Proof of Twin Prime Conjecture

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Author’s Biography

The author of this research paper is K.H.K. Geerasee Wijesuriya . And this proof of twin prime conjecture is completely K.H.K. Geerasee Wijesuriya's proof.

Geerasee is now 30 years old and she studied before at Faculty of Science, University of Colombo Sri Lanka. And she graduated with BSc (Hons) in Physics and Mathematics from the University of Colombo, Sri Lanka in 2014. And in March 2018, she completed her first Doctorate Degree in Physics with first class recognition. Now she is following her second PhD in Astrophysics with Belarusian National Technical University.

Geerasee has been invited by several Astronomy/Physics institutions and organizations world-wide, asking to get involve with them. Also, She has received several invitations from some private researchers around the world asking to contribute to their researches. She worked as Mathematics tutor/Instructor at Mathematics department, Faculty of Engineering, University of Moratuwa, Sri Lanka. Furthermore she has achieved several other scientific achievements already.

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I would be thankful to my parents who gave me the strength to go forward with mathematics and Physics knowledge and achieve my scientific goals.

Keywords: prime; contradiction; greater than ; natural number

Abstract

A twin prime numbers are two prime numbers which have the difference of 2 exactly. In other words, twin primes is a pair of prime that has a prime gap of two. Sometimes the term twin prime is used for a pair of twin primes; an alternative name for this is prime twin or prime pair. Up to date there is no any valid proof/disproof for twin prime conjecture. Through this research paper, my attempt is to provide a valid proof for twin prime conjecture.

Literature Review

The question of whether there exist infinitely many twin primes has been one of the great open questions in number theory for many years. This is the content of the twin prime conjecture, which states that there are infinitely many primes p such that p + 2 is also prime. In 1849, de Polignac made the more general conjecture that for every natural number k, there are infinitely many primes p such that p + 2k is also prime. The case k = 1 of de Polignac's conjecture is the twin prime conjecture.

A stronger form of the twin prime conjecture, the Hardy–Littlewood conjecture, postulates a distribution law for twin primes akin to the prime number theorem. On April 17, 2013, Yitang Zhang announced a proof that for some integer N that is less than 70 million, there are infinitely many pairs of primes that differ by N. Zhang's paper was accepted by Annals of Mathematics in early May 2013. Terence Tao subsequently proposed a Polymath Project collaborative effort to optimize Zhang's bound. As of April 14, 2014, one year after Zhang's announcement, the bound has been reduced to 246. Further, assuming the Elliott–Halberstam conjecture and its generalized form, the Polymath project wiki states that the bound has been reduced to 12 and 6, respectively. These improved bounds were discovered using a different approach that was simpler than Zhang's and was discovered independently by James Maynard and Terence Tao.

Assumption

Let's assume that there are finitely many twin prime numbers.

Therefore we proceed by considering that there are finitely many twin prime numbers. Then let the highest twin prime numbers are P_{n-1} and (P_{n-1} +2). Then for all prime numbers P_n greater than P_{n-1} , (P_n - 2) is not a prime number.

Methodology

With this mathematical proof, I use the contradiction method to prove the twin prime conjecture.

Let P_n is an arbitrary prime number greater than P_{n-1} (because there are infinite number of prime numbers). Then according to our consideration, (P_n - 2) is not a prime number. Since $P_n > 2$ and since P_n is a prime number and since P_n is an odd number, for all prime numbers P_i :

P_i (< P_n / 2): $P_n / P_i = r_1$

Thus $P_n = P_i * r_1$..................(01.0)

Where r_1 is a rational number (which is not a natural number)

But according to our consideration, (P_n - 2) is not a prime number. Also since P_n is a prime number greater than 2, (P_n - 2) is an odd number.

