On the equation: $x^x - x - 1 = 0$

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Abstract . This note presents some remarks on the equation:

$$x^x - x - 1 = 0 \quad , x > 0$$

Resumen . Esta nota muestra algunas observaciones sobre la ecuación:

$$x^x - x - 1 = 0 \quad , x > 0$$

1. Introducción

Para $x > 0$, la única raíz de la ecuación: $x^x - x - 1 = 0$ es:

$$x = \alpha = 1.77677504009705469747973074 4... \tag{1}$$

En esta nota mostramos algunas fórmulas iterativas relacionadas con el número α .

2. Fórmulas

Entry 1.

$$x_{n+1} = (1 + x_n)^{1/x_n} \quad , x_1 = 2 \Rightarrow x_n \to \alpha \tag{2}$$

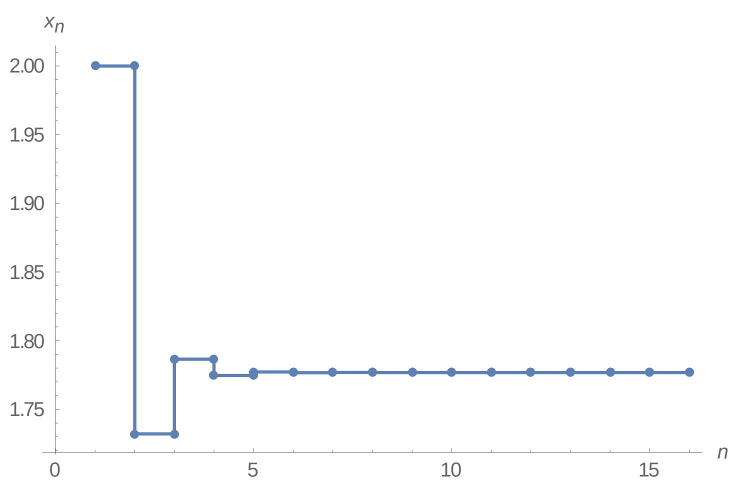

Figura 1.

- Si $f(x)=(1+x)^{1/x}$, entonces $f'(x)=(1+x)^{1/x}\left(\frac{1}{x(1+x)}-\frac{\ln(1+x)}{x^2}\right)$.

- Si $[a,b]=\left[\frac{3}{2},\frac{5}{2}\right]$, entonces se tiene:

$$|f'(x)|<1 \quad \forall\, x\in[a,b]=\left[\frac{3}{2},\frac{5}{2}\right] \tag{3}$$

$$\frac{3}{2}\le x\le\frac{5}{2}\Rightarrow\frac{3}{2}\le f(x)\le\frac{5}{2} \tag{4}$$

$$\lambda=\max_{a\le x\le b}|f'(x)|=0.25893... \tag{5}$$

$$|\alpha-x_n|\le\lambda^{n-1}\cdot 0.36157 \quad ,n\in\mathbb{N} \tag{6}$$

Entry 2.

$$x_{n+1}=2-\sqrt{(3-x_n)(2-x_n)^{x_n}} \quad ,x_1=1/4\Rightarrow x_n\to 2-\alpha \tag{7}$$

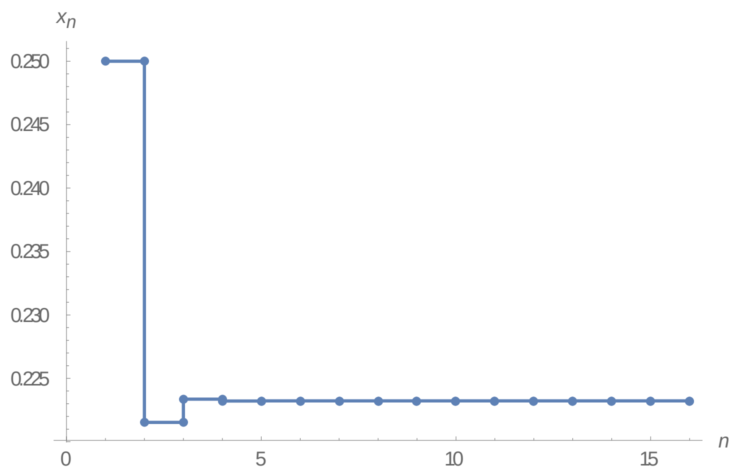

Figura 2.

- Si $f(x)=2-\sqrt{(3-x)(2-x)^x}$, entonces

$$f'(x)=-\frac{-(2-x)^x+(2-x)^x(3-x)\left(-\frac{x}{2-x}+\ln(2-x)\right)}{2\sqrt{(2-x)^x(3-x)}}$$

- Si $[a,b]=\left[0,\frac{1}{2}\right]$, entonces se tiene:

$$\left|f'(x)\right|<1 \ \ \forall x\in[a,b]=\left[0,\frac{1}{2}\right] \tag{8}$$

$$0\le x\le\frac{1}{2}\Rightarrow 0\le f(x)\le\frac{1}{2} \tag{9}$$

$$\lambda=\max_{a\le x\le b}\left|f'(x)\right|=0.311608... \tag{10}$$

$$\left|2-\alpha-x_n\right|\le\lambda^{n-1}\cdot 0.041354 \quad ,n\in\mathbb{N} \tag{11}$$

Entry 3.

$$x_{n+1}=\left(1+\frac{1}{x_n}\right)^{-x_n} \quad ,x_1=1/2\Rightarrow x_n\to\frac{1}{\alpha} \tag{12}$$

