Quantum chromodynamics on lattice: state-of-the-art and new methods with new results

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Abstract

This paper consists of two parts.

Part A is a state-of-the-art report in quantum chromodynamics.
Here is presented in a concise form:
the QCD gauge-theory, the standard model and its particles, the perturbative QCD (QCD/QED Feynman diagrams with results), QCD on-lattice with Wilson loops.

Part B describes a new numerical QCD calculation method (direct minimization of QCD-QED-action) and its results for the first-generation (u,d) hadrons.
Here we start with the standard color-Lagrangian LQCD=LDirac+Lgluon , model the quarks q_i as parameterized gaussians, and the gluons Ag_i as Ritz-Galerkin-expansion.
We minimize the Lagrangian with parameters $par=(par(q),\{\alpha_k\},par(Ag))$ for first-generation hadrons (nucleons, pseudo-scalar mesons, vector mesons).
The resulting parameters yield the correct masses, correct magnetic moments for the nucleons, the gluon-distribution and the quark-distribution with interesting insights into the hadron structure.

Part A Quantum chromodynamics theory

1. QCD gauge theory

Gauge theory

[1]

The gauge invariant QCD Lagrangian is

$$\boxed{\mathcal{L}_{\mathrm{QCD}} = \bar{\psi}_i \left(i(\gamma^\mu D_\mu)_{ij} - m\,\delta_{ij} \right) \psi_j - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a}$$

where $\psi_i(x)$ is the quark field, a dynamical function of spacetime, in the fundamental representation of the SU(3) gauge group, indexed by $i, j, \ldots$; $A^a_\mu(x)$ are the (color) gluon fields, also dynamical functions of spacetime, in the adjoint representation of the SU(3) gauge group, indexed by *a*, *b*,... The γ^μ are Dirac matrices connecting the spinor representation to the vector representation of the Lorentz group.

The total gluon-field is $A_\mu(x) \equiv A^a_\mu(x) \cdot \lambda_a/2$; and the Dirac-conjugate $\bar{\psi}_i(x) = \psi_i^c(x)\gamma^0$, where ψ_i^c is the complex-conjugate.

D_μ is the gauge covariant derivative $D_\mu := \partial_\mu - ig\, A^a_\mu\, \lambda_a$

where is the coupling constant, $A_\mu^a(x)$ is the (color) gluon gauge field, for eight different gluons is a four-component Dirac spinor, and where is one of the eight Gell-Mann matrices,

The symbol $G^a_{\mu\nu}$ represents the gauge invariant gluon field strength tensor, analogous to the electromagnetic field strength tensor, $F^{\mu\nu}$, in quantum electrodynamics. It is given by

$$G^a{}_{\mu\nu} = \partial_\mu A^a{}_\nu - \partial_\nu A^a{}_\mu + g\ f^{abc} A^b{}_\mu A^c{}_\nu$$

where f_{abc} are the structure constants of SU(3) : the generators T^a satisfy the commutator relations

$$[T^a, T^b] = i\ f^{abc} T^c$$

Note that the rules to move-up or pull-down the *a*, *b*, or *c* indexes are *trivial*, (+, ..., +), so that $f^{abc} = f_{abc} = f^a{}_{bc}$ whereas for the μ or ν indexes one has the non-trivial *relativistic* rules, corresponding e.g. to the metric signature (+ − − −).

The constants *m* and *g* control the quark mass and coupling constants of the theory, subject to renormalization in the full quantum theory.

Yang-Mills theory

Yang–Mills theories are a special example of gauge theory with a non-commutative symmetry group given by the Lagrangian

$$\mathcal{L}_{\mathrm{gf}} = -\frac{1}{2}\,\mathrm{Tr}(F^2) = -\frac{1}{4} F^{a\mu\nu} F^a_{\mu\nu}$$

, where for QCD with SU(3) $F = G^a{}_{\mu\nu}$

with the generators of the Lie algebra, indexed by *a*, corresponding to the *F*-quantities (the curvature or field-strength form) satisfying

$$\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}, \quad [T^a, T^b] = i f^{abc} T^c,$$

where the f^{abc} are structure constants of the Lie algebra, and the covariant derivative defined as

$$D_\mu = I\partial_\mu - igT^a A^a_\mu$$

where *I* is the identity matrix (matching the size of the generators), $A^a{}_\mu$ is the vector potential, and *g* is the coupling constant. In four dimensions, the coupling constant *g* is a pure number and for a SU(*N*) group one has

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The relation

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$$

follows from the commutator for the covariant derivative D_μ

$$[D_\mu, D_\nu] = -igT^a F^a_{\mu\nu}.$$

The field has the property of being self-interacting and equations of motion that one obtains are said to be semilinear, as nonlinearities are both with and without derivatives. This means that one can manage this theory only by perturbation theory, with small nonlinearities.
From the given Lagrangian one can derive the equations of motion given by

$$\partial^\mu F^a_{\mu\nu} + g f^{abc} A^{\mu b} F^c_{\mu\nu} = 0.$$

(Yang-Mills-equations), which correspond to the Maxwell equations in electrodynamics, where $f^{abc} = 0$
Putting these can be rewritten as

$$(D^\mu F_{\mu\nu})^a = 0.$$

The Bianchi identity holds

$$(D_\mu F_{\nu\kappa})^a + (D_\kappa F_{\mu\nu})^a + (D_\nu F_{\kappa\mu})^a = 0$$

which is equivalent to the Jacobi identity

$$[D_\mu, [D_\nu, D_\kappa]] + [D_\kappa, [D_\mu, D_\nu]] + [D_\nu, [D_\kappa, D_\mu]] = 0$$

for Lie-groups since Define the dual strength tensor $\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ then the Bianchi identity can be rewritten as

$$D_\mu \tilde{F}^{\mu\nu} = 0.$$

A source current $J^a{}_\nu$ enters into the equations of motion (eom) as

$$\partial^\mu F^a_{\mu\nu} + g f^{abc} A^{b\mu} F^c_{\mu\nu} = -J^a_\nu.$$

The Dirac part of the Lagrangian is

$$L_D = \overline{\psi}\left(i\hbar D_\mu \gamma^\mu - mc\right)\psi$$

with the resulting eom=gauge Dirac equation

$$\left(i\hbar D_\mu \gamma^\mu - mc\right)\psi = 0$$

Invariants

[6]
As usual, quantum states are characterized by their eigenvalues of a complete set of commuting observables. Out of the generators of the Poincare group, one can form the operator

$$P^2 = g_{\alpha\beta} P^\alpha P^\beta$$

that commutes with P^μ and $L^{\mu\nu}$. This can be checked explicitly and should also be intuitively clear since P^2 is a Lorentz-scalar. The eigenvalue of P^2 is the mass m^2.
A second commuting operator can be constructed using the Pauli-Lubanski-Vector

$$W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma} P^\nu M^{\rho\sigma}$$

that satisfies the commutator relations characteristic of a four vector:

$$[W^\mu, P^\nu] = 0,$$
$$[W^\mu, M^{\rho\sigma}] = \mathrm{i}\left(g^{\mu\rho} W^\sigma - g^{\mu\sigma} W^\rho\right).$$

As for P2, it follows that the operator W2 commutes with all generators of the Poincare group. In order to interpret W2, we use that it can be evaluated in any reference frame since it is a Lorentz scalar.

$$p_0^\mu = (m, \vec{0}).$$

$$W_0 = 0$$
$$W_i = \frac{m}{2}\epsilon_{ijk}L^{jk} = -mJ_i \qquad W^2 = -m^2\vec{J}^2$$

helicity

$$h = \frac{\vec{p}\cdot\vec{J}}{|\vec{p}|}$$

all eigenvalues for massive particles

$$P^2\,|m,p,s,s_p,\ldots\rangle = m^2\,|m,p,s,s_p,\ldots\rangle$$
$$P^\mu\,|m,p,s,s_p,\ldots\rangle = p^\mu\,|m,p,s,s_p,\ldots\rangle\ , \quad p^0 = \sqrt{\vec{p}^2+m^2}$$
$$W^2\,|m,p,s,s_p,\ldots\rangle = m^2 s(s+1)\,|m,p,s,s_p,\ldots\rangle$$
$$h\,|m,p,s,s_p,\ldots\rangle = s\,|m,p,s,s_p\ldots\rangle\ , \quad h = \frac{\vec{p}\cdot\vec{J}}{|\vec{p}|}$$

all eigenvalues for massless particles

$$W^2 = 0\,, \qquad W\cdot P = 0 \qquad W^\mu = hP^\mu$$

$$[h, P^\mu] = [h, M^{\mu\nu}] = 0$$

$$P^\mu\,|p,s,\ldots\rangle = p^\mu\,|p,s,\ldots\rangle\ , \quad p^0 = |\vec{p}|$$
$$h\,|p,s,\ldots\rangle = s\,|p,s,s\ldots\rangle\ , \qquad s = 0, \pm\frac{1}{2}, \pm 1, \ldots$$

Color gauge transformations [4, 1.1]

$$\psi_a \to\ e^{i\theta_C(x)t^C_{ab}}\psi_b$$
$$\mathcal{A}^C t^C \to\ e^{i\theta^D(x)t^D}\left(\mathcal{A}^C t^C - \frac{1}{g_s}\partial_\mu\theta^C(x)t^C\right)e^{-i\theta^E(x)t^E}$$

Dirac spinors

Spinors u, v: solution to Dirac equation in momentum space:

$$(\not{p} - m)u_\lambda(p) = 0$$
$$(\not{p} + m)v_\lambda(p) = 0$$

$$\not{p} = \gamma^\mu p_\mu$$

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

Dirac-representation (Bjorken-Drell) γ^μ

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Spinors can be chosen as helicity eigenstates:

$$h = \frac{\vec{p}\cdot\vec{S}}{|\vec{p}|} \qquad\qquad \vec{S} = \frac{1}{2}\begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix}$$

$$h\, u_{R/L}(p) = \pm\frac{1}{2}u_{R/L}(p)$$

$$h\, v_\lambda(p) = \lambda v_\lambda(p)$$

$\bar{u}_\sigma(p)\gamma^\mu u_{\sigma'}(p) = \bar{v}_\sigma(p)\gamma^\mu v_{\sigma'}(p) = 2p^\mu\,\delta_{\sigma\sigma'}$ where $\bar{u} = u^\dagger\gamma^0$

propagator

$$|\psi^{\lambda_1}_{k_1}, \psi^{\lambda_2}_{k_2}\rangle = -\,|\psi^{\lambda_2}_{k_2}, \psi^{\lambda_1}_{k_1}\rangle$$

$$\mathrm{i}S_F(x-y) = \langle 0|T[[\psi(x)\bar{\psi}(y)]]|0\rangle = \int\frac{\mathrm{d}^4k}{(2\pi)^4}e^{-ik(x-y)}S_F(p^2)$$

$$S_F(p^2) = \frac{\mathrm{i}(\not{p}+m)}{p^2-m^2+\mathrm{i}\epsilon}$$

Massless vector bosons

$$A^\mu(x) = \sum_\lambda\int\frac{\mathrm{d}^3p}{(2\pi)^3 2p^0}\left(a_\lambda(\vec{p})\epsilon^\mu_\lambda(p)e^{-ipx} + a^\dagger_\lambda(\vec{p})\epsilon^{\mu,*}_\lambda(p)e^{ipx}\right)$$

$$[a_\lambda(\vec{k}), a^\dagger_{\lambda'}(\vec{p})] = \delta_{\lambda,\lambda'}(2\pi)^3(2p^0)\delta^3(\vec{k}-\vec{p})$$

polarization and gauge

$$(\epsilon_\lambda(p)\cdot\epsilon^*_{\lambda'}(p)) = -\delta_{\lambda\lambda'}$$

$$\partial_\mu A^\mu = 0\ , \qquad p_\mu\epsilon^\mu = 0$$

In the frame where $p^\mu = (p,0,0,p)$ the polarization vectors can be chosen as

$$\epsilon^\mu_\pm = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\\pm\mathrm{i}\\0\end{pmatrix}$$

They are helicity eigenstates with eigenvalues s = ±1

$$h\epsilon^\mu_\pm = \frac{\vec{p}\cdot\vec{J}}{|\vec{p}|}\epsilon^\mu_\pm = J^3\epsilon^\mu_\pm = \begin{pmatrix}0&0&0&0\\0&0&-\mathrm{i}&0\\0&\mathrm{i}&0&0\\0&0&0&0\end{pmatrix}\epsilon^\mu_\pm = \pm\epsilon^\mu_\pm$$

$$\sum_\lambda\epsilon^\mu_\lambda\epsilon^{\nu*}_\lambda = \begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix} = -g^{\mu\nu} + n^\mu\bar{n}^\nu + \bar{n}^\mu n^\nu$$

with $\bar{n}^\mu = \frac{1}{\sqrt{2}}(1,0,0,1)\ ,\qquad n^\mu = \frac{1}{\sqrt{2}}(1,0,0,-1)\ ,\qquad n\cdot\bar{n} = 1$

the polarization vectors therefore form a complete basis of Minkowski space.