Thus for some prime number P_1 (< [(P_n - 2) / 2]) ; (P_n - 2) / P_1 = x_1 . Where we choose P_1 such that x_1 is a natural number. But since previously chose P_i is any arbitrary prime number less than (P_n / 2); now we consider $P_1 = P_i$

Then (P_n - 2) = $P_1 * x_1$ ....(02) and $P_n = P_1 * r_1$..................(01)

Let P_N is a prime number (greater than P_n). Then according to our assumption, (P_N + 2) is not a prime number. Here P_N is a prime number such that <u>(P_N + 2) is dividing by prime number P_2.</u> …………………(1.1)

Thus $(P_N + 2) = P_2 * x_2$ for some x_2 natural number. Because there are infinitely many prime numbers. But we choose x_2 such that x_2 does not equal to $[(5.k_0 + 4) / 3]$; for any integer k_0 . There exists such integer x_2 , since there are infinitely many prime numbers P_N . *** Refer the "Proof" below, to see the verification of the existance of such $x_2 \neq [(5.k_0 + 4) / 3]$; for any integer k_0 .

Since P_N is a prime number, for some r_2 (rational number which is not a natural number):

$P_N / r_2 = P_2$.

Thus $(P_N + 2) = P_2 * x_2$ ………………(03) and $P_N = r_2 * P_2$ ……………….(04)

x_1 and x_2 are natural numbers and P_1 and P_2 are prime numbers.

Since P_N is a prime number , $(P_N – 2)$ is also not a prime number (Since $P_N – 2 > P_{n-1}$)

Then for some prime P_3 , $(P_N – 2) / P_3 = x_3$

$(P_N – 2) = P_3 * x_3$ ……………………..(05)

By (04) and (05): $P_3 * x_3 = P_2 * r_2 - 2$ …………………..(06)

But according to the below induction method proof which is in the "Proof" below, there exists primes P_n and P_N such that $(P_N + 2)$ and $(P_n – 2)$ both are divisible by 3(where $P_1 = 3$).

***** To see the induction method proof, please refer the 'Proof ' below.**

Then $(P_N + 2) = (P_n – 2) + 3.l$ for some l even natural number…………….(06)'

Then $(P_N - 2) = (P_n – 2) + 3.l - 4 = P_n + 3.l - 6 = P_n + 3 . (l - 2)$.

Since $(P_N – 2)$ is divisible by P_3 , $[P_n + 3.(l - 2)]$ is divisible by P_3 . ...(6.1)

And we know that $(P_N + 2) = (P_n – 2) + 3.l \rightarrow P_N = P_n + 3.l - 4)$ ……………………(*)

By (*): $P_1. r_1 + 3.l - 4 = r_2 * P_2$. Thus by (06): $P_3 * x_3 + 2 = P_1. r_1 + 3.l - 4$

Thus $P_3 * x_3 - 3.l + 6 = P_1. r_1$

$P_3 * x_3 - 3.(l – 2) = P_1. r_1$ ……………(6.1.0) $P_3 * x_3 - 3.l + 6 = P_1. r_1 = P_n$.

Thus $P_3 * x_3 - 3.(l - 2) = P_n$. Then $P_3 * x_3 - 3.(l - 2) + q = P_n + q = P_0$ ……(06)'' Where we choose q (where $6 \mid q$) an integer number such that $P_n + q = P_0$ = integer number that does not divide by P_3.

Then $P_3 * x_3 - 3.(l - 2) + q = P_0 = P_3. r$

Then $P_3 * x_3 - [3.(l - 2) - q] = P_0 = P_3. r$ ………………..(6.1.0.1)

Here r is some non-natural number. Because since P_0 is an integer number that does not divide by P_3 , r is not an integer number, but r is a rational number.

But by (6.1.0): $P_3 * x_3 - 3.(l - 2) = P_1. r_1 = P_n$. Then **$[P_3 . x_3 - P_n] = 3.(l - 2)$; since l is an even number, $(3.l - 6)$ is divisible by 6. Thus $[P_3 . x_3 - P_n]$ is divisible by 6.....(6.1.1)**

But we choose $l = (q / 6) + 2 + \{ [P_3 .x_3 - P_n] / 6 \}$ = a natural number ......(6.1.1.2)

(because by 6.1.1). Here we can adjust the value of [q / 6] (integer number) such that the output of $[(q / 6) + 2 + \{ [P_3 .x_3 - P_n] / 6 \}]$ gives the value of l as in (06)'.

