $P_n \mid (4.\theta + 2)$ and P_n does not divide (θ – 1) both in the <u>proof 1</u>. Thus by (16.1): P_n does not divide (a + 1 - θ). Because ($P_n \mid a$).

Since P is a prime and does not divide by P_n , by (17): ($2.l + M$) = P_n . r' ; where r' is not an integer.

Thus ($2.l + M$) = P_n . r' ; where r' is not an integer…………………(ii)

But by (i) and (ii): r' = integer. Thus we have a contradiction……………………(18)

Therefore the only possibility is: our assumption (1.0) is false. Therefore there are infinitely many Twin Prime Numbers.

Proof

Let's prove the possibility of $P_n \mid (\alpha - 5)$ whenever $P_n \mid (4.\theta + 2)$ and (α - 1) = [θ. P] for (3 = P_n) as below.

Let's assume (α – 5) = r.P_n for non-integer r. Since 3 = P_n , (α + 1) = r''.P_n for some non integer r''………………….(19)

But we have $(\alpha - 1) = (\theta. P)$. Then $\alpha + 1 = [(\theta. P) + 2]$.

But we have $P - 4 = (P_n . P_L)$ for an integer P_L . Thus $\theta.P - 4.\theta = (\theta. P_n . P_L)$

Then $(\theta.P + 2) = (\theta. P_n .P_L) + 4.\theta + 2$ ………………..….(20)

But $P_n | (4.\theta + 2)$. Thus $P_n | (\theta.P + 2)$. Thus $[P_n | (\alpha + 1)]$. Thus $(\alpha + 1) = P_n.v$; v is an integer. But by (19): $(\alpha + 1) = r''.P_n$ for some non-integer r''. Thus v = r''. Thus we have a contradiction. Thus $[P_n | (\alpha -5)]$……………(21)

Proof 1

Let's prove the existence of some integer θ such that P_n does not divide $(\theta - 1)$ when we have $P_n | (4.\theta + 2)$.

Let's assume for all θ integer, $(\theta - 1) = P_n.D$, when $(4.\theta + 2) = P_n .E$; where D and E are integers.

Then $(5.\theta + 1) = P_n . G$; where G is an integer. Thus $(5.\theta + 1) = P_n . G$ for all integer θ.

But $P_n = 3$. Thus $(5.\theta + 1) = 3. G$ for all integer θ. Then put θ = 3.

Then $(5.\theta + 1) \neq 3. G$ for all integer G. Thus we have a contradiction.

Thus there exists θ such that P_n does not divide $(\theta - 1)$ when we have $P_n | (4.\theta + 2)$.

Proof 2

Let's prove that there exists infinite number of prime numbers P_N such that $3 | (P_N - 2)$ by using mathematical induction method in this proof 2 as below.

Let's consider the statement Q(n) : [P(n) - 2] / 3 = x(n) ; where P(n) is the nth prime number which obeys P(n) - 2 = 3. x(n). And therefore the meaning of x(n) is: x(n) is an integer which obeys those conditions.

Q(1): [5 - 2] / 3 = 1 = x(1) = a natural number. Thus for n =1 , the result holds.

Now assume for n = s, the result Q(s) holds. Then $[P_s - 2] / 3 = x(s)$ = natural number.

Here we must considered n = s part as below. Now please refer the 2nd reference below.

Let ϵ_s is a positive real number $\epsilon_s = [- A + P_s + C_s - 2 + 3.k''] / P_s > 0$, such that $g_s < P_s * \epsilon_s$ for all s > (L- 2) . (Here s is going from 1 to (L - 1). Then " for all s > (L - 2)" means s = (L - 1)). Where k'' is an integer number. Here the chosen k'' integer number is responsible for

$g_s < P_s * \epsilon_s$ for all s > (L - 2) (i.e. s = (L - 1)) and $\epsilon_{L-1} > 0$. That means here the value of k'' is responsible to say " ϵ_s is existing such that $g_s < P_s * \epsilon_s$, for s = (L-1) " . Here $g_j = a_j$ for all j < (L– 1) = s. And where $\Sigma a_j = A$ for j < (L – 1) = s. Then for some C_s , $g_s = P_s * \epsilon_s - C_s$; here s ≡ L– 1. *** the meaning of 'j' is the order number and g_i is the prime gap between P_{j+1} and P_j . Please refer the below content and the 2nd reference.

But s ≡ (L - 1). But here we chose C_{L-1} such that $g_{L-1} = P_{L-1} * \epsilon_{L-1} – C_{L-1}$

But $g_{L-1} = P_{L-1} * \epsilon_{L-1} – C_{L-1} = (P_s - A – 2 + 3.k'')$. Where k'' is an integer number.

Then let's show for n = s +1, Q(s+1) holds. We denote $P(s+1) = P_L$

But we know $[P_s - 2] / 3 = x(s)$ ……………..…..(22)

Now let's use the 2nd reference to proceed further.

By 2nd reference, $P_L = 2 + \sum_{j=1}^{L-1} gj$ ……………….(iii)

But we know already that for $\epsilon_{L-1} > 0$, $g_{L-1} < P_{L-1} * \epsilon_{L-1}$. Here s ≡ (L - 1)

(*** refer the 2nd reference below)

Then we already know that for some C_{L-1} positive number, $g_{L-1} = P_{L-1} * \epsilon_{L-1} – C_{L-1}$.

But $g_{L-1} = P_{L-1} * \epsilon_{L-1} – C_{L-1}$ for (L– 1) ≡ s

We know already that $\epsilon_{L-1} = [P_s - A + C_{L-1} – 2 + 3.k''] / P_{L-1} > 0$.

