A GEOMETRIC AND TOPOLOGICAL QUANTISATION OF MASS

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Abstract: In this work we discuss a geometric and topological quantisation of mass by extending our work on the principle of least action in which the quantisation of both the angular momentum of a Bohr hydrogen atom and the charge of an elementary particle can be shown to be quantised from the continuous deformation of a differentiable manifold. Similar to the case of the quantisation of charge using the two-dimensional Ricci scalar curvature, we show that in the case of three-dimensional differentiable manifolds we can apply the Yamabe problem, which states that any Riemannian metric on a compact smooth manifold of dimension greater or equal to three is conformal to a metric with constant scalar curvature, to show that mass can also be quantised by the deformation of differentiable manifolds. The Yamabe problem is a generalisation of the uniformisation theorem for two-dimensional differentiable manifolds. We also discuss whether quantum particles can be expressed as direct sums of quantum masses when they possess mathematical structures of differentiable manifolds, with the quantum masses are considered as prime manifolds. This may be regarded as a physical manifestation of an established mathematical proposition in differential geometry and topology that states that any compact, connected, and orientable differentiable manifold M can be decomposed into prime manifolds and the decomposition is unique up to an absorption or emission of 3-spheres S^3. This process of decomposition of differentiable manifolds into prime manifolds and the radiation of 3-spheres is similar to the radiation of the quanta of physical fields from a quantum system, such as the radiation of photons from a hydrogen atom. Even though our work is highly speculative and suggestive, we hope that it may pay way for further rigorous mathematical investigations into whether mass is also quantised like charge and other fundamental entities in physics.

Our physical existence as beings endowed with consciousness is so balanced, or rather not fully developed, that we are normally unable to perceive physical processes that are happening around us but beyond our physical ability. As a consequence, the shortcomings due to our limited physical structure may also prevent us from visualising physical phenomena that exist in higher spatial dimensions. In the spirit of scientific investigations, this poses a challenging problem of how we can possibly devise experiments that can be used to verify the existence of a fourth spatial dimension, despite the fact that the three-dimensional observable universe in which we are living is expanding and, conceivably, such an expansion may be seen as a bending of the boundary of a 3-sphere S^3, which is formed by two 3D solid balls, into the fourth spatial dimension, and our observable universe is one of the two 3D solid balls. This may also explain why we exist as 3D physical objects [1]. Now the most seemingly obvious perception that we take for granted in physical science is that we are formed from matter which is a substance that can be identified with mass. But how mass exists is a profound epistemological question that still puzzles our conscious mind. In physics, mass is a defined property of physical objects that can manifest in different ways depending on how it can be related to the dynamics of a physical system. In classical physics, mass can simply be perceived as a resistance of a body to its acceleration under an applied force, or a measure of the strength of active and passive gravitational attraction, or in a more abstract manner, it is considered as an amount of rest energy and is related directly to the curvature of spacetime as formulated in Einstein's theories of relativity. On the other hand, in quantum physics, mass manifests as a wave property of a quantum object. Despite the fact that mass has played such important role in various formulations of classical and quantum physics there is still a distinctive feature that involves with mass that is needed to be clarified. That is it is not quantised, even though mass seems to exist in discrete amounts associated with elementary particles that form the basis for the construction of all physical objects. In this work we will discuss a geometric and topological quantisation of mass by extending our work on the principle of least action in which the quantisation of the angular momentum of a Bohr hydrogen atom and the charge of an elementary particle can be shown to be quantised from a continuous deformation of a differentiable manifold. Similar to the case of the quantisation of charge in which the two-dimensional Ricci scalar curvature of surface is invoked, we show that in the case of three-dimensional differentiable manifolds we can apply the Yamabe problem, which states that any Riemannian metric on a compact smooth manifold of dimension greater or equal to three is conformal to a metric with constant scalar curvature, to show that mass can also be quantised by the deformation of differentiable manifolds. The Yamabe problem is a generalisation of the uniformisation theorem for two-dimensional differentiable manifolds. As a further discussion, we consider whether quantum particles can be expressed as direct sums of quantum masses when they possess the mathematical structures of differentiable manifolds with the quantum masses are regarded as prime manifolds. This may be regarded as a physical manifestation of an established mathematical proposition in differential geometry and topology that states that any compact, connected, and orientable differentiable manifold M can be decomposed into prime manifolds P_i as $M = P_1 \# \dots \# P_n$, and the decomposition is unique up to an absorption or emission of 3-spheres S^3 [2]. It is interesting to observe that the picture of decomposition of differentiable manifolds into prime manifolds and the radiation of 3-spheres is similar to the radiation of the quanta of physical fields from a quantum system, such as the radiation of photons from a hydrogen atom. However, the main focus in this work is to show that if quantum particles are endowed with geometric and topological structures of differentiable manifolds, then their physical property manifests as mass can also be quantised. For completeness, we first outline how physical entities such as angular momentum and charge can be quantised from a deformation of differentiable manifolds [3].