Weyl spinors

$$\langle pk\rangle \equiv \bar{u}_L(p)u_R(k) = u_-^\dagger(p)u_+(k) \quad [pk] \equiv \bar{u}_R(p)u_L(k) = u_+^\dagger(p)u_-(k)$$

$$u_R(p) = \begin{pmatrix} u_+(p) \\ 0 \end{pmatrix} \qquad u_L(p) = \begin{pmatrix} 0 \\ u_-(p) \end{pmatrix}$$

transformations are given in terms of the angles φ_i and rapidities ν_i as

$$\Lambda_L = \exp\left(-\frac{\mathrm{i}}{2}(\vec{\varphi} - \mathrm{i}\vec{\nu})\vec{\sigma}\right), \qquad \Lambda_R = \exp\left(-\frac{\mathrm{i}}{2}(\vec{\varphi} + \mathrm{i}\vec{\nu})\vec{\sigma}\right)$$

Weyl spinor index notation

$$u_+(p) \leftrightarrow p_A \qquad u_-(p) \leftrightarrow p^{\dot{A}}$$

$$\varepsilon^{AB}p_A k_B \to \underbrace{\det(\Lambda_R)}_{=1}\varepsilon^{AB}p_A k_B\,, \qquad \varepsilon_{\dot{A}\dot{B}}p^{\dot{A}}k^{\dot{B}} \to \underbrace{\det(\Lambda_L)}_{=1}\varepsilon_{\dot{A}\dot{B}}p^{\dot{A}}k^{\dot{B}}$$

$$\varepsilon^{AB} = \varepsilon_{AB} = \varepsilon_{\dot{A}\dot{B}} = \varepsilon^{\dot{A}\dot{B}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad p^A = \varepsilon^{AB}p_B, \qquad p_{\dot{B}} = p^{\dot{A}}\varepsilon_{\dot{A}\dot{B}}$$

$$\langle pk\rangle = p^A k_A = p_B k_A \varepsilon^{AB},$$
$$[pk] = p_{\dot{A}} k^{\dot{A}} = p^{\dot{B}}\varepsilon_{\dot{B}\dot{A}}k^{\dot{A}}.$$

$$p^A k_A = -p_A k^A \qquad p^{\dot{A}} k_{\dot{A}} = -p_{\dot{A}} k^{\dot{A}}$$

$$\langle pk\rangle \equiv \bar{u}_L(p)u_R(k) = u_-^\dagger(p)u_+(k) \quad [pk] \equiv \bar{u}_R(p)u_L(k) = u_+^\dagger(p)u_-(k)$$

$$\langle pk\rangle^* = p^{\dot{A}}k_{\dot{A}} = [kp]$$

braket notation: $u_+(p) \leftrightarrow p_A \quad u_-(p) \leftrightarrow p^{\dot{A}}$

$$\langle pk\rangle = \langle p - |k+\rangle = p^A k_A \qquad p_A \leftrightarrow |p+\rangle = |p\rangle \qquad p^{\dot{A}} \leftrightarrow |p-\rangle = |p],$$
$$[kp] = \langle k + |p-\rangle = k_{\dot{A}}p^{\dot{A}} \qquad p_{\dot{A}} \leftrightarrow \langle p+| = [p| \qquad p^A \leftrightarrow \langle p-| = \langle p|$$

Wilson loop

The **Wilson loop** variable is a quantity defined by the trace of a path-ordered exponential of a gauge field A_μ transported along a closed line C:

$$W_C := \mathrm{Tr}\,(\,\mathcal{P}\exp i\oint_C A_\mu dx^\mu\,)\,.$$

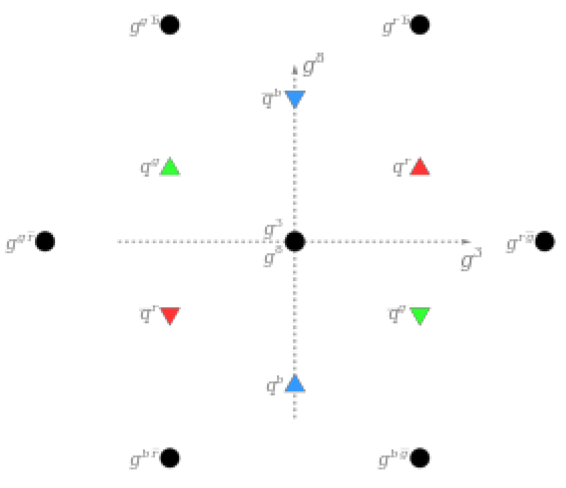
Here, C is a closed curve in space, $\mathcal{P}$ is the path-ordering operator. Under a gauge transformation

$$\mathcal{P}e^{i\oint_C A_\mu dx^\mu} \to g(x)\mathcal{P}e^{i\oint_C A_\mu dx^\mu}g^{-1}(x),$$

where x corresponds to the initial (and end) point of the loop (only initial and end point of a line contribute, whereas gauge transformations in between cancel each other). For SU(2) gauges, for example, one has $g^{\pm 1}(x) \equiv \exp\{\pm i\alpha^j(x)\frac{\sigma^j}{2}\}$; $\alpha^j(x)$ is an arbitrary real function of x, and σ^j are the three Pauli matrices; as usual, a sum over repeated indices is implied.

The invariance of the trace under cyclic permutations guarantees that W_C is invariant under gauge transformations.

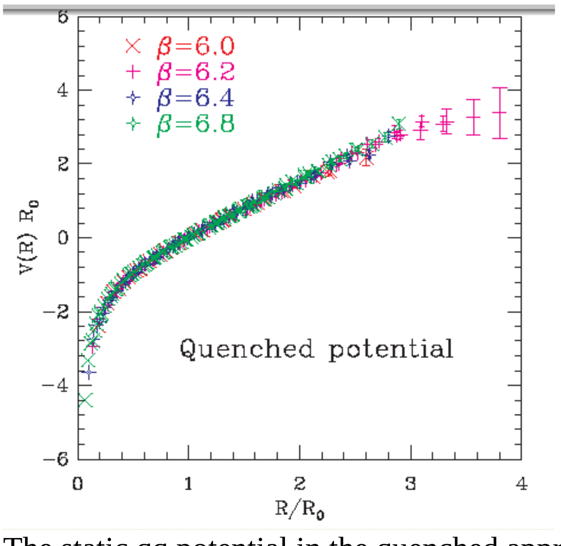
[10]



The pattern of strong charges for the three colors of quark, three antiquarks, and eight gluons (with two of zero charge overlapping).
Quarks are massive spin-1/2 fermions which carry a color charge whose gauging is the content of QCD. Quarks are represented by Dirac fields in the fundamental representation **3** of the gauge group SU(3). They also carry electric charge (either −1/3 or 2/3) and participate in weak interactions as part of weak isospin doublets. They carry global quantum numbers including the baryon number, which is 1/3 for each quark, hypercharge and one of the flavor quantum numbers.
Gluons are spin-1 bosons which also carry color charges, since they lie in the adjoint representation **8** of SU(3). They have no electric charge, do not participate in the weak interactions, and have no flavor. They lie in the singlet representation **1** of all these symmetry groups.
Every quark has its own antiquark. The charge of each antiquark is exactly the opposite of the corresponding quark.

The running coupling constant [9]

$$V(r) = -\frac{4}{3}\frac{\alpha_s \hbar c}{r} + kr \quad \text{potential} = \langle q\bar{q} \rangle$$



The static qq potential in the quenched approximation obtained by the Wuppertal collaboration. The data at β = 6.0, 6.2, 6.4 and 6.8 has been scaled by R_0, and normalized such that V (R_0) = 0. The collapse of the different sets of data on to a single curve after the rescaling by R_0 is evidence for scaling. The linear rise at large *r* implies confinement. [9]

The color confinement results from $\lim(V(r), r \to \infty) = \infty$

the running coupling is characterized by the β-function with colors $N=3$, flavors $n_f=4$, μ=transfer energy

$$\mu \frac{\partial g}{\partial \mu} = -\beta(g) = -(\beta_0 g^3 + \beta_1 g^5 + \ldots)$$

$$\beta_0 = (\frac{11N - 2n_f}{3})/16\pi^2$$

$$\beta_1 = (\frac{34N^2}{3} - \frac{10Nn_f}{3} - \frac{n_f(N^2-1)}{N})/(16\pi^2)^2 .$$

$$\alpha_s(\mu) = \frac{g^2(\mu)}{4\pi} = \frac{1}{8\pi\beta_0 \log\left(\frac{\mu}{\Lambda}\right)} = \frac{12\pi}{(33-2N_f)\ln(\frac{\mu^2}{\Lambda^2})}$$ coupling constant

where

$\Lambda \approx 220$ *MeV* “cutoff parameter”

$N_F = 3$: Number of quark flavours with 2m < Q

$k \approx 1 GeV/fm$

$\alpha_s(m_b) = 0.189$ (bottom quark)

$\alpha_s(m_c) = 0.173$ (charm quark)

for the cutoff Λ follows

$$\Lambda_{QCD} = \lim_{\mu \to \infty} \mu \left(\frac{1}{\beta_0 g^2(\mu)}\right)^{\frac{\beta_1}{2\beta_0^2}} \exp[-\frac{1}{2\beta_0 g^2(\mu)}] \equiv \mu\, f_P\big(g(\mu)\big)$$

The cut-off parameter on-lattice [9 14.8]

$$\Lambda_{QCD} = C\ \Lambda_{lattice}$$

n_f	0	1	2	3	4
$\Lambda_{\overline{MS}}/\Lambda_{latt}$	28.8	34.0	41.1	51.0	65.5
$\Lambda_{MOM}/\Lambda_{latt}$	83.4	89.4	96.7	105.8	117.4

The relation between Λ_{MOM} and Λ_{MS} and Λ_{latt} as a function of the number of active flavors. The results for $\Lambda_{MOM}/\Lambda_{latt}$ with g defined by the triple gluon vertex are in Feynman gauge [9]

Λ_{MOM}= one-loop renormalized coupling constant in momentum space subtraction procedure [14]

Λ_{MS} = one-loop renormalized coupling constant in minimal subtraction procedure

The evolution equation for the QCD coupling, $\alpha_s(Q^2)$, is:

$$Q^2 \frac{\partial \alpha_s}{\partial Q^2} = \beta(\alpha_s), \qquad \beta(\alpha_s) = -\alpha_s^2(b_0 + b_1\alpha_s + b_2\alpha_s^2 + \ldots),$$

$$b_0 = \frac{11C_A - 2n_f}{12\pi}, \qquad b_1 = \frac{17C_A^2 - 5C_A n_f - 3C_F n_f}{24\pi^2} = \frac{153 - 19n_f}{24\pi^2}$$

Gauge fixing

[6]

a gauge fixing term has to be added to the Lagrangian

$$\mathcal{L}_{gf} = -\frac{1}{2\xi}(\partial_\mu A^{a,\mu})^2$$

Fadeev-Popov ghost fields c_a and anti-ghost fields $\bar{c}_a$ are introduced:

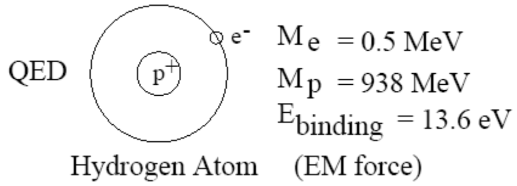
$$\mathcal{L}_{\rm FP} = (\partial^\mu \bar{c}_a) D^{\rm (ad)}_{ab,\mu} c_b = (\partial^\mu \bar{c}_a)(\partial_\mu \delta_{ab} + g_s f^{abc} A_{c,\mu}) c_b$$