(But q is (where $6 \mid q$) an integer number such that $P_n + q = P_0$ = integer number that does not divide by P_3).

i.e. we can adjust the value of [q / 6] as 1, 2, 3, 4, … or -1, -2, -3,… such that the output of $[(q / 6) + 2 + \{ [P_3 .x_3 - P_n] / 6 \}]$ gives the value of l as in (06)'.

Then: $6.l + P_n - 12 - q = P_3 . x_3$; Where x_3 is a natural number.

Then $[P_n + 3.(l -2)] + [3.(l - 2) - q] = P_3 .x_3$ ....(6.2)

By (6.1) we know that $(P_n + 3.(l - 2))$ is divisible by P_3 . Since x_3 is a natural number, by (6.2) : $(3. l - 6 - q)$ is divisible by P_3 ....(6.3). Thus we know that $[3.l - 6 - q]$ is divisible by P_3 (by (6.3)).

Thus by (6.1.0.1): $P_3 * [x_3 - l_0] = P_3 . r$; where $l_0 = (3.l - 6 - q) / P_3$ = integer number (by (6.3)). Thus, $x_3 - l_0 = r$ ....(07) where $l_0 = [3.l - 6 - q] / P_3$ = an integer number.

But $[x_3 - l_0]$ is an integer number. But r is not an integer number. Thus by (07), there is a contradiction. Therefore the only possibility is: our assumption is false.

Therefore there are infinitely many Twin Prime Numbers.

Proof

Now let's prove that there exists infinite number of Prime numbers P_n such that 3| (P_n - 2) , by using mathematical induction method as below.

Let's consider the statement Q(n) : [P(n) - 2] / 3 = x(n) ; where P(n) is the nth prime number which obeys P(n) - 2 = 3. x(n). And the meaning of x(n) is similar to that.

Q(1): [5 – 2] / 3 = 1 = x(1) = a natural number. Thus for n =1 , the result holds.

Now assume for n = s, the result Q(s) holds. Then [P_s - 2] / 3 = x(s) = natural number.

Here we must considered n = s part as below.

Let ϵ_s is a positive real number ϵ_s = [- B + P_s + C_s - 2 + 3.k'] / P_s > 0 for all s > (M - 2), h_s < P_s* ϵ_s (since the only existing s > (M - 2) is (M - 1); " for all s > (M - 2) means s = (M - 1))''. Where k' is an integer number. Here the chosen k' integer number is responsible for h_s < P_s* ϵ_s for all s > (M - 2) and k' is responsible for ϵ_{M-1} > 0. That means here the value of k' is responsible to say : " ϵ_s is existing such that h_s < P_s* ϵ_s , only for s = (M-1) ". Here h_j = b_j for all j < (M – 1) = s. And where Σ b_j = B for j < (M – 1) = s. Then for C_s , h_s = P_s* ϵ_s - C_s ; here s ≡ M – 1. *** the meaning of 'j' is the order number and h_j is the prime gap between P_{j+1} and P_j, please refer the below content and the 2nd reference.

But s ≡ (M - 1). But here we chose C_{M-1} such that h_{M-1} = P_{M-1} * ϵ_{M-1} – C_{M-1}

But h_{M-1} = P_{M-1} * ϵ_{M-1} – C_{M-1} = (P_s - B – 2 + 3.k'). Where k' is an integer number.

Then let's show for n = s +1, Q(s+1) holds. We denote P(s+1) = P_M

But we know [P_s - 2] / 3 = x(s) ……………..…..(8.1)

Now let's use the 2nd reference to proceed further.

By 2nd reference, $P_M = 2 + \sum_{j=1}^{M-1} hj$ ……………….(i)

But we know already that for ϵ_{M-1}> 0, h_{M-1}< P_{M-1} * ϵ_{M-1} for M - 1 = s.

Here s ≡ (M - 1)

(*** refer the 2^{nd} reference below)

Then we already know that for some C_{M-1} positive number, $h_{M-1} = P_{M-1} * Є_{M-1} – C_{M-1}$.

