Energy Stored in the Gravitational Field

V. Nardozza*

May. 2019†

Abstract

We evaluate the energy of the gravitational field.

Key Words: Newtonian Gravity.

1 Introduction

The attractive force between two charges is given by:

$$F = \frac{1}{4\pi\epsilon}\frac{q_1 q_2}{d^2} \tag{1}$$

By definition the electric field e is given by:

$$e = \frac{F}{q} \tag{2}$$

where F is the force experienced by a probe charge q. F and e are vectors. The energy density stored in the electric field is given by:

$$\hat{E} = \frac{1}{2}\epsilon |e|^2 \tag{3}$$

We turn now our attention to the Newtonian gravitational field. The attractive force between two masses is given by:

$$F = G\frac{m_1 m_2}{d^2} \tag{4}$$

By definition the gravitational field g is given by:

$$g = \frac{F}{m} \tag{5}$$

where F is the force experienced by a probe mass m. Once again F and g are vectors. The field g has units of acceleration and it is in fact the acceleration of the probe mass m if free to move.

*Electronic Engineer (MSc). Lancashire, UK. mailto: vinardo@nardozza.eu

†Posted at: http://vixra.org/abs/1905.0515 - Current version: v2 - Nov. 2019

By analogy with the electric field we feel safe to say that the energy density stored in the gravitational field is given by:

$$\hat{E} = \frac{1}{2}\left(\frac{1}{4\pi G}\right)|g|^2 \tag{6}$$

We should be satisfied by the above equation at this point! However, we want to prove the above equation by a full calculation method because it is instructive anyway.

To do that, we will assume that the energy density of the gravitational field is given by:

$$\hat{E} = \frac{1}{2}\Omega|g|^2 \tag{7}$$

In the next paragraph we will evaluate Ω.

2 Evaluation of Ω

Given Fig. 1, the potential of U the gravitational field of two point masses m_1 and m_2 is given by:

$$U = -G\left(\frac{m_2}{r_1} + \frac{m_1}{r_2}\right) \tag{8}$$

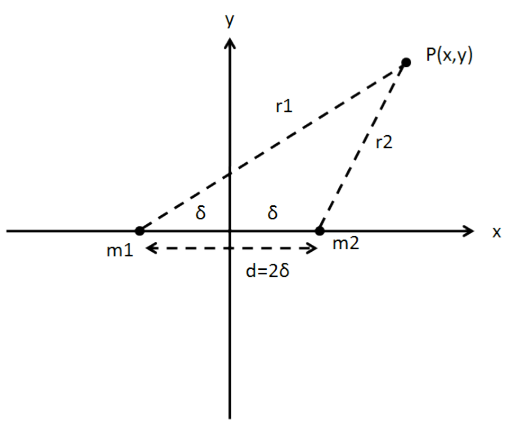

Figure 1: Definition of r_1 and r_2

with $d = 2\delta$ and:

$$r_1 = \sqrt{(x-\delta)^2 + y^2} \tag{9}$$

$$r_2 = \sqrt{(x+\delta)^2 + y^2} \tag{10}$$

we have:

$$U = -G\left(\frac{m_2}{\sqrt{(x+\delta)^2 + y^2}} + \frac{m_1}{\sqrt{(x-\delta)^2 + y^2}}\right) \tag{11}$$

The gravitational field is given by $g = -\nabla U$. The components of g are therefore:

$$g_x = -\frac{\partial U}{\partial x} = -G\left(\frac{m_2(x+\delta)}{\left((x+\delta)^2+y^2\right)^{\frac{3}{2}}} + \frac{m_1(x-\delta)}{\left((x-\delta)^2+y^2\right)^{\frac{3}{2}}}\right) \tag{12}$$

and:

$$g_y = -\frac{\partial U}{\partial y} = -G\left(\frac{m_2 y}{\left(y^2+(x+\delta)^2\right)^{\frac{3}{2}}} + \frac{m_1 y}{\left(y^2+(x-\delta)^2\right)^{\frac{3}{2}}}\right) \tag{13}$$

and we have:

$$|g|^2 = G^2\left[\left(\frac{m_2 y}{\left(y^2+(x+\delta)^2\right)^{\frac{3}{2}}} + \frac{m_1 y}{\left(y^2+(x-\delta)^2\right)^{\frac{3}{2}}}\right)^2 + \left(\frac{m_2(x+\delta)}{\left(y^2+(x+\delta)^2\right)^{\frac{3}{2}}} + \frac{m_1(x-\delta)}{\left(y^2+(x-\delta)^2\right)^{\frac{3}{2}}}\right)^2\right] \tag{14}$$

