Proof of Twin Prime Conjecture

K.H.K. Geerasee Wijesuriya

Research Scientist in Physics and Astronomy

PhD student in Astrophysics, Belarusian National Technical University

BSc (Hons) in Physics and Mathematics University of Colombo, Sri Lanka
Doctorate Degree in Physics

geeraseew@gmail.com , geerasee1@gmail.com

October 2019

Author's Biography

The author of this research paper is K.H.K. Geerasee Wijesuriya . And this proof of twin prime conjecture is completely K.H.K. Geerasee Wijesuriya's proof.

Geerasee she studied before at Faculty of Science, University of Colombo Sri Lanka. And she graduated with BSc (Hons) in Physics and Mathematics from the University of Colombo, Sri Lanka in 2014. And in March 2018, she completed her first Doctorate Degree in Physics with first class recognition. Now she is following her second PhD in Astrophysics with Belarusian National Technical University.

Geerasee has been invited by several Astronomy/Physics institutions and organizations world-wide, asking to get involve with them. Also, She has received several invitations from some private researchers around the world asking to contribute to their researches. She worked as Mathematics tutor/Instructor at Mathematics department, Faculty of Engineering, University of Moratuwa, Sri Lanka. Furthermore she has achieved several other scientific achievements already.

List of Affiliations

Faculty of Science, University of Colombo, Sri Lanka , Belarusian National Technical University Belarus

Acknowledgement

I would be thankful to my parents who gave me the strength to go forward with mathematics and Physics knowledge and achieve my scientific goals.

Keywords: prime; contradiction; greater than ; natural number

Abstract

Twin prime numbers are two prime numbers which have the difference of 2 exactly. In other words, twin primes is a pair of prime that has a prime gap of two. Sometimes the term twin prime is used for a pair of twin primes; an alternative name for this is prime twin or prime pair. Up to date there is no any valid proof/disproof for twin prime conjecture. Through this research paper, my attempt is to provide a valid proof for twin prime conjecture.

Literature Review

The question of whether there exist infinitely many twin primes has been one of the great open questions in number theory for many years. This is the content of the twin prime conjecture, which states that there are infinitely many primes p such that p + 2 is also prime. In 1849, de Polignac made the more general conjecture that for every natural number k, there are infinitely many primes p such that p + 2k is also prime. The case k = 1 of de Polignac's conjecture is the twin prime conjecture.

A stronger form of the twin prime conjecture, the Hardy–Littlewood conjecture, postulates a distribution law for twin primes akin to the prime number theorem. On April 17, 2013, Yitang Zhang announced a proof that for some integer N that is less than 70 million, there are infinitely many pairs of primes that differ by N. Zhang's paper was accepted by Annals of Mathematics in early May 2013. TerenceTao subsequently proposed a Polymath Project collaborative effort to optimize Zhang's bound. As of April 14, 2014, one year after Zhang's announcement, the bound has been reduced to 246. Further, assuming the Elliott–Halberstam conjecture and its generalized form, the Polymath project wiki states that the bound has been reduced to 12 and 6, respectively. These improved bounds were discovered using a different approach that was simpler than Zhang's and was discovered independently by James Maynard and Terence Tao.

Assumption

Let's assume that there are finitely many twin prime numbers.

Therefore we proceed by considering that there are finitely many twin prime numbers. Then let the highest twin prime numbers are P_{n-1} and (P_{n-1} +2). Then for all prime numbers P_n greater than P_{n-1} , (P_n - 2) is not a prime number.

Methodology

With this mathematical proof, I use the contradiction method to prove the twin prime conjecture.

Let P_n is an arbitrary prime number greater than P_{n-1} (because there are infinite number of prime numbers). Then according to our consideration, (P_n - 2) is not a prime number. Since $P_n > 2$ and since P_n is a prime number and since P_n is an odd number, for all prime numbers P_i :

P_i (< P_n / 2): $P_n / P_i = r_1$

Thus $P_n = P_i * r_1$..................(01.0)

Where r_1 is a rational number (which is not a natural number)

But according to our consideration, (P_n - 2) is not a prime number. Also since P_n is a prime number greater than 2, (P_n - 2) is an odd number.