In the case of the geometric and topological quantisation of angular momentum we consider Bohr's planar model of a hydrogen-like atom. As shown in our work on the principle of least action, from the Frenet system of equations in differential geometry, we are able to establish a relationship between the momentum p of a quantum particle and the curvature κ of its path through the relation

$$p = \hbar\kappa \tag{1}$$

The relationship given in Equation (1) can be shown to lead to the Bohr's postulate of the quantisation of angular momentum. According to the canonical formulation of classical physics, the particle dynamics is governed by the action principle $\delta S = \delta \int p ds = 0$. Using the relationship $p = \hbar\kappa$ and the expression of the curvature of a path $f(x)$ in a plane, $\kappa = f''/(1 + f'^2)^{3/2}$, the action integral $S = \int p ds$ takes the form $S = \int \hbar\kappa ds = \int \hbar f''/(1 + f'^2)^{3/2}\, dx$. It is shown in the calculus of variations that to extremise the integral $S = \int L(f, f', f'', x) dx$, the function $f(x)$ must satisfy the differential equation [4]

$$\frac{\partial L}{\partial f} - \frac{d}{dx}\frac{\partial L}{\partial f'} + \frac{d^2}{dx^2}\frac{\partial L}{\partial f''} = 0 \tag{2}$$

However, with the functional of the form $L = \hbar f''/(1 + f'^2)$, it is straightforward to verify that the differential equation (2) is satisfied by any function $f(x)$. This result may be considered as a foundation for the Feynman's path integral formulation of quantum mechanics, which uses all classical trajectories of a particle in order to calculate the transition amplitude of a quantum mechanical system. Since any path can be taken by a particle moving in a plane, if the orbits of the particle are closed, it is possible to represent each class of paths of the fundamental homotopy group of the particle by a circular path, since topologically, any path in the same equivalence class can be deformed continuously into a circular path. This validates Bohr's assumption of circular motion for the electron in a hydrogen-like atom. This assumption then leads immediately to the Bohr quantum condition

$$\oint p ds = \hbar \oint \kappa ds = \hbar \oint \frac{ds}{r} = \hbar \oint d\theta = nh \tag{3}$$

The Bohr quantum condition possesses a topological character in the sense that the principal quantum number n is identified with the winding number, which is used to represent the fundamental homotopy group of paths of the electron in the hydrogen atom.

In order to formulate a geometric and topological quantisation of charge, we need to extend Feynman's method of sum over random paths to sum over hypersurfaces in higher-dimensional spaces so that we can formulate physical theories in which the transition amplitude between states of a quantum mechanical system can be determined from such sum. For the case of two-dimensional differentiable manifolds, the method is simply the sum over random surfaces in three-dimensional Euclidean space. Consider a surface defined by the relation $x^3 = f(x^1, x^2)$ in terms of the Cartesian coordinates (x^1, x^2, x^3). The Gaussian curvature is given by $K = (f_{11}f_{22} - (f_{12})^2)/(1 + f_1^2 + f_2^2)^2$, where $f_\mu = \partial f/\partial x^\mu$ and $f_{\mu\nu} = \partial^2 f/\partial x^\mu \partial x^\nu$. Let P be a 3-dimensional physical quantity which plays the role of the momentum p in the 2-dimensional space action integral. The quantity P can be identified with the surface density of a physical quantity, such as charge. Since the momentum p is proportional to the curvature κ, which determines the planar path of a particle, it is seen that in the 3-dimensional space the quantity P should be proportional to the Gaussian curvature K, which is used to characterise a surface. If we consider a surface action integral of the form $S = \int PdA = \int (q/4\pi)\, KdA$, where q is a universal constant, then we have $S = (q/2\pi) \int \left((f_{11}f_{22} - (f_{12})^2)/\left((1 + f_1^2 + f_2^2)^{3/2}\right) \right) dx^1 dx^2$. According to the calculus of variations, similar to the case of path integral, to extremise the action integral $S = \int L(f, f_\mu, f_{\mu\nu}, x^\mu) dx^1 dx^2$, the functional $L(f, f_\mu, f_{\mu\nu}, x^\mu)$ must satisfy the differential equations [5]

$$\frac{\partial L}{\partial f} - \frac{\partial}{\partial x^\mu}\frac{\partial L}{\partial f_\mu} + \frac{\partial^2}{\partial x^\mu \partial x^\nu}\frac{\partial L}{\partial f_{\mu\nu}} = 0 \tag{4}$$