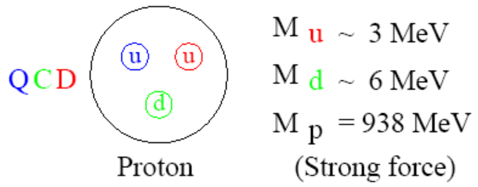
alternatively, an axial gauge-fixing term could be used,

$$\mathcal{L}_{\rm gf} \ = \ -\frac{1}{2\xi}(n^\mu A_\mu^a)^2.$$

in axial gauges, it can be shown that the ghost fields are not necessary

Comparison energy electromagnetic-strong





2. The standard model and QCD

Particles of the standard models

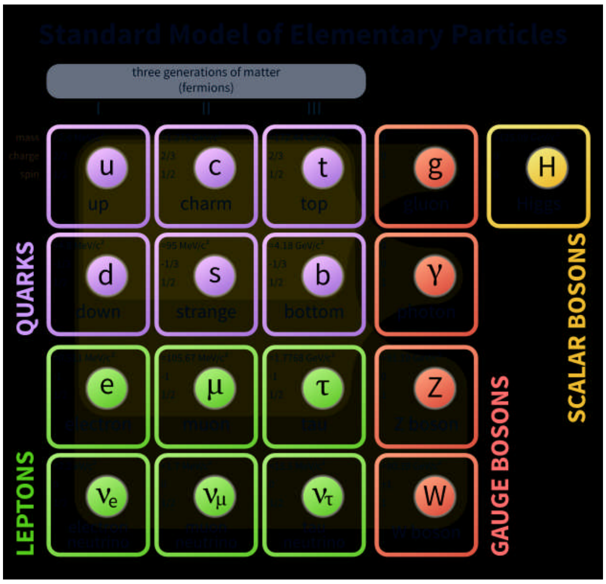
Quarks — 3 colours: $\psi_a = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$

Quark part of Lagrangian:

$$\mathcal{L}_q = \bar{\psi}_a (i\gamma^\mu \partial_\mu \delta_{ab} - g_s \gamma^\mu t^C_{ab} \mathcal{A}^C_\mu - m)\psi_b$$

$SU(3)$ local gauge symmetry $\leftrightarrow$ 8 $(= 3^2 - 1)$ generators $t^1_{ab} \dots t^8_{ab}$ corresponding to 8 gluons $\mathcal{A}^1_\mu \dots \mathcal{A}^8_\mu$.

A representation is: $t^A = \frac{1}{2}\lambda^A$,



Quark properties [16]

Particle		Mass (MeV/c^2)*	J	B	Q (e)	I_3	C	S	T	B'	Antiparticle	
Name	Symbol										Name	Symbol
First generation												
up	u	2.3 ±0.7 ± 0.5	½	+⅓	+⅔	+½	0	0	0	0	antiup	$\bar{u}$
down	d	4.8 ±0.5 ± 0.3	½	+⅓	−⅓	−½	0	0	0	0	antidown	$\bar{d}$
Second generation												
charm	c	1275 ±25	½	+⅓	+⅔	0	+1	0	0	0	anticharm	$\bar{c}$
strange	s	95 ±5	½	+⅓	−⅓	0	0	−1	0	0	antistrange	$\bar{s}$
Third generation												
top	t	173 210 ±510 ± 710	½	+⅓	+⅔	0	0	0	+1	0	antitop	$\bar{t}$
bottom	b	4180 ±30	½	+⅓	−⅓	0	0	0	0	−1	antibottom	$\bar{b}$

quark radius: as of 2014, experimental evidence indicates they are no bigger than 10^{-4} times the size of a proton, i.e. less than 10^{-19} metres [16]

Quark Model [6]

Gell-Mann, Ne'eman, Zweig (1961-64) classified the hadron spectrum and proposed that hadrons are composed of spin 1=2 quarks with fractional electric charges:

$$u(I = +\tfrac{1}{2}, S = 0, Q = \tfrac{2}{3}), \qquad d(-\tfrac{1}{2}, 0, -\tfrac{1}{3}), \qquad s(0, -1 - \tfrac{1}{3}),$$

Baryons

$|p\rangle \sim |uud\rangle$ $\qquad$ $|n\rangle \sim |udd\rangle$

Particle name	Symbol	Quark content	Rest mass (MeV/c^2)	I	J^P	Q (e)	S	C	B'	Mean lifetime (s)	Commonly decays to
proton[8]	p / p^+ / N^+	uud	938.272 046(21)[a]	½	$\frac{1}{2}^+$	+1	0	0	0	Stable[b]	Unobserved
neutron[9]	n / n^0 / N^0	udd	939.565 379(21)[a]	½	$\frac{1}{2}^+$	0	0	0	0	$(8.800 \pm 0.009) \times 10^{+2}$[c]	$p^+ + e^- + \bar{\nu}_e$
Lambda[10]	Λ^0	uds	1 115.683 ± 0.006	0	$\frac{1}{2}^+$	0	−1	0	0	$(2.632 \pm 0.020) \times 10^{-10}$	$p^+ + \pi^-$ or $n^0 + \pi^0$
charmed Lambda[11]	Λ_c^+	udc	2 286.46 ± 0.14	0	$\frac{1}{2}^+$	+1	0	+1	0	$(2.00 \pm 0.06) \times 10^{-13}$	See Λ_c^+ decay modes
bottom Lambda[12]	Λ_b^0	udb	5 619.4 ± 0.6	0	$\frac{1}{2}^+$	0	0	0	−1	$(1.429 \pm 0.024) \times 10^{-12}$	See Λ_b^0 decay modes
Sigma[13]	Σ^+	uus	1 189.37 ± 0.07	1	$\frac{1}{2}^+$	+1	−1	0	0	$(8.018 \pm 0.026) \times 10^{-11}$	$p^+ + \pi^0$ or $n^0 + \pi^+$
Sigma[14]	Σ^0	uds	1 192.642 ± 0.024	1	$\frac{1}{2}^+$	0	−1	0	0	$(7.4 \pm 0.7) \times 10^{-20}$	$\Lambda^0 + \gamma$
Sigma[15]	Σ^-	dds	1 197.449 ± 0.030	1	$\frac{1}{2}^+$	−1	−1	0	0	$(1.479 \pm 0.011) \times 10^{-10}$	$n^0 + \pi^-$
charmed Sigma[16]	Σ_c^{++}	uuc	2 453.98 ± 0.16	1	$\frac{1}{2}^+$	+2	0	+1	0	$(2.91 \pm 0.32) \times 10^{-22}$[d]	$\Lambda_c^+ + \pi^+$
charmed Sigma[16]	Σ_c^+	udc	2 452.9 ± 0.4	1	$\frac{1}{2}^+$	+1	0	+1	0	$>1.43 \times 10^{-22}$[d]	$\Lambda_c^+ + \pi^0$
charmed Sigma[16]	Σ_c^0	ddc	2 453.74 ± 0.16	1	$\frac{1}{2}^+$	0	0	+1	0	$(3.05 \pm 0.37) \times 10^{-22}$[d]	$\Lambda_c^+ + \pi^-$
bottom Sigma[17]	Σ_b^+	uub	$5\,811.3\,^{+0.9}_{-0.8} \pm 1.7$	1	$\frac{1}{2}^+$	+1	0	0	−1	$6.8\,^{+2.7}_{-3.5} \times 10^{-23}$[d]	$\Lambda_b^0 + \pi^+$
bottom Sigma†	Σ_b^0	udb	Unknown	1	$\frac{1}{2}^+$	0	0	0	−1	Unknown	Unknown
bottom Sigma[17]	Σ_b^-	ddb	$5\,815.5\,^{+0.6}_{-0.5} \pm 1.7$	1	$\frac{1}{2}^+$	−1	0	0	−1	$1.34\,^{+0.87}_{-1.15} \times 10^{-22}$[d]	$\Lambda_b^0 + \pi^-$
Xi[18]	Ξ^0	uss	1 314.86 ± 0.20	½	$\frac{1}{2}^+$	0	−2	0	0	$(2.90 \pm 0.09) \times 10^{-10}$	$\Lambda^0 + \pi^0$
Xi[19]	Ξ^-	dss	1 321.71 ± 0.07	½	$\frac{1}{2}^+$	−1	−2	0	0	$(1.639 \pm 0.015) \times 10^{-10}$	$\Lambda^0 + \pi^-$
charmed Xi[20]	Ξ_c^+	usc	$2\,467.8\,^{+0.4}_{-0.6}$	½	$\frac{1}{2}^+$	+1	−1	+1	0	$(4.42 \pm 0.26) \times 10^{-13}$	See Ξ_c^+ decay modes
charmed Xi[21]	Ξ_c^0	dsc	$2\,470.88\,^{+0.34}_{-0.80}$	½	$\frac{1}{2}^+$	0	−1	+1	0	$1.12\,^{+0.13}_{-0.10} \times 10^{-13}$	See Ξ_c^0 decay modes
charmed Xi prime[22]	$\Xi_c'^+$	usc	2 575.6 ± 3.1	½	$\frac{1}{2}^+$	+1	−1	+1	0	Unknown	$\Xi_c^+ + \gamma$ (seen)
charmed Xi prime[23]	$\Xi_c'^0$	dsc	2 577.9 ± 2.9	½	$\frac{1}{2}^+$	0	−1	+1	0	Unknown	$\Xi_c^0 + \gamma$ (seen)
double charmed Xi[24]	Ξ_{cc}^{++}	ucc	3621.40 ± 0.72 ± 0.27 ± 0.14	½	$\frac{1}{2}^+$	+2	0	+2	0	Unknown	$\Lambda_c^+ + K^- + \pi^+ + \pi^+$ (seen)
double charmed Xi[e]	Ξ_{cc}^+	dcc	Unknown	½	$\frac{1}{2}^+$	+1	0	+2	0	Unknown	Unknown

bottom Xi[25] (or Cascade B)	Ξ_b^0	usb	5 787.8 ± 5.0 ± 1.3	½	½+	0	−1	0	−1	Unknown	See Ξ_b decay modes
bottom Xi[25] (or Cascade B)	Ξ_b^-	dsb	5 791.1 ± 2.2	½	½+	−1	−1	0	−1	$(1.56^{+0.27}_{-0.25} \pm 0.02) \times 10^{-12}$	See Ξ_b decay modes
bottom Xi prime†	Ξ'^0_b	usb	Unknown	½	½+	0	−1	0	−1	Unknown	Unknown
bottom Xi prime†	Ξ'^-_b	dsb	Unknown	½	½+	−1	−1	0	−1	Unknown	Unknown
double bottom Xi†	Ξ^0_{bb}	ubb	Unknown	½	½+	0	0	0	−2	Unknown	Unknown
bottom Xi prime†	Ξ'^-_b	dsb	Unknown	½	½+	−1	−1	0	−1	Unknown	Unknown
double bottom Xi†	Ξ^0_{bb}	ubb	Unknown	½	½+	0	0	0	−2	Unknown	Unknown
double bottom Xi†	Ξ^-_{bb}	dbb	Unknown	½	½+	−1	0	0	−2	Unknown	Unknown
charmed bottom Xi†	Ξ^+_{cb}	ucb	Unknown	½	½+	+1	0	+1	−1	Unknown	Unknown
charmed bottom Xi†	Ξ^0_{cb}	dcb	Unknown	½	½+	0	0	+1	−1	Unknown	Unknown
charmed bottom Xi prime†	Ξ'^+_{cb}	ucb	Unknown	½	½+	+1	0	+1	−1	Unknown	Unknown
charmed bottom Xi prime†	Ξ'^0_{cb}	dcb	Unknown	½	½+	0	0	+1	−1	Unknown	Unknown
charmed Omega[26]	Ω^0_c	ssc	2 695.2 ± 1.7	0	½+	0	−2	+1	0	$(6.9 \pm 1.2) \times 10^{-14}$	See Ω^0_c decay modes
bottom Omega[27]	Ω^-_b	ssb	6071 ± 40	0	½+	−1	−2	0	−1	$(1.13^{+0.55}_{-0.42} \pm 0.02) \times 10^{-12}$	Ω^- + J/ψ (seen)
double charmed Omega†	Ω^+_{cc}	scc	Unknown	0	½+	+1	−1	+2	0	Unknown	Unknown
charmed bottom Omega†	Ω^0_{cb}	scb	Unknown	0	½+	0	−1	+1	−1	Unknown	Unknown
charmed bottom Omega prime†	Ω'^0_{cb}	scb	Unknown	0	½+	0	−1	+1	−1	Unknown	Unknown
double bottom Omega†	Ω^-_{bb}	sbb	Unknown	0	½+	−1	−1	0	−2	Unknown	Unknown

double charmed bottom Omega†	Ω^{+}_{ccb}	ccb	Unknown	0	$\frac{1}{2}^{+}$	+1	0	+2	−1	Unknown	Unknown
charmed double bottom Omega†	Ω^{0}_{cbb}	cbb	Unknown	0	$\frac{1}{2}^{+}$	0	0	+1	−2	Unknown	Unknown