Thus for some prime number P_1 (< [(P_n - 2) / 2]) ; (P_n - 2) / P_1 = x_1 . Where we choose P_1 such that x_1 is a natural number. But since previously chose P_i is any arbitrary prime number less than (P_n / 2); now we consider $P_1 = P_i$

Then (P_n - 2) = $P_1 * x_1$ ....(02) and $P_n = P_1 * r_1$..................(01)

Let P_N is a prime number (greater than P_n). Then according to our assumption, (P_N + 2) is not a prime number. Here P_N is a prime number such that <u>(P_N + 2) is dividing by prime number P_2.</u> …………………(1.1)

Thus $(P_N + 2) = P_2 * x_2$ for some x_2 natural number. Because there are infinitely many prime numbers. Since P_N is a prime number, for some r_2 (rational number which is not a natural number): $P_N / r_2 = P_2$.

Thus $(P_N + 2) = P_2 * x_2$ ………………(03) and $P_N = r_2 * P_2$ ……………….(04)

x_1 and x_2 are natural numbers and P_1 and P_2 are prime numbers.

Since P_N is a prime number , $(P_N – 2)$ is also not a prime number (Since $P_N – 2 > P_{n-1}$)

Then for some prime P_3 , $(P_N – 2) / P_3 = x_3$

$(P_N – 2) = P_3 * x_3$ ……………………..(05)

By (04) and (05): $P_3 * x_3 = P_2 * r_2 - 2$ …………………..(06)

But according to the below induction method proof which is in the "Proof" below, there exists an integer 'M' such that ($P_n – 6 - M) = x_3 .m_1$ for some integer m_1 (m_1 is divisible by P_3) and (P_n - 2) divides by x_3. And 'M' is an integer such that $(4 + M) = x_3.m_3$; for an integer m_3 . Those facts have been proven in the 'Proof' below.

And here we should consider that x_3 divides by P_3. But $P_3 \neq x_3$. Then $x_3.P_3 = P_3^2 .x''$ for some integer x''. Then $(P_N – 2) = P_3^2.x''$. Refer the 'Proof' below to see the verification of existence of prime number P_N (greater than P_{n-1}) such that $(P_N – 2) = P_3^2 .x''$; for the prime number P_3 and integer x''.

***** To see the induction method proof, please refer the 'Proof ' below.**

But $(P_N + 2)$, $(P_n – 2)$ both are odd numbers. Thus $(P_N + 2) = (P_n – 2) + 2.l$ for some l natural number…………….(06)'

Then $(P_N - 2) = (P_n – 2) + 2.l - 4 = P_n +2.l - 6 = P_n + 2 . (l - 3)$ ....(6.1)'

Since $(P_N – 2)$ is divisible by P_3 , [$P_n + 2.(l - 3)$] is divisible by P_3 . ...(6.1)

And we know that $(P_N + 2) = (P_n – 2) + 2.l \rightarrow P_N = P_n + 2.l - 4$ ……………………(*)

Thus by (*): $P_1. r_1 + 2.l - 4 = r_2 * P_2$. Thus by (06): $P_3 * x_3 + 2 = P_1. r_1 + 2.l - 4$

Thus $P_3 * x_3 - 2.l + 6 = P_1. r_1 = P_n$ ....(6.1.0)

Thus $P_3 * x_3 + 2.(l - 3) = P_n + 4.(l - 3) = P_n + 2.P_N - 4 - 2.P_n = 2.P_N – 4 – P_n$ (by (6.1)')

Thus $P_3 * x_3 + 2.(l - 3) = 2.P_N – 4 – P_n = P_{n''}$

Thus $P_3 * x_3 + 2.(l - 3) = P_{n''}$ ....(A)

Thus $P_3 * x_3 + 2.l = 6 + 2.P_3 * x_3 - P_n \rightarrow P_3*(x_3 + 1)+(2.l – P_3) = (6 - P_n) + 2.P_3 * x_3$

$P_3*(x_3 + 1) +(2.l – P_3 + M) = (6 + M - P_n) + 2.P_3 * x_3$; Where M is an integer………(A.2)

But we chose M such that ($P_n – 6 - M) = x_3 .m_1$ for some integer m_1 (m_1 is divisible by P_3). And we chose P_n such that $(P_n - 2)$ divides by x_3…………(A.3) But here $P_3 \neq x_3$.