Also as in the case of path integral, it is straightforward to verify that with the functional of the form $L = (q/2\pi)\, (f_{11}f_{22} - (f_{12})^2)/(1 + f_1^2 + f_2^2)^{3/2}$ the differential equations given by Equation (4) are satisfied by any surface. Hence, we can generalise Feynman's postulate to formulate a quantum theory in which the transition amplitude between states of a quantum mechanical system is a sum over random surfaces, provided the functional P in the action integral $S = \int PdA$ is taken to be proportional to the Gaussian curvature K of a surface. Consider a closed surface and assume that we have many such different surfaces which are described by the higher dimensional homotopy groups. As in the case of the fundamental homotopy group of paths, we choose from among the homotopy class a representative spherical surface, in which case we can write

$$\oint (q/4\pi) K dx^1 dx^2 = \frac{q}{4\pi} \oint d\Omega = nq \tag{5}$$

where $d\Omega$ is an element of solid angle. Since $\oint d\Omega$ depends on the homotopy class of the sphere that it represents, we have $\oint d\Omega = 4\pi n$, where n is the topological winding number of the higher dimensional homotopy group. This result can be regarded as a generalised Bohr quantum condition. From the result obtained in Equation (5), as in the case of Bohr's theory of quantum mechanics, we may consider a quantum process in which a physical entity transits from one surface to another with some radiation-like quantum created in the process. Since this kind of physical process can be considered as a transition from one homotopy class to another, the radiation-like quantum may be the result of a change of the topological structure of the physical system, and so it can be regarded as a topological effect. Furthermore, it is interesting to note that the action integral $(q/4\pi) \oint KdA$ is identical to Gauss's law in electrodynamics. In this case the constant q can be identified with the charge of a particle. In this case the charge q represents the topological structure of a physical system, and must exist in multiples of q. Hence, the charge of a physical system, such as an elementary particle, may depend on the topological structure of the system and is classified by the homotopy group of closed surfaces. This result may shed some light on why charge is quantised even in classical physics.

From the outline of the geometric and topological method of quantisation for the angular momentum and the charge of an elementary particle, we now extend our discussion to the case of quantum particles which are considered as differentiable manifolds and we show that masses can also be quantised as a result of a deformation of differentiable manifolds and the geometrical object that involves is also the Ricci scalar curvature. We anticipate that if for two-dimensional manifolds the Ricci scalar curvature is used to represent the charge of a quantum particle then in three-dimensional manifolds it should be associated with mass. In fact, a convincing reason for such identification comes from our works on the spacetime structures of quantum particles which showed that geometrical substances can also be described in terms of the Ricci scalar curvature obtained from the Riemannian tensor defined on differentiable manifolds [6-9]. We have shown that the three main dynamical descriptions of physical events in classical physics, namely Newton mechanics, Maxwell electromagnetism and Einstein gravitation, can be formulated in the same general covariant form and they can be represented by the general equation $\nabla_\beta M = kJ$, where M is a mathematical object that represents the corresponding physical system, J is a four-current, k is a dimensional constant, and $\nabla_\beta$is a covariant derivative. For Newton mechanics, we have $M = E = \frac{1}{2}m\sum_{\mu=1}^{3}(dx^\mu/dt)^2 + V$. For Maxwell electromagnetism, $M = F^{\alpha\beta} = \partial^\mu A^\nu - \partial^\nu A^\mu$ with the four-vector potential $A^\mu \equiv (V, \mathbf{A})$. And for Einstein gravitation $M = R^{\alpha\beta}$, and in this case J can be defined in terms of a metric $g_{\alpha\beta}$ and the Ricci scalar curvature as $j^\alpha = \frac{1}{2}g^{\alpha\beta}\nabla_\beta R$. By comparing this current with the Poisson equation for a potential V in classical physics $\nabla^2 V = 4\pi\rho$, we can identify the scalar potential V with the Ricci scalar curvature R and then obtain a diffusion equation $\partial_t R = k\nabla^2 R$. Solutions to the diffusion equation can be found to take the form $R(x,y,z,t) = \left(M/\left(\sqrt{4\pi kt}\right)^3\right)e^{-\left(x^2+y^2+z^2\right)/(4kt)}$. The diffusion equation determines the probabilistic distribution of an amount of geometrical substance M which is defined via the Ricci scalar curvature R and manifests as observable matter. Let P be a dimensional physical quantity which determines the mass of a quantum particle then the quantity P should be proportional to the Ricci scalar curvature R, which is used to characterise the geometric and topological structure of a three-dimensional differentiable manifold associated with the quantum particle. If we consider an action integral of the form $S = \int PdV = \int(\mu/2\pi^2)\, Rdx^1dx^2dx^3$, where μ is a constant, then we would expect that this integral should be extremised by the method of calculus of variations and then it would satisfy the differential equation given in Equation (4) if the Ricci scalar curvature could be expressed in terms of a function as $R = L(f, f_\mu, f_{\mu\nu}, x^\mu)$. However, in the following, instead, we will apply the Yamabe problem on a deformation of Riemannian structures on compact manifolds to argue that the present situation should also be established similar to the case of using Gaussian curvature to quantise the charge of quantum particles as discussed above. Yamabe's and subsequent works in differential geometry have shown that every compact C^∞ Riemannian manifold of dimensions greater than or equal to three can be deformed conformally to a C^∞ Riemannian structure of constant scalar curvature [10-11]. Consider a closed hypersurface and assume that we have many such different hypersurfaces which are described by the higher dimensional homotopy groups. As in the case of the homotopy groups of paths and surfaces, we choose from among the higher dimensional homotopy class a representative spherical hypersurface, then the action integral can be rewritten as $\oint(\mu/2\pi^2)Rdx^1dx^2dx^3 = (\mu/2\pi^2)\oint d\Omega_4$, where $d\Omega_4$ is an element of solid angle and the integral $\oint d\Omega_4$ depends on the homotopy class of the hyperspheres that it represents. It is shown that the solid angle Ω_n subtended by the $(n-1)$-dimensional unit hypersphere in n-dimensional Euclidean space R^n is given by the formula $\Omega_n = 2\pi^{n/2}/\Gamma(n/2)$, and for the case of $n = 4$ we have $\Omega_4 = 2\pi^2$. Then we arrive at the quantisation of mass