Vector mesons

Particle name	Particle symbol	Antiparticle symbol	Quark content	Rest mass (MeV/c^2)	I^G	J^{PC}	S	C	B'	Mean lifetime (s)	Commonly decays to (>5% of decays)
Charged rho meson[23]	$\rho^+(770)$	$\rho^-(770)$	$u\bar{d}$	775.11 ± 0.34	1^+	1^-	0	0	0	$(4.41 \pm 0.02) \times 10^{-24}$[f][g]	$\pi^\pm + \pi^0$
Neutral rho meson[23]	$\rho^0(770)$	Self	$\frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$	775.26 ± 0.25	1^+	1^{--}	0	0	0	$(4.45 \pm 0.03) \times 10^{-24}$[f][g]	$\pi^+ + \pi^-$
Omega meson[24]	$\omega(782)$	Self	$\frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$	782.65 ± 0.12	0^-	1^{--}	0	0	0	$(7.75 \pm 0.07) \times 10^{-23}$[f]	$\pi^+ + \pi^0 + \pi^-$ or $\pi^0 + \gamma$
Phi meson[25]	$\varphi(1020)$	Self	$s\bar{s}$	1 019.461 ± 0.019	0^-	1^{--}	0	0	0	$(1.54 \pm 0.01) \times 10^{-22}$[f]	$K^+ + K^-$ or $K^0_S + K^0_L$ or $(\rho + \pi) / (\pi^+ + \pi^0 + \pi^-)$
J/Psi[26]	J/ψ	Self	$c\bar{c}$	3 096.916 ± 0.011	0^-	1^{--}	0	0	0	$(7.09 \pm 0.21) \times 10^{-21}$[f]	See J/ψ(1S) decay modes
Upsilon meson[27]	$\Upsilon(1S)$	Self	$b\bar{b}$	9 460.30 ± 0.26	0^-	1^{--}	0	0	0	$(1.22 \pm 0.03) \times 10^{-20}$[f]	See ϒ(1S) decay modes
Kaon[28]	K^{*+}	K^{*-}	$u\bar{s}$	891.66 ± 0.026	½	1^-	1	0	0	$(3.26 \pm 0.06) \times 10^{-23}$[f][g]	See $K^*(892)$ decay modes
Kaon[28]	K^{*0}	$\bar{K}^{*0}$	$d\bar{s}$	895.81 ± 0.19	½	1^-	1	0	0	$(1.39 \pm 0.02) \times 10^{-23}$[f]	See $K^*(892)$ decay modes
D meson[29]	$D^{*+}(2010)$	$D^{*-}(2010)$	$c\bar{d}$	2 010.26 ± 0.07	½	1^-	0	+1	0	$(7.89 \pm 0.17) \times 10^{-21}$[f]	$D^0 + \pi^+$ or $D^+ + \pi^0$
D meson[30]	$D^{*0}(2007)$	$\bar{D}^{*0}(2007)$	$c\bar{u}$	2 006.96 ± 0.10	½	1^-	0	+1	0	$>3.1 \times 10^{-22}$[f]	$D^0 + \pi^0$ or $D^0 + \gamma$
Strange D meson[31]	D^{*+}_s	D^{*-}_s	$c\bar{s}$	2 112.1 ± 0.4	0	1^-	+1	+1	0	$>3.4 \times 10^{-22}$[f]	$D^{*+} + \gamma$ or $D^{*+} + \pi^0$
B meson[32]	B^{*+}	B^{*-}	$u\bar{b}$	5 325.2 ± 0.4	½	1^-	0	0	+1	Unknown	$B^+ + \gamma$
B meson[32]	B^{*0}	$\bar{B}^{*0}$	$d\bar{b}$	5 325.2 ± 0.4	½	1^-	0	0	+1	Unknown	$B^0 + \gamma$
Strange B meson[33]	B^{*0}_s	$\bar{B}^{*0}_s$	$s\bar{b}$	$5\,415.4^{+2.4}_{-2.1}$	0	1^-	−1	0	+1	Unknown	$B^0_s + \gamma$
Charmed B meson†	B^{*+}_c	B^{*-}_c	$c\bar{b}$	Unknown	0	1^-	0	+1	+1	Unknown	Unknown

first generation vector mesons

rho+ $u\bar{d}$ m=775.1MeV, Q=1, charge radius r0=0.748fm+-0.02fm [18]

rho0 $(u\bar{u} - d\bar{d})/\sqrt{2}$ m=775.3MeV, Q=0

omega0 $(u\bar{u} + d\bar{d})/\sqrt{2}$ m=782.6MeV, Q=0

Pseudoscalar mesons

$$|\pi^+\rangle \sim |u\bar{d}\rangle \qquad |\pi^-\rangle \sim |d\bar{u}\rangle \qquad |\pi^0\rangle \sim |d\bar{d}\rangle - |u\bar{u}\rangle\,,$$
$$|K^+\rangle \sim |u\bar{s}\rangle \qquad |K^-\rangle \sim |s\bar{u}\rangle \qquad |K^0\rangle \sim |d\bar{s}\rangle \qquad |\bar{K}^0\rangle \sim |s\bar{d}\rangle$$

Particle name	Particle symbol	Antiparticle symbol	Quark content	Rest mass (MeV/c^2)	I^G	J^{PC}	S	C	B'	Mean lifetime (s)	Commonly decays to (>5% of decays)
Pion[6]	π^+	π^-	$u\bar{d}$	139.570 18 ± 0.000 35	1^-	0^-	0	0	0	$(2.6033 \pm 0.0005) \times 10^{-8}$	$\mu^+ + \nu_\mu$
Pion[7]	π^0	Self	$\frac{u\bar{u}-d\bar{d}}{\sqrt{2}}$ [a]	134.9766 ± 0.0006	1^-	0^{-+}	0	0	0	$(8.52 \pm 0.18) \times 10^{-17}$	$\gamma + \gamma$
Eta meson[8]	η	Self	$\frac{u\bar{u}+d\bar{d}-2s\bar{s}}{\sqrt{6}}$ [a]	547.862 ± 0.018	0^+	0^{-+}	0	0	0	$(5.02 \pm 0.19) \times 10^{-19}$[b]	$\gamma + \gamma$ or $\pi^0 + \pi^0 + \pi^0$ or $\pi^+ + \pi^0 + \pi^-$
Eta prime meson[9]	$\eta'(958)$	Self	$\frac{u\bar{u}+d\bar{d}+s\bar{s}}{\sqrt{3}}$ [a]	957.78 ± 0.06	0^+	0^{-+}	0	0	0	$(3.32 \pm 0.15) \times 10^{-21}$[b]	$\pi^+ + \pi^- + \eta$ or $(\rho^0 + \gamma) / (\pi^+ + \pi^- + \gamma)$ or $\pi^0 + \pi^0 + \eta$
Charmed eta meson[10]	$\eta_c(1S)$	Self	$c\bar{c}$	2 983.6 ± 0.7	0^+	0^{-+}	0	0	0	$(2.04 \pm 0.05) \times 10^{-23}$[b]	See η_c decay modes
Bottom eta meson[11]	$\eta_b(1S)$	Self	$b\bar{b}$	9 398.0 ± 3.2	0^+	0^{-+}	0	0	0	Unknown	See η_b decay modes
Kaon[12]	K^+	K^-	$u\bar{s}$	493.677 ± 0.016	½	0^-	1	0	0	$(1.2380 \pm 0.0021) \times 10^{-8}$	$\mu^+ + \nu_\mu$ or $\pi^+ + \pi^0$ or $\pi^0 + e^+ + \nu_e$ or $\pi^+ + \pi^+ + \pi^-$
Kaon[13]	K^0	$\bar{K}^0$	$d\bar{s}$	497.614 ± 0.024	½	0^-	1	0	0	[c]	[c]
K-Short[14]	K^0_S	Self	$\frac{d\bar{s}+s\bar{d}}{\sqrt{2}}$ [e]	497.614 ± 0.024[d]	½	0^-	(*)	0	0	$(8.954 \pm 0.004) \times 10^{-11}$	$\pi^+ + \pi^-$ or $\pi^0 + \pi^0$
K-Long[15]	K^0_L	Self	$\frac{d\bar{s}-s\bar{d}}{\sqrt{2}}$ [e]	497.614 ± 0.024[d]	½	0^-	(*)	0	0	$(5.116 \pm 0.021) \times 10^{-8}$	$\pi^\pm + e^\mp + \nu_e$ or $\pi^\pm + \mu^\mp + \nu_\mu$ or $\pi^0 + \pi^0 + \pi^0$ or $\pi^+ + \pi^0 + \pi^-$
D meson[16]	D^+	D^-	$c\bar{d}$	1 869.61 ± 0.10	½	0^-	0	+1	0	$(1.040 \pm 0.007) \times 10^{-12}$	See D^+ decay modes
D meson[17]	D^0	$\bar{D}^0$	$c\bar{u}$	1 864.84 ± 0.07	½	0^-	0	+1	0	$(4.101 \pm 0.015) \times 10^{-13}$	See D^0 decay modes
strange D meson[18]	D^+_s	D^-_s	$c\bar{s}$	1 968.30 ± 0.11	0	0^-	+1	+1	0	$(5.00 \pm 0.07) \times 10^{-13}$	See D^+_s decay modes
B meson[19]	B^+	B^-	$u\bar{b}$	5 279.26 ± 0.17	½	0^-	0	0	+1	$(1.638 \pm 0.004) \times 10^{-12}$	See B^+ decay modes
B meson[20]	B^0	$\bar{B}^0$	$d\bar{b}$	5 279.58 ± 0.17	½	0^-	0	0	+1	$(1.519 \pm 0.009) \times 10^{-12}$	See B^0 decay modes
Strange B meson[21]	B^0_s	$\bar{B}^0_s$	$s\bar{b}$	5 366.77 ± 0.24	0	0^-	−1	0	+1	$(1.512 \pm 0.007) \times 10^{-12}$	See B^0_s decay modes
Charmed B meson[22]	B^+_c	B^-_c	$c\bar{b}$	6 275.6 ± 1.1	0	0^-	0	+1	+1	$(4.52 \pm 0.33) \times 10^{-13}$	See B^+_c decay modes

Pion

In particle physics, a **pion** is any of three subatomic particles: π^0, π^+, and π^- .
Each pion consists of a quark and an antiquark and is therefore a meson. Pions are the lightest mesons and, more generally, the lightest hadrons. They are unstable, with the charged pions $\pi^+$and π^- decaying with a mean lifetime of 26.033 nanoseconds (2.6033×10^{-8} seconds), and the neutral pion π^0 decaying with a much shorter lifetime of 8.4×10^{-17} seconds. Charged pions most often decay into muons and muon neutrinos, while neutral pions generally decay into gamma rays.
The exchange of virtual pions, along with the vector, rho and omega mesons, provides an explanation for the residual strong force between nucleons. Pions are not produced in radioactive decay, but are commonly produced in high energy accelerators in collisions between hadrons. All types of pions are also produced in natural processes when high energy cosmic ray protons and other hadronic cosmic ray components interact with matter in the Earth's atmosphere. Recently, the detection of characteristic gamma rays originating from the decay of neutral pions in two supernova remnants has shown that pions are produced copiously after supernovas, most probably in conjunction with production of high energy protons that are detected on Earth as cosmic rays
The concept of mesons as the carrier particles of the nuclear force was first proposed in 1935 by Hideki Yukawa. While the muon was first proposed to be this particle after its discovery in 1936, later work found that it did not participate in the strong nuclear interaction. The pions, which turned out to be examples of Yukawa's proposed mesons, were discovered later: the charged pions in 1947, and the neutral pion in 1950.
Pions, which are mesons with zero spin, are composed of first-generation quarks. In the quark model, an up quark and an anti-down quark make up a π^+ , whereas a down quark and an anti-up quark make up the π^- , and these are the antiparticles of one another. The neutral pion π^0 is a combination of an up quark with an anti-up quark or a down quark with an anti-down quark. The two combinations have identical quantum numbers, and hence they are only found in superpositions. The lowest-energy superposition of these is the π^0 , which is its own antiparticle. Together, the pions form a triplet of isospin. Each pion has isospin (I=1) and third-component isospin equal to its charge (I_z=+1, 0 or −1).