Thus ($P_n – 6 - M) = x_3 .m_1$ and $(P_n - 2) = x_3.m_0$ for some integer m_1 and m_0 .

But $(P_N – 2)$ divides by x_3. Thus according to our choice, [$(P_N – 2) – (P_n - 2)$] divides by x_3. i.e. $(P_N – P_n)$ divides by x_3 . Thus $P_N – P_n = x_3.m_2$; for some integer m_2 .

Thus according to our choice: [($P_n – 6 - M) = x_3 .m_1$] and [$(P_N – P_n) = x_3.m_2$] for some integer m_1 and m_2 . But ($P_n – 6 - M) = x_3 .m_1$ and $(P_n - 2) = x_3.m_0$ for some integer m_1 and m_0 .

Thus $(4 + M) = x_3.m_3$; $m_3 = (m_0 – m_1)$ for an integer m_3 …………………(A.4)

By (*): $P_N – P_n + 4 = 2.l$

Thus $2.l – P_3 + M = (P_N – P_n) + 4 – P_3 + M = x_3.m_2 + (M + 4 – P_3) = x_3 .r'$ (since [M + 4] divisible by x_3 and since P_3 is a prime number which is not equal to x_3). And here r' is not an integer.

Thus by (A.2): $(x_3 +1)* P_3 + x_3.r' = - x_3 .m_1 + 2.P_3 * x_3$ ……………..(A.5)

But m_1 divisible by P_3 . Thus $x_3 .m_1 = x_3.P_3 .m_4$; m_4 is an integer.

Thus by (A.5): $(x_3 +1)* P_3 + x_3.r' = - x_3 .P_3.m_4 + 2.P_3 * x_3$

Thus $P_3. \{ (x_3 + 1) + [x_3.r' / P_3] \} = P_3.[2.x_3 – x_3.m_4]$………(A.6)

Then [$x_3 + 1 + (x_3.r' / P_3)$] = [$2.x_3 – x_3.m_4$] ; here r' is not an integer. And here x_3 divisible by P_3 (But $P_3 \neq x_3$). Thus $(x_3.r' / P_3) = r''$ is not an integer. But $(x_3 + 1)$ and [$2.x_3 – x_3.m_4$] both are integers. Thus by (A.6) , we have a contradiction. Therefore the only possibility is: our assumption is false. Therefore there are infinitely many Twin Prime Numbers.

Proof

Now let's prove that there exists infinite number of Prime numbers P_n such that $(P_n – 6 - M) = x_3 .m_1$ for some integer m_1 (m_1 is divisible by P_3) and $(P_n - 2)$ divides by x_3 whenever $P_3 \neq x_3$ (by using mathematical induction method as below).

(Here 'M' is an integer such that $M = [x_3.m_3 – 4]$; for an integer m_3).

But if we can prove $(P_n – 6 - M) = x_3 .m_1$ for some integer m_1 (m_1 is divisible by P_3) ; where M is an integer such that $M = [x_3.m_3 – 4]$; for an integer m_3 , then it is automatically proven that $(P_n - 2)$ divides by x_3.

Thus only thing that needs to prove is : we have to prove that $(P_n – 6 - M) = (x_3. P_3 .m_4)$ for some integer m_4 ; when M is an integer such that $M = [x_3.m_3 – 4]$; for an integer m_3 , when we considered $x_3 \neq P_3$.