And $g_{L-1} = P_{L-1} * \epsilon_{L-1} – C_{L-1} = (- A + P_s – 2 + 3.k'')$. Where k'' is an integer number. We know already that the chosen k'' integer number is responsible for $\epsilon_{L-1} > 0$.

We know that $g_j = a_j$ for all j < (L– 1). Where a_j is a natural number. Also we know that $\Sigma a_i = A$ for j < L – 1.

Thus by (iii): $P_L = 2 + P_s + 3.k'' - A - 2 + A = 3.k'' + P_s$

Thus $(P_L - 2) = (P_s - 2) + 3.k''$ ……………..(23)

But $[P_s - 2] = 3. x(s)$. Thus by (23): $(P_L - 2) = 3. x(s) + 3.k'' = 3. [x(s) + k'']$.

Thus $(P_L - 2)$ is divisible by 3. i.e. $[P(s+1) - 2]$ is divisible by 3.

Thus for n = s + 1 , the result Q(n +1) holds. Thus by mathematical induction method:

There exists infinite number of prime numbers P_L such that $3| (P_L - 2)$. Where $(P_L - 2) = 3.x_L$ Thus there exists $(P_N – 2)$ integer (where we consider them as prime numbers greater than $[P_{n-1} + 2]$) such that $(P_N - 2)$ is divisible by 3. Thus $(P_N - 2)$ is divisible by 3.

Proof 3

Let's prove that $[3 | (A_1 – 2)]$ when there exist consecutive prime numbers P_N and $(P_N + A_1)$ which both are greater than $(P_{n-1} + 2)$ in this proof 3 as below.

By 2nd reference: $(P_N + A_1) = 2 + \sum_{j=1}^{N} hj$, where $h_j = P_{j+1} – P_j$ for all $j \in \{1 , 2, ……., (N- 1)\}$

Then $(A_1 – 2) = - P_N + \sum_{j=1}^{N} hj$ ……………………(24)

But by 2nd reference: for all $Є > 0$, there is a natural number 'm' such that for all $N > m$;

$g_N < P_N . Є$

Let $Є_s$ is a positive real number $Є_s = [- B + C_s + 2 + 3.k'] / P_s > 0$, such that $h_s < P_s* Є_s$ for all $s > (N - 1)$. Let here the chosen $Є_s$ implies that m = (N - 1) (Here s is going from 1 to N. Then " for all s > (N - 1)" means s = N. Where k' is an integer number. Here the chosen k' integer number is responsible for $h_s < P_s* Є_s$ for all s > (N - 1) (i.e. s = N) and $Є_N > 0$. That means here the value of k' is responsible to say " $Є_s$ is existing such that $h_s < P_s* Є_s$, for s = N " . Here $h_j = b_j$ for all j < N = s. And where $\Sigma b_j = B$ for j < N = s. Then for some C_s , $h_s = P_s* Є_s - C_s$; here $s \equiv N$. *** the meaning of 'j' is the order number and h_j is the prime gap between P_{j+1} and P_j . Please refer the below content and the 2nd reference. But here we chose C_N such that $h_N = P_N * Є_N – C_N$

But $h_N = P_N * Є_N – C_N = (- B + 2 + 3.k')$. Where k' is an integer number.

Now let's use the 2nd reference to proceed further. By (24):

$(A_1 – 2) = - P_N + \sum_{j=1}^{N} hj = - P_N + (- B + 2 + 3.k') + B = (2 – P_N) + 3.k'$…………….(25)

But $3 | (P_N - 2)$. Thus by (25): [$3 | (A_1 – 2)$]. Thus there exist consecutive prime numbers P_N and $(P_N + A_1)$ both greater than $(P_{n-1} + 2)$ where [$3 | (A_1 – 2)$].

Discussion

We assumed initially that there are finitely many twin primes. After proceeding with that, I ended up with a contradiction. But to get the contradiction, I used that P_N as a prime number greater than $(P_{n-1} + 2)$. And we chose P_n (= 3) odd integer such that $P_n | (P_N - 2)$ also and we chose an integer M such that $P_n | (M + 4)$ and also we chose an integer A_1 such that $P_n | (A_1 - 2)$. Also to get the contradiction, I used the facts that $(P_N + 2)$ and $(P_N - 2)$ as non-primes since $P_N - 2 > (P_{n-1} + 2)$. And also I have used that x_2 and x_3 as natural numbers (since, $(P_N + 2)$ and $(P_N – 2)$ are not prime numbers). And also I have used the fact (to get the contradiction as in (18)): The difference between any two consecutive prime numbers (which are greater than $(P_{n-1} + 2)$) is greater than 2. Therefore to get the contradiction, I have used the facts got from our assumption (1.0). Then the only possibility is our assumption is false.

Results

Therefore I have used our assumption to get the contradiction finally, as showed in (18). Therefore it is possible to conclude that our assumption (1.0) is false. Thus the negation of the assumption (1.0) is true.

Thus there are infinitely many twin prime numbers.

Appendix

Prime number: A natural number which divides by 1 and itself only.

Twin Prime Numbers: Two prime numbers which have the difference exactly 2.

We denote 'i' th prime gap $g_i = P_{i+1} – P_i$

Then according to the 2nd reference; Prime number $P_N = 2 + \sum_{j=1}^{N-1} gj$

Also by 2nd reference: for all $Є > 0$, there is a natural number 'n' such that for all $N -1 > n$;

$g_{N-1} < P_{N-1} . Є$

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