Pion decays

The $\pi^\pm$ mesons have a mass of 139.6MeV/c^2 and a mean lifetime of 2.6033×10^{-8}s. They decay due to the weak interaction. The primary decay mode of a pion, with a branching fraction of 0.999877, is a leptonic decay into a muon and a muon neutrino:

$$\pi^+ \to \mu^+ + \nu_\mu$$
$$\pi^- \to \mu^- + \bar{\nu}_\mu$$

The second most common decay mode of a pion, with a branching fraction of 0.000123, is also a leptonic decay into an electron and the corresponding electron antineutrino. This "electronic mode" was discovered at CERN in 1958:

$$\pi^+ \to \mu^+ + \nu_\mu$$
$$\pi^- \to \mu^- + \bar{\nu}_\mu$$

The suppression of the electronic decay mode with respect to the muonic one is given approximately (up to a few percent effect of the radiative corrections) by the ratio of the half-widths of the pion–electron and the pion–muon decay reactions:

$$R_\pi = (m_e/m_\mu)^2 \left(\frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2}\right)^2 = 1.283 \times 10^{-4}$$

The π^0 meson has a mass of 135.0 MeV/c^2 and a mean lifetime of 8.4×10^{-17} s. It decays via the electromagnetic force, which explains why its mean lifetime is much smaller than that of the charged pion (which can only decay via the weak force).
The dominant π^0 decay mode, with a branching ratio of BR=0.98823, is into two photons:

$$\pi^0 \to 2\,\gamma.$$

The second largest π^0 decay mode (BR=0.01174) is the Dalitz decay (named after Richard Dalitz), which is a two-photon decay with an internal photon conversion resulting a photon and an electron-positron pair in the final state:

$$\pi^0 \to \gamma + e^- + e^+.$$

Particle name	Particle symbol	Antiparticle symbol	Quark content[10]	Rest mass (MeV/c^2)	I^G	J^{PC}	S	C	B'	Mean lifetime (s)	Commonly decays to (>5% of decays)
Pion[8]	π^+	π^-	$u\bar{d}$	139.570 18 ± 0.000 35	1^-	0^-	0	0	0	$2.6033 \pm 0.0005 \times 10^{-8}$	$\mu^+ + \nu_\mu$
Pion[11]	π^0	Self	$\frac{u\bar{u}-d\bar{d}}{\sqrt{2}}$ [a]	134.976 6 ± 0.000 6	1^-	0^{-+}	0	0	0	$8.4 \pm 0.6 \times 10^{-17}$	$\gamma + \gamma$

Electromagnetic charge radius [17]
R(π+)=0.657+-0.003fm

Nucleons

Particle name	Symbol	Quark content	Mass (MeV/c^2)	Mass (u)[a]	I_3	J^P	Q (e)	Magnetic moment	Mean lifetime (s)	Commonly decays to
proton[PDG 1]	p / p^+ / N^+	uud	938.272 013 ±0.000 023	1.007 276 466 77 ±0.000 000 000 10	+½	½⁺	+1	2.792 847 356 ±0.000 000 023	Stable[b]	Unobserved
neutron[PDG 2]	n / n^0 / N^0	udd	939.565 346 ±0.000 023	1.008 664 915 97 ±0.000 000 000 43	-½	½⁺	0	−1.913 042 73 ±0.000 000 45	$(8.857 \pm 0.008) \times 10^{+2}$ [c]	$p + e^- + \bar{\nu}_e$
antiproton	$\bar{p}$ / p^- / N^-	$\overline{uud}$	938.272 013 ±0.000 023	1.007 276 466 77 ±0.000 000 000 10	-½	½⁺	−1	−2.793 ±0.006	Stable[b]	Unobserved
antineutron	$\bar{n}$ / $\bar{n}^0$ / $\bar{N}^0$	$\overline{udd}$	939.485 ±0.051	1.008 664 915 97 ±0.000 000 000 43	+½	½⁺	0	?	$(8.857 \pm 0.008) \times 10^{+2}$ [c]	$\bar{p} + e^+ + \nu_e$

The mass of the proton and neutron is quite similar: The proton is 1.6726×10^{-27}kg or $938.27 MeV/c^2$, while the neutron is 1.6749×10^{-27}kg or $939.57 MeV/c^2$. The neutron is roughly 0.13% heavier. The similarity in mass can be explained roughly by the slight difference in masses of up quarks and down quarks composing the nucleons. However, a detailed explanation remains an unsolved problem in particle physics.
The spin of both protons and neutrons is ½, which means they are fermions and, like electrons (and unlike bosons), are subject to the Pauli exclusion principle.
The **proton radius puzzle** is an unanswered problem in physics relating to the size of the proton. Historically the proton radius was measured via two independent methods, which converged to a value of about 0.8768 femtometres (1 fm = 10^{-15} m). This value was challenged by a 2010 experiment utilizing a third method, which produced a radius about 5% smaller than this. The discrepancy remains unresolved, and is a topic of ongoing research.
spectroscopy (Lamb shift): 0.8768±0.0069fm
nuclear scattering: 0.8775±0.0005fm
myonic hydrogen: 0.8751±0.0061fm

Parameters Standard model [9]

Parameters	Value	Comments
m_u(MS, 2 GeV)	3-4.5 MeV	χPT, sum-rules
m_d(MS, 2 GeV)	5-8 MeV	χPT, sum-rules
m_s(MS, 2 GeV)	100-140 MeV	χPT, sum-rules
m_c(MS,m_c)	1.0-1.6 GeV	J/ψ sprectra
m_b(MS,m_b)	4.1-4.6 GeV	´Y spectra
M_{Higgs}	125GeV	LHC 2016
$\alpha_s(M_Z)$	0.118 ± 0.003	World average
$\|V_{ud}\|$	0.9736 ± 0.0010	$n \to pe\nu_e$ and $\mu \to e\nu_e\nu_\mu$
$\|V_{us}\|$	0.2196 ± 0.0023	$K^+ \to \pi^0 e^+ \nu_e$ and $K^0_L \to \pi^\pm e^\mp \nu_e$
$\|V_{ub}\|$	$(3.3 \pm 0.2 \pm 0.4 \pm 0.7)10^{-3}$	$B \to \pi l \nu$; $B \to \rho l \nu$
$\|V_{cb}\|$	0.040 ± 0.003	$B \to X_c l \nu$ and $B \to D^* l \nu$
$\|\varepsilon\|$	2.26×10^{-3}	$K^0 \leftrightarrow K^0$ mixing
$\|\varepsilon/\varepsilon'\|$	$0 - 30 \times 10^{-4}$	$K \to \pi\pi$ decays

3. Perturbative QCD

Quark processes

neutron decay

$$d \to u + e^- + \overline{\upsilon}_e$$

lambda decay

$$\begin{array}{lll} uds & uud & \bar{u}d \\ \Lambda^0 \to & p + & \pi^- \\ s = -1 \neq & 0 + & 0 \end{array} \qquad \begin{array}{lll} uds & udd & \dfrac{\bar{u}u + \bar{d}d}{\sqrt{2}} \\ \Lambda^0 \to & n + & \pi^0 \\ s = -1 \neq & 0 + & 0 \end{array}$$

Conservation of strangeness is not in fact an independent conservation law, but can be viewed as a combination of the conservation of charge, isospin, and baryon number. It is often expressed in terms of hypercharge Y, defined by:

$$Y = S + B = 2(Q - I)$$

S = Strangeness

B = Baryon number

Q = Electric charge

I = Isopspin

Feynman rules

Lepton propagators [11, Fig.9.1]

Photon propagator	ν μ, q	$\dfrac{-\mathrm{i}g_{\mu\nu}}{q^2 + \mathrm{i}\varepsilon}$
W$^{\pm}$, Z propagators	ν μ, q	$\dfrac{\mathrm{i}}{q^2 - M^2 + \mathrm{i}\varepsilon}[-g_{\mu\nu} + \dfrac{q_\mu q_\nu}{M^2}]$
Higgs propagator	p	$\dfrac{\mathrm{i}}{p^2 - M_{\mathrm{H}}^2 + \mathrm{i}\varepsilon}$
Neutrino propagator	p	$\dfrac{1-\gamma_5}{2}\dfrac{\mathrm{i}}{\not{p} + \mathrm{i}\varepsilon}\dfrac{1+\gamma_5}{2}$
Lepton propagator	p	$\dfrac{\mathrm{i}}{\not{p} - m_\ell + \mathrm{i}\varepsilon}$

QCD propagators and vertices [11, Fig.8.2]

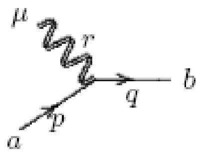
Quark propagator

$$\frac{\mathrm{i}}{\not p - m_A + \mathrm{i}\varepsilon}\,\delta_{ba}\,\delta_{BA}$$

Gluon propagator

$$\frac{\mathrm{i}\delta_{ij}}{q^2+\mathrm{i}\varepsilon}\left[-g_{\mu\nu} + \frac{(1-\xi)q_\mu q_\nu}{q^2+\mathrm{i}\varepsilon}\right]$$

Quark–gluon vertex



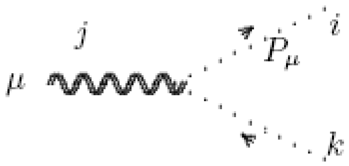
$$-\mathrm{i}g_s\gamma_\mu(T_i)_{ba}$$
$$(p+r-q=0;\ T_i=\lambda_i/2)$$

Three-gluon vertex

$$-g_s f_{ijk}[\,g_{\lambda\mu}(p-q)_\nu + g_{\mu\nu}(q-r)_\lambda + g_{\nu\lambda}(r-p)_\mu\,]$$
$$(p+q+r=0)$$

Four-gluon vertex

$$-\mathrm{i}g_s^2\,[f_{ijm}f_{k\ell m}(g_{\lambda\nu}g_{\mu\rho} - g_{\mu\nu}g_{\lambda\rho}) + f_{ikm}f_{j\ell m}(g_{\lambda\mu}g_{\nu\rho} - g_{\mu\nu}g_{\lambda\rho}) + f_{kjm}f_{i\ell m}(g_{\lambda\nu}g_{\mu\rho} - g_{\lambda\mu}g_{\nu\rho})]$$
$$(p+q+r+s=0)$$



$\mathrm{i}\,\delta_{ik}/P^2$ $\qquad$ $g_s f_{ijk} P_\mu$

Two-ghosts vertex

incoming [6, 6.73]