That means we have to prove that :

$(P_n – 6 - M) = P_n – 6 - x_3 .m_3 + 4 = (P_n – 2) - x_3 .m_3 = (x_3. P_3 .m_4)$. That means we have to prove that $(P_n – 2) = x_3 .m_3 + (x_3. P_3 .m_4) = x_3 .(m_3 + P_3 .m_4)$.

i.e. **we have to prove that $(P_n – 2) = x_3.(m_3 + P_3 .m_4)$ for some integer m_3 and m_4 ; when $x_3 \neq P_3$ and whenever P_3 and x_3 obey $(P_N – 2) = x_3. P_3$.**

Now let's consider $x_3 \neq P_3$.

Let's consider the statement Q(n) : $[P(n) - 2] / x_3 = x(n)$; where P(n) is the nth prime number which obeys $[P(n) – 2] = x_3. x(n)$. And x(n) is the n^{th} integer which is in the form of $(m_3 + P_3 .m_4)$ for some integer m_3 and m_4.

For n = 1, L.H.S. of $Q(1) = [2 – 2] / x_3 = 0$. But for $m_3 = - P_3 .m_4$ (which is an integer), R.H.S. of Q(1): 0. Thus for n =1 , R.H.S. of Q(1) = L.H.S. of Q(1).Thus for n = 1, the result holds.

Now assume for n = s, the result Q(s) holds. Then $[P_s - 2] / x_3 = x(s)$ = natural number , where $x(s) = (m_3 + P_3 .m_4)$ for some m_3 and m_4 integer numbers.

Here we must considered n = s part as below.

Let ϵ_s is a positive real number $\epsilon_s = [- B + P_s + C_s - 2 + x_3.k'] / P_s > 0$ for all $s > (L - 2)$, $h_s < P_s * \epsilon_s$ (since the only existing $s > (L - 2)$ is $(L - 1)$; " for all $s > (L -2)$ means $s = (L - 1)$)''. Where k' is an integer number. Here the chosen k' integer number is responsible for $h_s < P_s * \epsilon_s$ for all $s > (L - 2)$ and k' is responsible for $\epsilon_{L-1} > 0$. That means here the value of k' is responsible to say : " ϵ_s is existing such that $h_s < P_s * \epsilon_s$, for $s = (L-1)$ ". Here $h_j = b_j$ for all $j < (L – 1) = s$. And where $\Sigma b_j = B$ for $j < (L – 1) = s$. Then for C_s , $h_s = P_s * \epsilon_s - C_s$; here $s \equiv L – 1$. *** the meaning of 'j' is the order number and h_j is the prime gap between P_{j+1} and P_j, please refer the below content and the 2nd reference. And let $k' = k''.(m_3 + P_3 .m_4)$ for some integer k'' and m_3 and m_4 integers.

But $s \equiv (L - 1)$. But here we chose C_{L-1} such that $h_{L-1} = P_{L-1} * \epsilon_{L-1} – C_{L-1}$

But $h_{L-1} = P_{L-1} * \epsilon_{L-1} – C_{L-1} = (P_s - B – 2 + x_3 .k')$. Where k' is an integer number.

Then let's show for $n = s +1$, $Q(s+1)$ holds. We denote $P(s+1) = P_L$

But we know $[P_s - 2] / x_3 = x(s)$ ………………..…..(8.1)

Now let's use the 2nd reference to proceed further.

By 2nd reference, $P_L = 2 + \sum_{j=1}^{L-1} h_j$ ……………….(i)

But we know already that for $\epsilon_{L-1} > 0$, $h_{L-1} < P_{L-1} * \epsilon_{L-1}$ for $L - 1 = s$.

Here $s \equiv (L - 1)$

(*** refer the 2nd reference below)

Then we already know that for some C_{L-1} positive number, $h_{L-1} = P_{L-1} * \epsilon_{L-1} – C_{L-1}$.

But $h_{L-1} = P_{L-1} * \epsilon_{L-1} – C_{L-1}$ for $(L – 1) \equiv s$

We know already that $\epsilon_{L-1} = [P_s - B + C_{L-1} – 2 + x_3 .k'] / P_{L-1} > 0$.