$$\oint(\mu/2\pi^2)Rdx^1dx^2dx^3 = \frac{\mu}{2\pi^2}\oint d\Omega_4 = n\mu \tag{6}$$

where n is the topological winding number of the corresponding homotopy group.

Since quantum particles are considered as differentiable manifolds, which are assumed to be formed from mass-points joined together by contact forces, we may suggest that the quantum mass μ is the amount of mass-points required to form an elementary particle. This leads to a further suggestion that the most fundamental mass must be the mass of a mass-point. Another feature that is related to mass-points is that although they should be defined in terms of a mathematical property of a differentiable manifold, physically it can be seen as an inertial property associated with the motion of a physical body in space. Furthermore, if elementary particles are endowed with the geometric and topological structures of differentiable manifolds then we may also suggest that mass-points join together to form topological structures with particular patterns that give rise to the quanta of mass, therefore the problem of geometric and topological quantisation of mass reduces to the problem of forming quantum geometric and topological structures of differentiable manifolds. Then from these quantum masses endowed with prime geometric and topological structures all other quantum particles should be expressed as their direct sums. Despite we seem to only have the ability to perceive that we exist as three-dimensional physical objects that reside on the boundary, which is a 3D solid ball, of a 3-sphere, on the microscopic scale quantum particles could still possess the geometric and topological structures of three-dimensional differentiable manifolds, therefore we may suggest that quantum particles are formed as direct sums of the quantum masses which have the prime geometric and topological structures, including those of a 3-sphere. It should be mentioned here that a 3-sphere S^3 can also be topologically constructed from two solid tori $S^1 \times E^2$, where S^1 is a circle and $E^2 = \{(x,y) \in R^2 : x^2 + y^2 \leq r^2\}$ [12]. It is interesting to note that the problem of constructing quantum particles from the quantum masses may be related to the problem of hypersphere packing to form the configuration space for a three-dimensional manifold, and remarkably it may also be related to the problem of hypersphere packings in information theory [13-14]. Now, as an example of a decomposition of a differentiable manifold into prime manifolds we consider a hydrogen atom in four spatial dimensions. We have shown that quanta of the electromagnetic field may exist as 4D physical objects, therefore the radiation of such physical objects would require physical structures of four spatial dimensions. In particular, the radiation of quanta of the electromagnetic field from a four-dimensional hydrogen atom is similar to the decomposition of differentiable manifolds into prime manifolds with an absorption or emission of 3-spheres. If we regard a hydrogen atom consisting of a proton and an electron as a three-dimensional differentiable manifold in which the proton and the electron are observed as two 3D objects which form the boundary of the hydrogen manifold then we can use a four-dimensional Schrödinger wave equation to describe its quantum dynamics. Since both Bohr model and Schrödinger model in spaces of different dimensions use the same form of the potential for the hydrogen atom, we may adopt the same form of potential in four-dimensional Euclidean space to describe a four-dimensional hydrogen atom. To describe the wave dynamics on a hypersurface embedded or immersed in four-dimensional Euclidean space R^4 we need a four-dimensional time-independent Schrödinger wave equation of the form