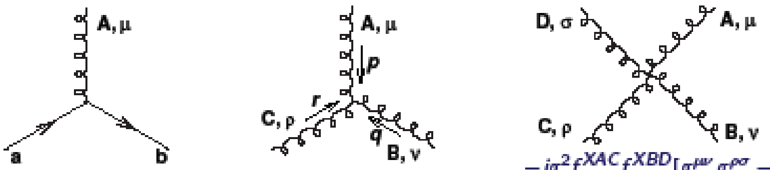
$$
\begin{aligned}
q_L(p) &: u_L(p) \to p^A \to |p-\rangle = |p] \\
q_R(p) &: u_R(p) \to p_A \to |p+\rangle = |p\rangle \\
\bar q_L(p) &: \bar v_R(p) \to p_{\dot A} \to \langle p+| = [p| \\
\bar q_R(p) &: \bar v_L(p) \to p^{\dot A} \to \langle p-| = \langle p| \\
g_-(p) &: \epsilon_-^\mu(p,q) = \frac{\langle q|\,\gamma^\mu\,|k]}{\sqrt{2}\,\langle qk\rangle} \\
g_+(p) &: \epsilon_+^\mu(p,q) = \frac{[q|\gamma^\mu\,|k\rangle}{\sqrt{2}\,[kq]}
\end{aligned}
$$

outgoing

$$
\begin{aligned}
&q_L(k) : \bar{u}_L(k) \to k^A \to \langle k-| = \langle k| \\
&q_R(k) : \bar{u}_R(k) \to k_{\dot{A}} \to \langle k+| = [k| \\
&\bar{q}_L(k) : v_R(k) \to k_A \to |k+\rangle = |k\rangle \\
&\bar{q}_R(k) : v_L(k) \to k^{\dot{A}} \to |k-\rangle = |k] \\
&g_-(p) : \epsilon_-^{\mu,*}(p,q) = \frac{[q|\gamma^\mu|k\rangle}{\sqrt{2}\,[kq]} \\
&g_+(p) : \epsilon_+^{\mu,*}(p,q) = \frac{\langle q|\gamma^\mu|k]}{\sqrt{2}\,\langle qk\rangle}
\end{aligned}
$$

[3]

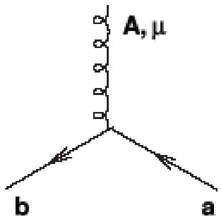
Interaction vertices of Feynman rules:



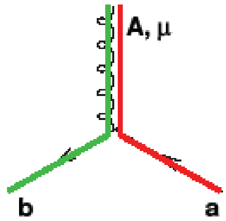
$-ig_s t^A_{ba}\gamma^\mu$

$-g_s f^{ABC}[(p-q)^\rho g^{\mu\nu} + (q-r)^\mu g^{\nu\rho} + (r-p)^\nu g^{\rho\mu}]$

$-ig_s^2 f^{XAC} f^{XBD}[g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\gamma}] + (C,\gamma) \leftrightarrow (D,\rho) + (B,\nu) \leftrightarrow (C,\gamma)$

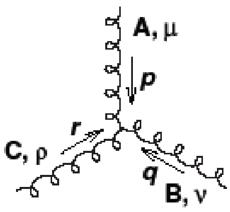


$\bar\psi_b(-ig_s t^A_{ba}\gamma^\mu)\psi_a$

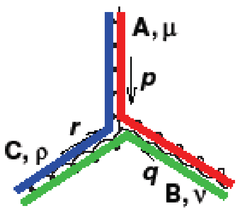


$$
\underbrace{\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}}_{\bar\psi_b}
\underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{t^1_{ab}}
\underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\psi_a}
$$

A gluon emission **repaints** the quark colour.
A gluon itself carries colour and anti-colour.



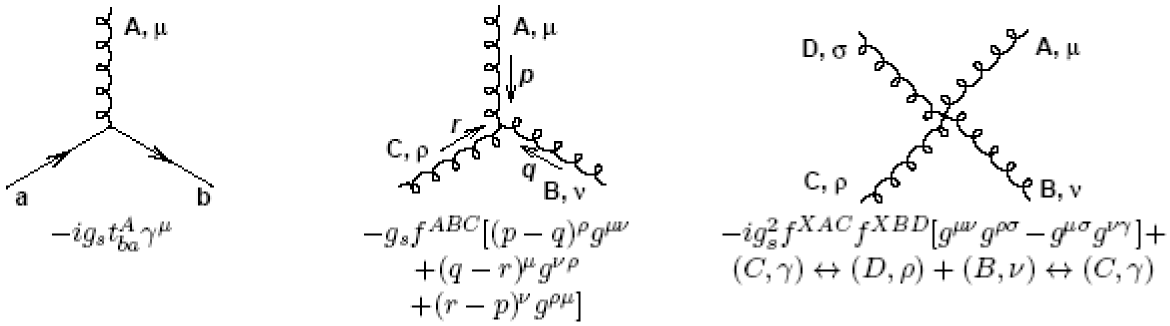
$-g_s f^{ABC}[(p-q)^\rho g^{\mu\nu} + (q-r)^\mu g^{\nu\rho} + (r-p)^\nu g^{\rho\mu}]$



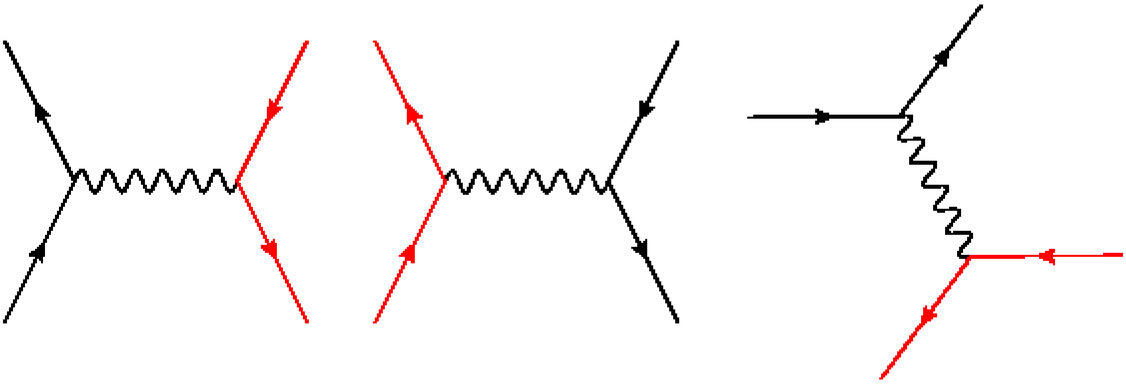
A gluon emission also repaints the gluon colours.

Because a gluon carries colour + anti-colour, it emits ∼ twice as strongly as a quark (just has colour)

[4]



[5]



Hadronic Z decay	Drell-Yan	DIS
$e^-e^+ \to \gamma^*/Z^0 \to q\bar{q}$	$q\bar{q} \to \gamma^*/Z^0 \to \ell^+\ell^-$	$\ell\bar{q} \xrightarrow{\gamma^*/Z^*} \ell\bar{q}$
$\propto N_C$	$\propto 1/N_C$	$\propto 1$

Hadronic decays

[6] R-ratio

$$R = \frac{\sigma(e^+e^- \to \text{Hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)},$$

$$\sigma(e^+e^- \to q\bar{q}) = N_c Q_q^2 \sigma(e^+e^- \to \mu^+\mu^-)$$

ECM < 2:5 GeV: production of u,d,s quarks:

$$R_{\text{quark}} = N_c((\tfrac{2}{3})^2 + (\tfrac{1}{3})^2 + (\tfrac{1}{3})^2 = N_c\frac{2}{3} = 2$$

$$R_{\text{exp}} \approx 2.2$$

4 GeV < ECM < 9 GeV: production of u,d,s,c quarks:

$$R_{\text{quark}} = N_c 2((\tfrac{2}{3})^2 + (\tfrac{1}{3})^2) = N_c\frac{10}{9} = 3.33\ldots$$

$$R_{\text{exp}} \approx 3.6$$

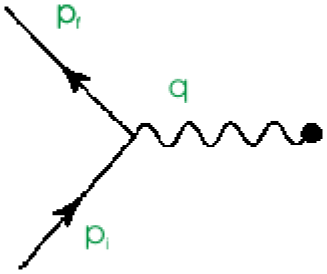
11 GeV < ECM < 90 GeV: production of u,d,s,c,b quarks:

$$R_{\text{quark}} = N_c(2(\tfrac{2}{3})^2 + 3(\tfrac{1}{3})^2) = N_c\frac{11}{9} = 3.66\ldots$$

$$R_{\text{exp}} \approx 4$$

Feynman cross-sections

Rutherford electron scattering [12, 5.58]

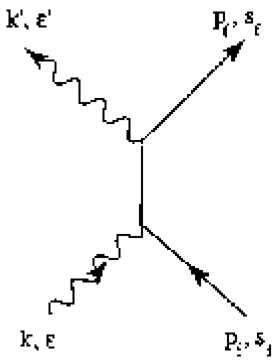


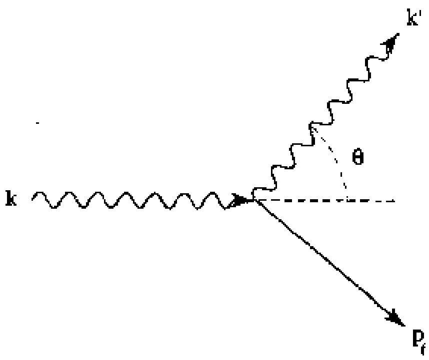
$$\frac{d\sigma}{d\Omega} = m^2 \frac{Z^2\alpha^2}{4|\mathbf{p}|^2 \sin^4(\theta/2)} \left[1 - \beta^2 \sin^2\left(\frac{\theta}{2}\right)\right]$$ Mott

where
$$p_i \cdot p_f = m^2 + 2\beta^2 E^2 \sin^2(\theta/2)$$
$$|\mathbf{q}|^2 \simeq 4|\mathbf{p}|^2 \sin^2(\theta/2)$$

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2}{4|\mathbf{p}|^2 \sin^4(\theta/2)} m^2$$ Rutherford $\beta \to 0$

Compton effect [12, 6.22]





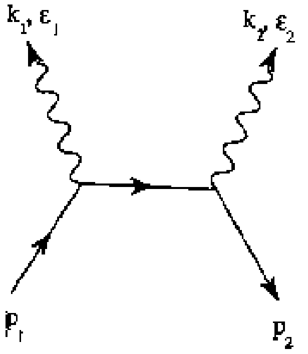
lab frame $\vec{p}_i = 0$ electron at rest

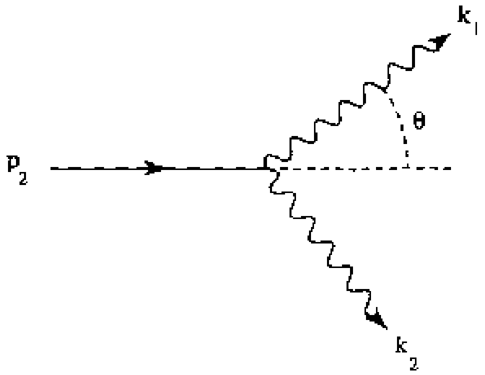
$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4m^2}\left(\frac{k'}{k}\right)^2 \left(\frac{k'}{k} + \frac{k}{k'} + 4(\epsilon \cdot \epsilon')^2 - 2\right)$$ Klein-Nishiima

$$\sigma = \left(\frac{8\pi\alpha^2}{3m^2}\right)(3/4)\left[\frac{1+a}{a^3}\left(\frac{2a(1+a)}{1+2a} - \log(1+2a)\right) + \frac{\log(1+2a)}{2a} - \frac{1+3a}{(1+2a)^2}\right]$$

where $a=k/m$

pair annihilation [12, 6.42]



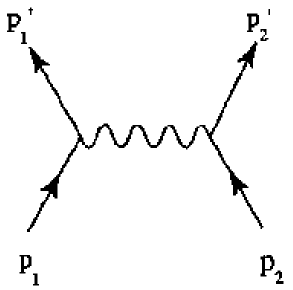


lab frame : $\vec{p}_1 = 0$ electron at rest

$$\sigma = \frac{\pi\alpha^2}{m^2(1+\gamma)}\left(\frac{\gamma^2+4\gamma+1}{\gamma^2-1}\log\left(\gamma+\sqrt{\gamma^2-1}\right)-\frac{\gamma+3}{\sqrt{\gamma^2-1}}\right)$$

where $\gamma = E_2/m$

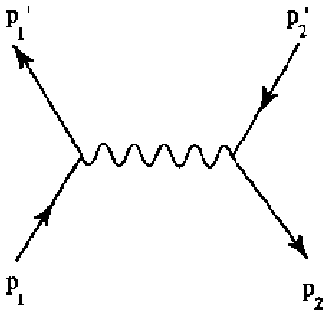
electron-electron (Moeller) scattering [12, 6.49]