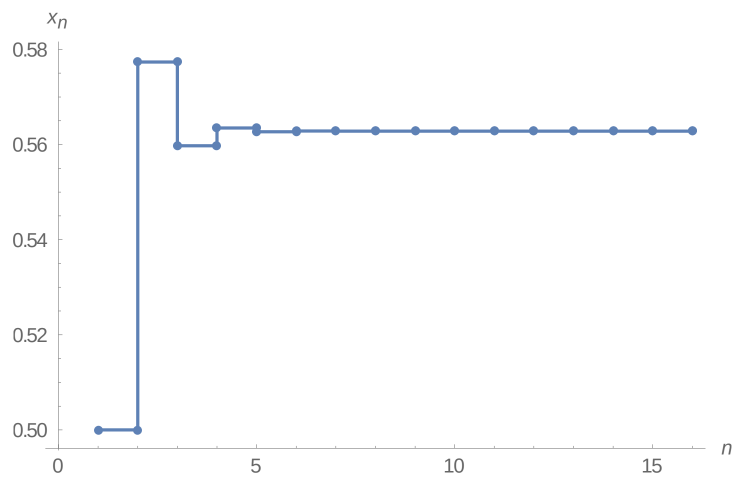

Figura 3.

- Si $f(x)=\left(1+\frac{1}{x}\right)^{-x}$, entonces $f'(x)=\left(1+\frac{1}{x}\right)^{-x}\left(\frac{1}{1+x}-\ln\left(1+\frac{1}{x}\right)\right)$.
- Si $[a,b]=\left[\frac{1}{3},1\right]$, entonces se tiene:

$$\left|f'(x)\right|<1 \quad \forall x\in[a,b]=\left[\frac{1}{3},1\right] \tag{13}$$

$$\frac{1}{3}\le x\le 1\Rightarrow\frac{1}{3}\le f(x)\le 1 \tag{14}$$

$$\lambda=\max_{a\le x\le b}\left|f'(x)\right|=0.40084... \tag{15}$$

$$\left|\frac{1}{\alpha}-x_n\right|\le\lambda^{n-1}\cdot 0.12909 \quad ,n\in\mathbb{N} \tag{16}$$

Entry 4.

$$x_{n+1}=e^{-x_n}\ln\left(1+e^{x_n}\right)\quad ,x_1=0\Rightarrow x_n\to\ln\alpha \tag{17}$$

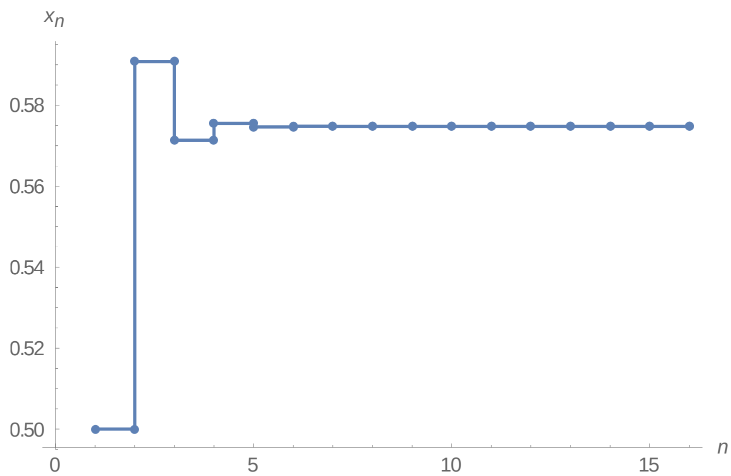

Figura 4.

- Si $f\left(x\right)=e^{-x}\ln\left(1+e^{x}\right)$, entonces $f'\left(x\right)=\dfrac{1}{1+e^{x}}-e^{-x}\ln\left(1+e^{x}\right)$.
- Si $\left[a,b\right]=\left[0,1\right]$, entonces se tiene:

$$\left|f'\left(x\right)\right|<1\quad\forall\, x\in\left[a,b\right]=\left[0,1\right] \tag{18}$$

$$0\le x\le 1\Rightarrow 0\le f\left(x\right)\le 1 \tag{19}$$

$$\lambda=\max_{a\le x\le b}\left|f'\left(x\right)\right|=0.216217... \tag{20}$$

$$\left|\ln\alpha-x_n\right|\le\lambda^{n-1}\cdot 0.115858\quad ,n\in\mathbb{N} \tag{21}$$

3. Observaciones Finales

Entry 5.

$$x^x-x-1=0\Rightarrow x^x=1+x\Rightarrow x=\left(1+x\right)^{1/x} \tag{22}$$

Entry 6.

$$x^x-x-1=0\wedge x=\frac{1}{y}\Rightarrow\left(1/y\right)^{1/y}=1+\frac{1}{y}\Rightarrow y=\left(1+\frac{1}{y}\right)^{-y} \tag{23}$$

Entry 7.

$$\begin{aligned} x^x - x - 1 = 0 \wedge x = 2 - y &\Rightarrow (2-y)^{2-y} = 3 - y \\ &\Rightarrow (2-y)^2 = (3-y)(2-y)^y \\ &\Rightarrow 2 - y = \sqrt{(3-y)(2-y)^y} \\ &\Rightarrow y = 2 - \sqrt{(3-y)(2-y)^y} \end{aligned} \tag{24}$$

Entry 8.

$$\begin{aligned} x^x - x - 1 = 0 \wedge y = \ln x &\Rightarrow x \ln x = \ln(1+x) \wedge x = e^y \\ &\Rightarrow y e^y = \ln(1+e^y) \\ &\Rightarrow y = e^{-y} \ln(1+e^y) \end{aligned} \tag{25}$$

Referencias

1. Frank W.J. Olver , Daniel W. Lozier , Ronald F. Boisvert , and Charles W. Clark : NIST Handbook of Mathematical Functions, Cambridge University Press , 2010.
2. Siegel, C.L.: Iteration of analytic functions. Ann. Math. (2) 43, 1942.