And $h_{L-1} = P_{L-1} * \epsilon_{L-1} – C_{L-1} = (- B + P_s – 2 + x_3 .k')$. Where k' is an integer number. We know already that the chosen k' integer number is responsible for $\epsilon_{L-1} > 0$.

We know that $h_j = b_j$ for all $j < (L - 1)$. Where b_j is a natural number. Also we know that $\Sigma\, b_j = B$ for $j < L - 1$.

Thus by (i): $P_L = 2 + P_s + x_3.k' - B - 2 + B = x_3.k' + P_s$

Thus $(P_L - 2) = (P_s - 2) + x_3.k'$ ……………..(8.2)

But $(P_s - 2)$ is divisible by x_3 and $(P_s - 2) / x_3 = x(s)$ according to (8.1). Thus $(P_L - 2)$ is divisible by x_3 according to (8.2), since $x_3.k'$ is divisible by x_3.

Thus $(P_L - 2)$ is divisible by x_3. i.e. $[P(s+1) - 2]$ is divisible by x_3.

By (8.2): $(P_L - 2) = (P_s - 2) + x_3.k'$. But $x(s) = (m_3 + P_3.m_4)$ for some m_3 and m_4 integer numbers and $(P_s - 2) = x_3.(m_3 + P_3.m_4)$. But $k' = k''.(m_3 + P_3.m_4)$ for some integer k''.

Thus by (8.2): $(P_L - 2) = x_3.(m_3 + P_3.m_4) + x_3.k''.(m_3 + P_3.m_4)$

$= (m_3 + P_3.m_4).x_3.[1 + k''] = x_3.[m_3.k''' + k'''.P_3.m_4]$; where $(1 + k'') = k'''$.

Thus $(P_L - 2) = x_3.[m'_3 + P_3.m'_4]$ for some integers m'_3 and m'_4.

Thus for $n = s + 1$ ($= L$) , the result $Q(n+1)$ holds. Thus by mathematical induction method:

There exists infinite number of prime numbers P_L such that $(P_L - 2) = x_3.[m_3 + P_3.m_4]$ for some integer numbers m_3 and m_4.

Thus there exists P_n prime (where we consider them as prime numbers greater than P_{n-1}) such that $(P_n - 2)$ is divisible by x_3 and $(P_n - 2) = x_3.[m_3 + P_3.m_4]$ for some integer numbers m_3 and m_4, whenever $P_3 \neq x_3$.

***Also we can say that there exists infinite number of primes P_n such that $(P_n - 2)$ is divisible by x_3 and $(P_n - 2) = x_3.[m_3 + P_3.m_4]$ for some integer numbers m_3 and m_4.

Verification of existence of prime number P_N (greater than P_{n-1}) such that $(P_N - 2) = P_3^2 .x''$; for the prime number P_3 and integer x''.

Let ϵ_s is a positive real number $\epsilon_s = [- A + C_s + (P_3)^2.t_s] / P_s > 0$ for all s > (R - 2), $g_s < P_s * \epsilon_s$ (since the only existing s > (R - 2) is (R - 1); " for all s > (R -2) means s = (R - 1))''. Where t_s is an integer number. Here the chosen t_s integer number is responsible for $g_s < P_s * \epsilon_s$ for all s > (R - 2) and t_s is responsible for $\epsilon_{R-1} > 0$. That means here the value of t_s is responsible to say : " ϵ_s is existing such that $g_s < P_s * \epsilon_s$, for s = (R-1) ". Here $g_j = a_j$ for all j < (R – 1) = s. And where $\Sigma a_j = A$ for j < (R – 1) = s. Then for C_s , $g_s = P_s * \epsilon_s - C_s$; here s ≡ R – 1. *** the meaning of 'j' is the order number and g_j is the prime gap between P_{j+1} and P_j, please refer the below content and the 2nd reference.