Moeller

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(2E^2-m^2)^2}{4E^2(E^2-m^2)^2}\left[\frac{4}{\sin^4\theta}-\frac{3}{\sin^2\theta}+\frac{(E^2-m^2)^2}{(2E^2-m^2)^2}\left(1+\frac{4}{\sin^2\theta}\right)\right]$$

electron-positron (Bhabha) scattering [12,6.54]



$$\frac{-i(ie)^2}{(p_1-p_2)^2}\left(\bar{\mathrm{v}}(p_1')\gamma^\mu \mathrm{v}(p_2')\right)\left(\bar{\mathrm{u}}(p_2)\gamma^\mu \mathrm{u}(p_1)\right)$$

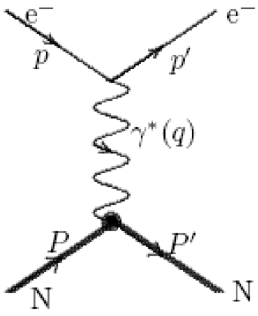
attractive potential in q-space: $V(\vec{q}) = \frac{-e^2}{|\vec{q}|^2}$

Bhabha

$$\frac{d\sigma}{d\Omega} = \frac{\alpha}{2E^2}\Big(\frac{5}{4} - \frac{8E^4 - m^4}{E^2(E^2-m^2)(1-\cos\theta)} + \frac{(2E^2-m^2)^2}{2(E^2-m^2)^2(1-\cos\theta)^2}$$
$$+ [16E^4]^{-1}\Big[2E^4(-1+2\cos\theta+\cos^2\theta) + 4E^2m^2(1-\cos\theta)(2+\cos\theta) + 2m^4\cos^2\theta\Big]\Big)$$

where $\theta = \theta(p_1, p_1')$

electron-nucleon scattering [11, 10.34]



lab frame: $P = (M, \vec{0})$, $p = (E, \vec{p})$, $p' = (E', \vec{p}\,')$ nucleon at rest

Rosenbluth

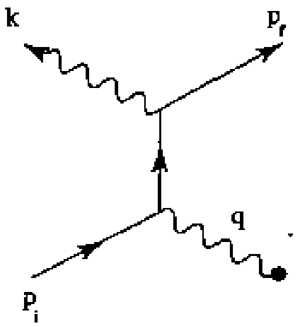
$$\frac{d\sigma}{d\Omega_{\mathrm{lab}}} = \left(\frac{d\sigma}{d\Omega_{\mathrm{lab}}}\right)_{\mathrm{NS}} \left[\frac{G_{\mathrm{E}}^2(q^2) - \frac{q^2}{4M^2}G_{\mathrm{M}}^2(q^2)}{1 - \frac{q^2}{4M^2}} - \frac{q^2}{2M^2}G_{\mathrm{M}}^2(q^2)\tan^2\frac{\theta}{2}\right]$$

$$\left(\frac{d\sigma}{d\Omega_{\mathrm{lab}}}\right)_{\mathrm{NS}} \equiv \frac{\sigma_{\mathrm{Mott}}}{1 + \frac{2E}{M}\sin^2\frac{\theta}{2}} \qquad \sigma_{\mathrm{Mott}} \equiv \left(\frac{\alpha\ \cos\frac{\theta}{2}}{2E\sin^2\frac{\theta}{2}}\right)^2$$

where $\theta = \theta(p, p')$, G_E and G_M are the electric and magnetic form factor of the nucleon *N=p* or *N=n*

$$G_E^{\mathrm{p}}(0) = 1 \ , \quad G_E^{\mathrm{n}}(0) = 0 \ , \quad G_M^{\mathrm{p}}(0) = 2.79 \ , \quad G_M^{\mathrm{n}}(0) = -1.91$$

bremsstrahlung [12, 6.68]



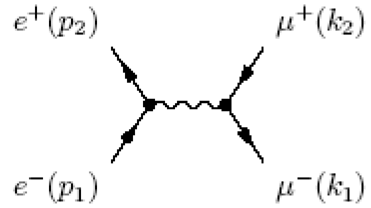
Bethe-Heitler

$$d\sigma = \frac{Z^2\alpha^3}{(2\pi)^2}\frac{p_f}{p_i q^4}\frac{d\omega}{\omega}d\Omega_\gamma d\Omega_e \Big(\frac{p_f^2\sin^2\theta_f}{(E_f - p_f\cos\theta_f)^2}(4E_i^2 - q^2) + \frac{p_i^2\sin\theta_i}{(E_i - p_i\cos\theta_i)^2}(4E_f^2 - q^2)$$
$$+ 2\omega^2\frac{p_i^2\sin^2\theta_i + p_f^2\sin^2\theta_f}{(E_f - p_f\cos\theta_f)(E_i - p_i\sin\theta_i)} - 2\frac{p_f p_i\sin\theta_i\sin\theta_f\cos\phi}{(E_f - p_f\cos\theta_f)(E_i - p_i\cos\theta_i)}(4E_iE_f - q^2 + 2\omega^2)\Big)$$

neutrino-electron scattering [11, 12.43]

$$\sigma(\nu_\mu + \mathrm{e}^- \to \nu_\mu + \mathrm{e}^-) \approx \frac{G_{\mathrm{F}}^2\ s}{4\pi}\left[(g_{\mathrm{V}}^{\mathrm{e}} + g_{\mathrm{A}}^{\mathrm{e}})^2 + \frac{1}{3}(g_{\mathrm{V}}^{\mathrm{e}} - g_{\mathrm{A}}^{\mathrm{e}})^2\right]$$

electron-positron muon-antimuon production [6]



$$(-\mathrm{i}e)^2(\bar{u}_{\sigma_1}(k_1)\gamma_\mu v_{\sigma_2}(k_2))\frac{-\mathrm{i}}{p^2+\mathrm{i}\epsilon}\left(g^{\mu\nu}-\frac{p^\mu p^\nu}{p^2+\mathrm{i}\epsilon}(1-\xi)\right)(\bar{v}_{\lambda_2}(p_2)\gamma_\nu u_{\lambda_1}(p_1))$$

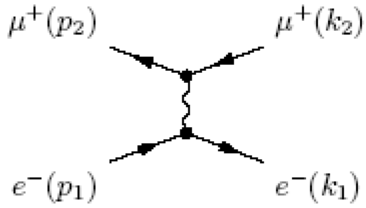
$p = k_1 + k_2 = p_1 + p_2.$

$m_e \to 0$: use cms frame: $p^\mu_{1,2} = (p, 0, 0, \pm p)$, $k^\mu_1 = (E, \pm k\cos\phi\sin\theta, \pm k\sin\phi\sin\theta, \pm k\cos\theta)$

$s = 4p^2 = 4E^2$.

$$\sigma = \frac{4\pi\alpha^2}{3s}\sqrt{1-\frac{4m_\mu^2}{s}}\left[1+\frac{2m_\mu^2}{s}\right]$$ (for $m_e << m_\mu$)

electron-muon scattering [6]



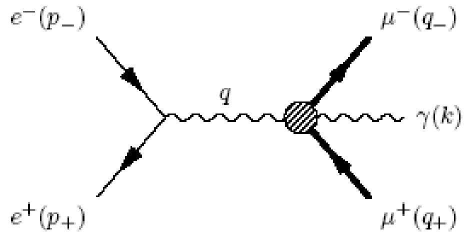
$$(-\mathrm{i}e)^2(\bar{v}_{\sigma_2}(p_2)\gamma_\mu v_{\lambda_2}(k_2))\frac{-\mathrm{i}g^{\mu\nu}}{(p_1-k_1)^2+\mathrm{i}\epsilon}(\bar{u}_{\sigma_1}(k_1)\gamma_\nu u_{\lambda_1}(p_1))$$

$p = k_1 + k_2 = p_1 + p_2.$

$s = 4p^2$

$$\sigma = \frac{4\pi\alpha^2}{3s}\sqrt{1-\frac{4m_\mu^2}{s}}\left[1+\frac{2m_\mu^2}{s}\right]$$

electron-positron muon-antimuon production [13]



m, M– are electron and muon masses

$$\sigma_{FSR}^{e^+e^-\to\mu^+\mu^-\gamma} = \frac{2\alpha}{\pi}\sigma_B(s)\Delta_{FSR}^{\mu^+\mu^-}(\beta),$$

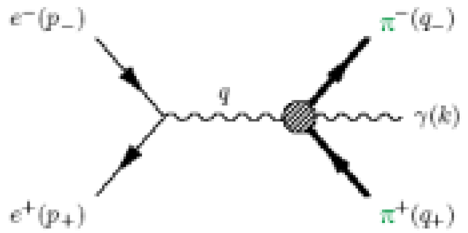
$$\begin{aligned}
\Delta_{FSR}^{\mu^+\mu^-}(\beta) &= \frac{3(5-3\beta^2)}{8(3-\beta^2)} + \frac{(1-\beta)(33-39\beta-17\beta^2+7\beta^3)}{16\beta(3-\beta^2)}L_\beta + \\
&+ 3\ln\left(\frac{1+\beta}{2}\right) - 2\ln\beta + \frac{1+\beta^2}{2\beta}F(\beta), \\
F(\beta) &= -2\mathrm{Li}_2(\beta) + 2\mathrm{Li}_2(-\beta) - 2\mathrm{Li}_2(1+\beta) + 2\mathrm{Li}_2(1-\beta) \\
&+ 3\mathrm{Li}_2\left(\frac{1+\beta}{2}\right) - 3\mathrm{Li}_2\left(\frac{1-\beta}{2}\right) + 3\ \xi_2.
\end{aligned}$$

$\sigma_B(s) = 2\pi\alpha^2\beta(3-\beta^2)/(3s)$ Born approximation

$$s=(p_+ + p_-)^2 \quad \beta^2 = 1-4\sigma, \qquad \sigma=\frac{M^2}{s}.$$

$$L_\beta = \ln\frac{1+\beta}{1-\beta}; \quad \xi_2=\pi^2/6$$ dilogarithm $$\mathrm{Li}_2(z) = -\int_0^z \frac{\ln(1-u)}{u}\,\mathrm{d}u$$

electron-positron pion production [13], [11]



m, *M*– are electron and pion masses

$$\sigma_{FSR}^{v+s} = \frac{2\alpha}{\pi}\sigma_B^{\pi^+\pi^-}(s)\left[\left(\frac{1+\beta^2}{2\beta}L_\beta - 1\right)\ln\Delta + b(s)\right],$$

$$\sigma_B^{\pi^+\pi^-}(s) \;=\; (\pi\alpha^2\beta^3)/(3s)|F_\pi(s)|^2$$ Born approximation

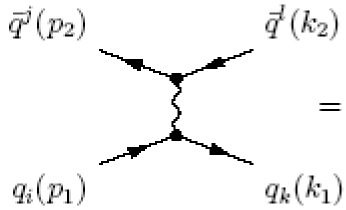
$$F_\pi(q^2) = 1 + \frac{q^2}{m_\rho - q^2}\frac{g_{\rho\pi\pi} f_\rho}{m_\rho}$$ pion form factor [11] with $g_{\rho\pi\pi} = 6.01$ $f_\rho = 0.15GeV$ $m_\rho = 0.902GeV$

$$s=(p_+ + p_-)^2 \quad \beta^2 = 1-4\sigma, \qquad \sigma=\frac{M^2}{s}. \quad \varepsilon=\frac{\sqrt{s}}{2}. \quad \xi_2=\pi^2/6 \qquad L_\beta=\ln\frac{1+\beta}{1-\beta};$$

$$b(s) = -1 + \frac{1-\beta}{2\beta}\rho + \frac{2+\beta^2}{\beta}\ln\frac{1+\beta}{2} + \frac{1+\beta^2}{2\beta}\left[\rho+\xi_2+L_\beta\ln\frac{1+\beta}{2\beta^2}+2\mathrm{Li}_2\left(\frac{1-\beta}{1+\beta}\right)\right],$$

$$\rho=\ln\frac{4}{1-\beta^2}, \qquad \Delta=\frac{\Delta\varepsilon}{\varepsilon}.$$

quark-antiquark scattering [6, 4.109]