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi(\mathbf{r})-\frac{kq^2}{r}\psi(\mathbf{r})=E\psi(\mathbf{r}) \tag{7}$$

Consider a d-dimensional hypersphere S_r^d of radius r embedded in the ambient $(d+1)$-dimensional Euclidean space R^{d+1}. If spherical coordinates $(r,\theta,,\theta_1,\dots,\theta_{d-2},\phi)$ are defined in terms of the Cartesian coordinates $(x_1,x_2,\dots,x_{d+1})$ as $x_1=r\cos\theta$, $x_2=r\sin\theta\cos\theta_1$,..., $x_{d+1}=r\sin\theta\dots\sin\theta_{d-2}\sin\phi$ then the Laplacian $\nabla_{S^d}^2$ on the hypersphere S_r^d is given as $\nabla_{S^d}^2\psi=(1/r^2)\left(\partial^2\psi/\partial\theta^2+(d-1)\cot\theta\,\partial\psi/\partial\theta+(1/\sin^2\theta)\nabla_{S^{d-1}}^2\psi\right)$ [15]. For the case of a 3-sphere S^3 embedded in four-dimensional Euclidean space R^4, the four-dimensional time-independent Schrödinger wave equation becomes

$$-\frac{\hbar^2}{2\mu}\left(\frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2}+2\cot\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\nabla_{S^2}^2\right)\right)\psi-\frac{kq^2}{r}\psi=E\psi \tag{8}$$

Even though rigorous solutions to Equation (8) cannot be obtained, we can still discuss the possibility to observe the total energy contained in the system. If we consider the four-dimensional region bounded by the proton and the electron as a 3-sphere of radius r then the four-dimensional volume is given as $V=\pi^2r^4/2$. As discussed in our work on the Olbers paradox, there is a chance that there is trapped energy inside the 3-sphere and this energy, even though which cannot be observed and measured because it is associated with the fourth dimension, may be required to balance the hydrogen atom as a stable physical system. However, when the atom absorbs a photon the system becomes unstable which results in the expansion of the 3-sphere, which causes the system to change its stable state in the form of a deformation of the region of the manifold associated with the electron. Despite the four-dimensional trapped energy E_D cannot be observed or measured, it can be calculated as follows. If ρ_D is a four-dimensional volume energy density then the trapped energy E_D stored in the 3-sphere is given by $E_D=\rho_D V$. Let E is the observable energy that is responsible for the supposed rate of expansion of the observable atom. However, as in the case of the accelerating expansion of the observable universe, if we are able to observe that the rate of expansion is greater than the expected rate that caused by the observable energy E then we would speculate that there must exist some form of unknown energy resides inside the atom. If we let σ is the ratio between the unknown energy E_D and the observed energy E, i.e., $E_D=\sigma E$, then the total energy would be $E_T=2(E+E_D)=2(1+\sigma)E$. If R is the radius of the 3-sphere that is formed by the hydrogen atom then the four-dimensional energy density ρ_D is given as

$$\rho_D = \frac{4(1+\sigma)E}{\pi^2 R^4} \tag{9}$$

The radius R of the 3-sphere S^3 can be determined from a physical theory such as Einstein theory of general relativity $R_{\alpha\beta} - (1/2)g_{\alpha\beta}R - \Lambda g_{\alpha\beta} = --(8\pi G/c^4)T_{\alpha\beta}$, together with the Robertson-Walker metric $ds^2 = c^2dt^2 - R^2(t)\big((1/(1-kr^2))dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\big)$. Furthermore, if we assume an energy-momentum tensor of the form $T_{\alpha\beta} = diag(-g_{ii}p, c^2\rho)$ for the Einstein field equations then the radius R can be shown to satisfy the following system of equations [16]

$$\frac{2}{R}\frac{d^2R}{dt^2} + \frac{1}{R^2}\left(\frac{dR}{dt}\right)^2 + \frac{kc^2}{R^2} - \Lambda c^2 = -\frac{8\pi G\rho}{c^4} \tag{10}$$

$$\frac{1}{R^2}\left(\frac{dR}{dt}\right)^2 + \frac{kc^2}{R^2} - \frac{\Lambda c^2}{3} = -\frac{8\pi G\rho}{3} \tag{11}$$

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