But $h_{M-1} = P_{M-1} * Є_{M-1} – C_{M-1}$ for $(M – 1) \equiv s$

We know already that $Є_{M-1} = [P_s - B + C_{M-1} – 2 + 3.k'] / P_{M-1} > 0$.

And $h_{M-1} = P_{M-1} * Є_{M-1} – C_{M-1} = (- B + P_s – 2 + 3.k')$. Where k' is an integer number. We know already that the chosen k' integer number is responsible for $Є_{M-1} > 0$.

We know that $h_j = b_j$ for all $j < (M – 1)$. Where b_j is a natural number. Also we know that $\Sigma b_j = B$ for $j < M – 1$.

Thus by (i): $P_M = 2 + P_s + 3.k'- B - 2 + B = 3.k' + P_s$

Thus $(P_M - 2) = (P_s - 2) + 3.k'$ ……………..(8.2)

But $(P_s - 2)$ is divisible by 3 (= P_1) according to (8.1). Thus $(P_M - 2)$ is divisible by 3 (=P_1) according to (8.2), since 3.k' is divisible by 3.

Thus $(P_M – 2)$ is divisible by 3 (= P_1). i.e. $[P(s+1) – 2]$ is divisible by 3 (= P_1).

Thus for n = s + 1 , the result Q(n +1) holds. Thus by mathematical induction method:

There exists infinite number of prime numbers P_M such that $3| (P_M - 2)$.

Thus there exists P_n prime (where we consider them as prime numbers greater than P_{n-1}) such that $(P_n – 2)$ is divisible by 3 (=P_1). Thus there exists P_n prime (greater than P_{n-1}) such that $(P_n - 2)$ is divisible by P_1 (=3).

Now let's prove that there exists infinite number of Prime numbers P_N such that $3| (P_N + 2)$, by using mathematical induction method as below. Where $(P_N + 2) = P_2 . x_2$; $x_2 \neq [(5.k_0 + 4) / 3]$ for any k_0 integer.

Let's consider the statement Q(n) : $[P(n) + 2] / 3 = x(n)$; where P(n) is the nth prime number which obeys $P(n) + 2 = 3. x(n)$. And $x(n) \neq [(5.k_0 + 4) / 3]$ and therefore the meaning of x(n) is:

x(n) is an integer which obeys those conditions.

Q(1): [13 + 2] / 3 = 5 = x(1) = a natural number. But x_2 [≡ x(1)] = 5 ≠ [(5.k_0 + 4) / 3] for any k_0 integer. Thus for n =1 , the result holds.

Now assume for n = s, the result Q(s) holds. Then [P_s + 2] / 3 = x(s) = natural number.

But here, x(s) ≠ [(5.k_0 + 4) / 3] for any k_0 integer.

Here we must considered n = s part as below.

Let $Є_s$ is a positive real number $Є_s$ = [- A + P_s + C_s - 2 + 3.k''] / P_s > 0 , such that g_s < P_s* $Є_s$ for all s > (L- 2) . (Here s is going from 1 to (L - 1). Then " for all s > (L - 2)" means s = (L - 1)). Where k'' is an even integer number and divisible by 5. That means k'' is divisible by 10. Here the chosen k'' integer number is responsible for g_s < P_s* $Є_s$ for all s > (L - 2) (i.e. s = (L - 1)) and $Є_{L-1}$ > 0. That means here the value of k'' is responsible to say " $Є_s$ is existing such that g_s < P_s* $Є_s$, only for s = (L-1) " . Here g_j = a_j for all j < (L– 1) = s. And where Σ a_j = A for j < (L – 1) = s. Then for some C_s , g_s = P_s* $Є_s$ - C_s ; here s ≡ L– 1. *** the meaning of 'j' is the order number and g_i is the prime gap between P_{j+1} and P_j . Please refer the below content and the 2nd reference.

But s ≡ (L - 1). But here we chose C_{L-1} such that g_{L-1} = P_{L-1} * $Є_{L-1}$ – C_{L-1}

But g_{L-1} = P_{L-1} * $Є_{L-1}$ – C_{L-1} = (P_s - A – 2 + 3.k''). Where k'' is an integer number.