But s ≡ (R - 1). But here we chose C_{R-1} such that $g_{R-1} = P_{R-1} * \epsilon_{R-1} - C_{R-1}$

But $g_{R-1} = P_{R-1} * \epsilon_{R-1} - C_{R-1} = (- A + P^2{}_3.t_s)$. Where t_s is an integer number.

Now let's use the 2nd reference to proceed further.

By 2nd reference, $P_R = 2 + \sum_{j=1}^{R-1} gj$ ……………….(ii)

But we know already that for $\epsilon_{R-1} > 0$, $g_{R-1} < P_{R-1} * \epsilon_{R-1}$ for R - 1 = s.

Here s ≡ (R - 1)

(*** refer the 2nd reference below)

Then we already know that for some C_{R-1} positive number, $g_{R-1} = P_{R-1} * \epsilon_{R-1} - C_{R-1}$.

But $g_{R-1} = P_{R-1} * \epsilon_{R-1} - C_{R-1}$ for (R – 1) ≡ s

We know already that $\epsilon_{R-1} = [- A + C_{R-1} + P^2{}_3.t_s] / P_{R-1} > 0$.

And $g_{R-1} = P_{R-1} * \epsilon_{R-1} - C_{R-1} = (- A + P^2{}_3.t_s)$. Where t_s is an integer number. We know already that the chosen t_s integer number is responsible for $\epsilon_{R-1} > 0$.

We know that $g_j = a_j$ for all $j < (R - 1)$. Where a_j is a natural number. Also we know that $\Sigma a_j = A$ for $j < R - 1$.

Thus by (ii): $P_R = 2 + P^2{}_3.t_s - A + A = P^2{}_3.t_s + 2$

Thus there exists prime number P_R such that $(P_R - 2) = P^2{}_3.t_s$ ……………..(9.1)

Now put $N \equiv R$. Then we can state that $(P_N - 2) = P^2{}_3.x''$ for some integer x''.

Discussion

We assumed initially that there are finitely many twin primes. After proceeding with that, I ended up with a contradiction. But to get the contradiction, I used that P_n and P_N as primes numbers greater than P_{n-1} . Also to get the contradiction, I used the facts that $(P_n - 2)$ and $(P_N + 2)$ and $(P_N - 2)$ as non-primes. And also I have used that x_1 , x_2 and x_3 as natural numbers (since $(P_n - 2)$, $(P_N + 2)$ and $(P_N - 2)$ are not prime numbers). Therefore to get the contradiction, I have used the facts got from our assumption. Then the only possibility is our assumption is false.

Results

Therefore I have used our assumption to get a contradiction finally as showed in (A.6). Therefore it is possible to conclude that our assumption is false.

Thus there are infinitely many twin prime numbers.

Appendix

Prime number: A natural number which divides by 1 and itself only.

Twin Prime Numbers: Two prime numbers which have the difference exactly 2.

We denote 'i' th prime gap $g_i = P_{i+1} - P_i$

Then according to the 2nd reference; Prime number $P_N = 2 + \sum_{j=1}^{N-1} gj$

Also by 2nd reference: for all $\epsilon > 0$, there is a natural number 'n' such that for all $N - 1 > n$;

$g_{N-1} < P_{N-1} . \epsilon$

References

1. Zhang, Yitang (2014). "Bounded gaps between primes". Annals of Mathematics. 179 (3): 1121–1174.

2. https://ipfs.io/ipfs/QmXoypizjW3WknFiJnKLwHCnL72vedxjQkDDP1mXWo6uco/wiki/Prime_gap.html

3. Terry Tao, Small and Large Gaps between the Primes

4. Maynard, James (2015), "Small gaps between primes", Annals of Mathematics, Second Series, **181** (1): 383–413

5. Tchudakoff, N. G. (1936). "On the difference between two neighboring prime numbers". Math. Sb. **1**: 799–814.

6. **I**ngham, A. E. (1937). "On the difference between consecutive primes" Quarterly Journal of Mathematics. Oxford Series. **8** (1): 255–266.