Let Γ to be the semi-plane of the (x, y) plane with $y > 0$. We have that:

$$E = \frac{1}{2}\Omega\int_V |g|^2 dV = \frac{1}{2}\Omega\int_V |g(x,y)|^2 dx\,dy\,y d\theta$$

$$= \frac{G^2}{2}\Omega\int_0^{2\pi} d\theta\int_\Gamma \left[\frac{y|g(x,y|^2}{G^2}\right] dx\,dy = \pi\Omega G^2\int_\Gamma \Lambda(\delta)\,dx\,dy \tag{15}$$

where:

$$\Lambda(x,y,\delta) = \frac{y|g(x,y|^2}{G^2} dx\,dy \tag{16}$$

With some manipulation we have:

$$\Lambda = \frac{m_1^2 y}{\left(y^2+(x-\delta)^2\right)^2} + \frac{2m_1 m_2 y\left(y^2+x^2-\delta^2\right)}{\left(y^2+(x-\delta)^2\right)^{\frac{3}{2}}\left(y^2+(x+\delta)^2\right)^{\frac{3}{2}}} + \frac{m_2^2 y}{\left(y^2+(x+\delta)^2\right)^2}$$

$$= m_1^2\lambda_1 + 2m_1 m_2\lambda_{12} + m_2^2\lambda_2 \tag{17}$$

with obvious meaning of the λ symbols.

This result was expected because the energy given by the field $|u_1 + u_2|$ generated by the two masses has two components depending by the two masses and a cross-component. The two components depending by the masses should not depend on $d = 2\delta$. This is obvious by the fact that with a simple change of coordinates which has no effect on dx we can make the parameter δ to disappear. If we try to evaluate the integrals relevant to λ_1 and λ_2 they diverge:

$$\int_\Gamma \lambda_{1,2}\,dx\,dy = \int_\Gamma \frac{m_1^2 y}{\left(y^2+(x\pm\delta)^2\right)^2}\,dx\,dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{2x^2} dx = \infty \tag{18}$$

This is also expected since the energy associated with a point mass is infinite. However they do not depend on $d = 2\delta$ and therefore since we are interested by the derivative of the energy with respect to d, as it will be clear later, we can ignore them for the purpose of our calculations. We are more interested in the mixed term. To evaluate the integral of $\lambda_{1,2}$ it helps to assume that $x > 0$. This is not a problem since we know from the problem at hand that the energy is an even function with respect to x and therefore we can evaluate it in one quadrant and double the result. We have:

$$\int_\Gamma \lambda_{12}\, dx\, dy = 2 \int_0^\infty dx \int_0^\infty \lambda_{12}\, dx\, dy$$

$$= 2 \int_0^\infty dx \int_0^\infty \frac{y\left(y^2 + x^2 - \delta^2\right)}{\left(y^2 + (x-\delta)^2\right)^{\frac{3}{2}} \left(y^2 + (x+\delta)^2\right)^{\frac{3}{2}}} dy$$

$$= 2 \int_0^\infty \frac{|x-\delta| + x - \delta}{4x^2\,(x-\delta)} dx$$

$$= 2 \int_0^\delta 0\, dx + 2 \int_\delta^\infty \frac{1}{2x^2} dx = \frac{1}{\delta} = \frac{2}{d} \tag{19}$$

If we do not take into account the terms that go to infinite we have therefore:

$$E = \pi \Omega G^2 \int_\Gamma \Lambda(\delta)\, dx\, dy = \pi \Omega G^2 2 m_1 m_2 \int_\Gamma \lambda_{12}\, dx\, dy \tag{20}$$

which is:

$$E = \frac{4\pi\Omega G^2 m_1 m_2}{d} \tag{21}$$

Moreover:

$$F = -\frac{\partial E}{\partial d} = \frac{4\pi\Omega G^2 m_1 m_2}{d^2} \tag{22}$$

By equating the above with the attractive force between two masses:

$$F = G\frac{m_1 m_2}{d^2} = \frac{4\pi\Omega G^2 m_1 m_2}{d^2} \tag{23}$$

we find:

$$\Omega = \frac{1}{4\pi G} \tag{24}$$

As expected. Note that we get a $8\pi G$ factor that is the same present in Einstein field equations and that relate gravitational fields and sources.