Then let's show for n = s +1, Q(s+1) holds. We denote P(s+1) = P_L

But we know [P_s + 2] / 3 = x(s) ………………..…..(9.1)

Now let's use the 2nd reference to proceed further.

By 2nd reference, $P_L = 2 + \sum_{j=1}^{L-1} gj$ ……………….(ii)

But we know already that for $Є_{L-1}$> 0, g_{L-1} < P_{L-1} * $Є_{L-1}$. Here s ≡ (L - 1)

(*** refer the 2nd reference below)

Then we already know that for some C_{L-1} positive number, g_{L-1} = P_{L-1} * $Є_{L-1}$ – C_{L-1} .

But g_{L-1} = P_{L-1} * $Є_{L-1}$ – C_{L-1} for (L– 1) ≡ s

We know already that $\epsilon_{L-1} = [P_s - A + C_{L-1} – 2 + 3.k''] / P_{L-1} > 0$.

And $g_{L-1} = P_{L-1} * \epsilon_{L-1} – C_{L-1} = (- A + P_s – 2 + 3.k'')$. Where k'' is an integer number. We know already that the chosen k'' integer number is responsible for $\epsilon_{L-1} > 0$.

We know that $g_j = a_j$ for all $j < (L– 1)$. Where a_j is a natural number. Also we know that $\Sigma a_i = A$ for $j < L – 1$.

Thus by (ii): $P_L = 2 + P_s + 3.k''- A - 2 + A = 3.k'' + P_s$

Thus $(P_L + 2) = (P_s + 2) + 3.k''$ ……………..(9.2)

But $[P_s + 2] = 3. x(s)$. Thus by (9.2): $(P_L + 2) = 3. x(s) + 3.k'' = 3. [x(s) + k'']$.

Then $3. [x(s) + k''] \neq [5.k_0 + 4] + 3.k''$ ....(9.3). But k'' is divisible by 5.

Thus $[5.k_0 + 4] + 3.k'' = 5.k_1 + 4$; where $k_1 = [5.k_0 + 3.k''] / 5$ = integer

Then by (9.3): $3. [x(s) + k''] = (P_L + 2) \neq 5.k_1 + 4$ for any integer k_1 .

Furthermore, $(P_s + 2)$ is divisible by 3 $(= P_1)$ according to (9.1). Thus $(P_L + 2)$ is divisible by 3 $(=P_1)$ according to (9.2), since 3.k'' is divisible by 3.

Thus $(P_L + 2)$ is divisible by 3 $(= P_1)$. i.e. $[P(s+1) + 2]$ is divisible by 3 $(= P_1)$.

Also $(P_L + 2) \neq 5.k_1 + 4$ for any integer k_1 .

Thus for $n = s + 1$, the result $Q(n +1)$ holds. Thus by mathematical induction method:

There exists infinite number of prime numbers P_L such that $3| (P_L + 2)$. Where $(P_L + 2) = 3.x_L$ for $x_L \neq [(5.k_2 + 4) / 3]$ for any k_2 integer.

Thus there exists P_N prime (where we consider them as prime numbers greater than P_{n-1}) such that $(P_N + 2)$ is divisible by 3 $(=P_1)$. Thus $(P_N + 2)$ is divisible by P_1 (=3).

Where $(P_N + 2) = P_2 . x_2$; $x_2 \neq [(5.k_0 + 4) / 3]$ for any k_0 integer.

Also then we can say there exists infinite number of primes P_n and P_N such that $3| (P_n - 2)$ and $3| (P_N + 2)$.

Explanation on how the change of the value of q (that is required to equalize the value of *l* in (06)' to the value of *l* that we define in the research paper) has considered in the research paper "Proof of Twin Prime Conjecture"

The value Y is the value (equivalant to q/6 in 6.1.1.2) which generates the chosen value l in (6.1.1.2) equals to l in (06)'. But in (06)' , $l = (P_N - P_n + 4) / 3$

Then $Y = [(P_N - P_n + 4) / 3] - [(P_3 .x_3 - P_n) / 6] - 2$

Then $Y = [P_3 .x_3 - P_n] / 6$ (Since $(P_N - 2) = P_3 .x_3$)

But by (6.1.1), Y is an integer. But by (6.1.0): $Y = (l - 2) / 2$.

Then $6.Y = 3. (l - 2)$. By (6.1): $P_n + 3.(l - 2)$ is divisible by P_3 .

Then $P_n + 6.Y = P_3 .x_4$ for some integer x_4 . Then $P_n + Y = P_3 .x_4 - 5.Y$ ....(10)

But $Y = [P_3 .x_3 - P_n] / 6$. But since $(P_3 . x_3)$ is divisible by P_3 , $[P_3 .x_3 - P_n]$ is not divisible by P_3 . Thus $[P_3 .x_3 - P_n] / 6$ (= Y) is not divisible by P_3 ....(11)

But $(P_N + 2) = 3. x_2$ for some natural number x_2 .

Then $(P_N - 2) = [3. x_2 - 4]$. But we considered x_2 such that $x_2 \neq [(5.k_0 + 4) / 3]$ for any k_0 integer. Thus $(P_N - 2)$ is not divisible by 5. Thus P_3 is not 5. By (11): Y is not divisible by P_3 . But P_3 is not 5. Thus (5.Y) does not divisible by P_3 . Then by 10 : $(P_n + Y)$ does not divisible by P_3 . Thus $(Y+ P_n) = P_3 .r'$; where r' is not an integer.

But $Y + 2 + (P_3 .x_3 - P_n) / 6 = (P_N - P_n + 4) / 3$

Thus we can adjust the value [q / 6] (= Y) integer number in (6.1.1.2) such that the output of $[(q / 6) + 2 + \{ [P_3 .x_3 - P_n] / 6 \}]$ gives the value of *l* as in (06)' . And $(Y+ P_n) = P_3 .r'$; where r' is not an integer.

Discussion

We assumed initially that there are finitely many twin primes. After proceeding with that, I ended up with a contradiction. But to get the contradiction, I used that P_n and P_N as primes numbers greater than P_{n-1} . Also to get the contradiction, I used the facts that (P_n - 2) and (P_N + 2) and (P_N - 2) as non-primes. And also I have used that x_1 , x_2 and x_3 as natural numbers (since (P_n – 2) , (P_N + 2) and (P_N – 2) are not prime numbers). Therefore to get the contradiction, I have used the facts got from our assumption. Then the only possibility is our assumption is false.

Results

Therefore I have used our assumption to get a contradiction finally as showed in (07). Therefore it is possible to conclude that our assumption is false.

Thus there are infinitely many twin prime numbers.

Appendix

Prime number: A natural number which divides by 1 and itself only.

Twin Prime Numbers: Two prime numbers which have the difference exactly 2.

We denote 'i' th prime gap $g_i = P_{i+1} - P_i$

Then according to the 2nd reference; Prime number $P_N = 2 + \sum_{j=1}^{N-1} gj$

Also by 2nd reference: for all $\epsilon > 0$, there is a natural number 'n' such that for all N -1 > n;

$g_{N-1} < P_{N-1} . \epsilon$

References

1. Zhang, Yitang (2014). "Bounded gaps between primes". Annals of Mathematics. **179** (3): 1121–1174.

2. https://ipfs.io/ipfs/QmXoypizjW3WknFiJnKLwHCnL72vedxjQkDDP1mXWo6uco/wiki/Prime_gap.html

3. Terry Tao, Small and Large Gaps between the Primes

4. Maynard, James (2015), "Small gaps between primes", Annals of Mathematics, Second Series, **181** (1): 383–413

5. Tchudakoff, N. G. (1936). "On the difference between two neighboring prime numbers". Math. Sb. **1**: 799–814.

6. **I**ngham, A. E. (1937). "On the difference between consecutive primes" Quarterly Journal of Mathematics. Oxford Series. **8** (1): 255–266.