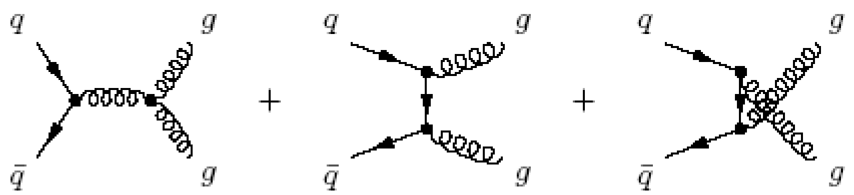
$=$

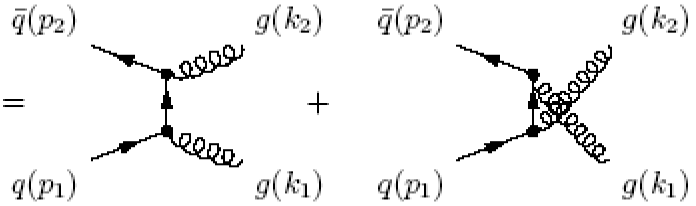
$$\frac{-\mathrm{i}(\mathrm{i}g_s)^2}{(p_1-k_1)^2}(\bar{v}(p_2)T_j^{a,l}\gamma^\mu v(k_2))(\bar{u}(k_1)T_k^{a,i}\gamma_\mu v(p_1))$$

potential $$V^{(R)}(\vec{q}) = \frac{-g_s^2}{|\vec{q}|^2}C^{(R)}$$ with R=1 (singlet) or R=8 (octet) since $3\otimes\bar{3}=1\oplus 8$

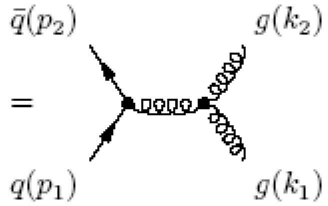
$$C^{(R)} = \begin{cases} C_F & R=1\,(\text{attractive}) \\ -\frac{1}{2N_c} & R=8\,(\text{repulsive}) \end{cases}$$

quark-antiquark annihilation [6,]



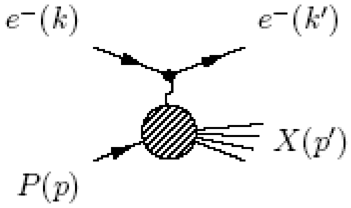


$$(\mathrm{i}g_s)^2\bar{v}(p_2)\left[T_j^{b,k}\not\epsilon_2^*\frac{i}{\not p_1-\not k_1-m_e}T_k^{a,i}\not\epsilon_1^*+\not\epsilon_1^*T_j^{a,k}\frac{i}{\not p_1-\not k_2-m_e}T_k^{b,i}\not\epsilon_2^*\right]u(p_1)$$



$$:\bar{v}(p_2)(-\mathrm{i}g_s)T_j^{c,i}\gamma^\mu u(p_1)\frac{-\mathrm{i}}{(p_1+p_2)^2}\times$$
$$g_s f^{abc}[(\epsilon_1^*\cdot\epsilon_2^*)(k_1-k_2)^\mu+2\epsilon_2^{\mu*}(k_2\cdot\epsilon_1^*)-2\epsilon_1^{\mu*}(k_1\cdot\epsilon_2^*)]]$$

DIS cross-section electron-parton [6, 5.2]



$$Q^2=-(k-k')^2 \qquad x=\frac{Q^2}{2(p\cdot q)} \qquad y=\frac{p\cdot(k-k')}{(k\cdot p)}$$

with

$$\frac{\mathrm{d}\sigma_{eP}}{\mathrm{d}Q^2\mathrm{d}x}=\frac{2\pi\alpha}{Q^4}\,\frac{1}{x}\,\left[xy^2F_1(x,Q^2)+F_2(x,Q^2)(1-y)\right].$$

where F_1 and F_2 are structure functions and $F_2(x,Q^2)=2xF_1(x,Q^2)$ (Callen-Gross relation)

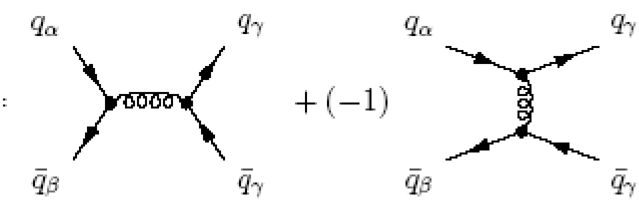
$$F_2^{em}=x\sum_{i=q,\bar{q}}e_i^2f_{i/p}(x)$$

, where e_i are charges, e.g.

$$F_2^{\text{proton}}=x(e_u^2u_p(x)+e_d^2d_p(x))=x\left(\frac{4}{9}u_p(x)+\frac{1}{9}d_p(x)\right)$$

[4, (34)]

four-quark cross-section [6, 5.85]



$$\frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}\hat{t}}=\frac{2}{\hat{s}}\frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}\cos\theta}=\frac{1}{\hat{s}^2}\frac{1}{8(2\pi)}\overline{|\mathcal{M}|^2}=\frac{2}{9}\frac{2\pi\alpha_s^2}{\hat{s}^2}\left\{\frac{s^2+u^2}{t^2}+\delta_{\alpha\beta}\left[n_\ell\left(\frac{t^2+u^2}{s^2}\right)-\frac{2}{3}\frac{u^2}{st}\right]\right\}$$

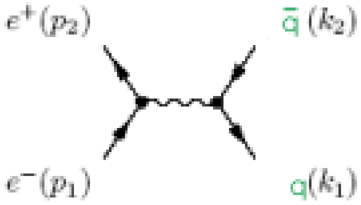
where

$$\hat{t}=(p_1-k_2)^2=2EE'(1-\cos\theta)$$

$t^2 = (p_1 \cdot k_1)(p_2 \cdot k_2)$ $u^2 = (p_1 \cdot k_2)(k_1 \cdot p_2)$ $s^2 = (p_1 \cdot p_2)(k_1 \cdot k_2)$

electron-positron quark-antiquark [6, 6.88]

$e^- e^+ \to q\bar{q}$



$$(-\mathrm{i}e)^2 Q_q (\bar{u}_{\sigma_1}(k_1)\gamma_\mu v_{\sigma_2}(k_2)) \frac{-\mathrm{i}g^{\mu\nu}}{(p_1+p_2)^2 + \mathrm{i}\epsilon} (\bar{v}_{\lambda_2}(p_2)\gamma_\nu u_{\lambda_1}(p_1)).$$

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = = 2\frac{e^4}{s^2}(u^2 + t^2)$$

$t^2 = (p_1 \cdot k_1)(p_2 \cdot k_2)$ $u^2 = (p_1 \cdot k_2)(k_1 \cdot p_2)$ $s^2 = (p_1 \cdot p_2)(k_1 \cdot k_2)$

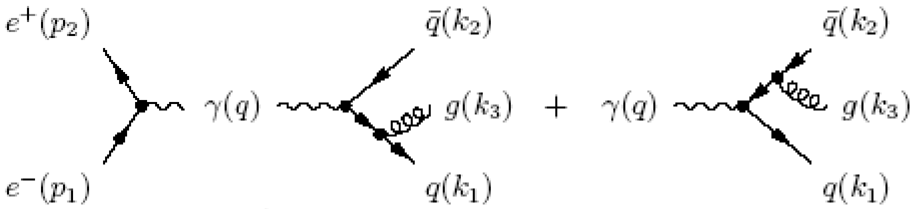
$$\frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}\hat{t}} = \frac{2}{\hat{s}}\frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}\cos\theta} = \frac{1}{\hat{s}^2}\frac{1}{8(2\pi)}\overline{|\mathcal{M}|^2} = \frac{2}{9}\frac{2\pi\alpha_s^2}{\hat{s}^2}\left\{\frac{s^2+u^2}{t^2} + \delta_{\alpha\beta}\left[n_\ell\left(\frac{t^2+u^2}{s^2}\right) - \frac{2}{3}\frac{u^2}{st}\right]\right\}$$

$$\frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}\hat{t}} = \frac{2}{\hat{s}}\frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}\cos\theta} = \frac{1}{\hat{s}^2}\frac{1}{8(2\pi)}\overline{|\mathcal{M}|^2} = 4\;\frac{2}{9}\frac{2\pi\alpha_s^2}{\hat{s}^2}\left\{\frac{s^2+u^2}{t^2} + \delta_{\alpha\beta}\left[n_\ell\left(\frac{t^2+u^2}{s^2}\right) - \frac{2}{3}\frac{u^2}{st}\right]\right\}$$

$$\hat{t} = (p_1 - k_2)^2 = 2EE'(1-\cos\theta) = \tfrac{\hat{s}}{2}(1-\cos\theta)$$

electron-positron quark-antiquark-gluon [6, 95]

$e^- e^+ \to q\bar{q}g$



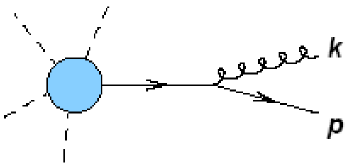
$$(-\mathrm{i}g_s)T^{a,i_1}_{i_2}Q_q$$

$$\bar{u}_{\sigma_1}(k_1)\left[\not{\epsilon}^*_{\sigma_3}(k_3)\frac{\mathrm{i}(\not{k}_1+\not{k}_3)}{(k_1+k_3)^2}\gamma^\mu + \gamma^\mu\frac{\mathrm{i}(\not{k}_2+\not{k}_3)}{(k_2+k_3)^2}\not{\epsilon}^*_{\sigma_3}(k_3)\right] v_{\sigma_2}(k_2)$$

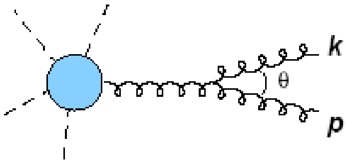
$$\mathrm{i}\mathcal{M}(e^{-,L}_{p_1}e^{+,R}_{p_2} \to q^R_{k_1}\bar{q}^L_{k_2}g^+_{k_3}) = (-\mathrm{i}e)^2\langle q^{\sigma_1}_{k_1}\bar{q}^{\sigma_2}_{k_2}g^{\sigma_3}_{k_3}|j_{q,\mu}(0)|0\rangle\frac{-\mathrm{i}g^{\mu\nu}}{(p_1+p_2)^2}(\bar{v}_L(p_2)\gamma_\nu u_L(p_1))$$

$$\frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}\hat{t}} = \frac{2}{\hat{s}}\frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}\cos\theta} = \frac{1}{\hat{s}^2}\frac{1}{8(2\pi)}\overline{|\mathcal{M}|^2}$$

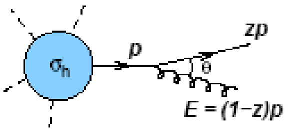
Soft gluon emission and DGLAP [4]



$$\simeq \quad \frac{2\alpha_s C_F}{\pi}\frac{dE}{E}\frac{d\theta}{\theta}\,,$$

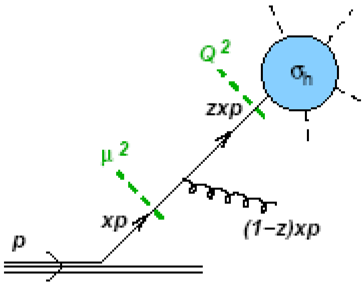


$$\simeq \quad \frac{2\alpha_s C_A}{\pi}\frac{dE}{E}\frac{d\theta}{\theta}\,.$$



$$\sigma_{h+g} \simeq \sigma_h \frac{\alpha_s C_F}{\pi}\frac{dz}{1-z}\frac{dk_t^2}{k_t^2}\,,$$

DGLAP ansatz



$$\sigma_0 = \int dx\; \sigma_h(xp)\; q(x,\mu_F^2)\,,$$

initial cross-section

$$\sigma_1 \simeq \frac{\alpha_s C_F}{\pi} \underbrace{\int_{\mu_F^2}^{Q^2} \frac{dk_t^2}{k_t^2}}_{\text{finite (large?)}} \underbrace{\int \frac{dx\,dz}{1-z}\left[\sigma_h(zxp) - \sigma_h(xp)\right] q(x,\mu_F^2)}_{\text{finite}}\,,$$

correction with one initial-emission

$$\frac{dq(x,\mu_F^2)}{d\ln \mu_F^2} = \frac{\alpha_s}{2\pi}\int_x^1 dz\, p_{qq}(z)\,\frac{q(x/z,\mu_F^2)}{z} \;-\; \frac{\alpha_s}{2\pi}\int_0^1 dz\, p_{qq}(z)\, q(x,\mu_F^2)$$

$$p_{qq}(z) = P_{qq} = C_F\left(\frac{1+z^2}{1-z}\right)$$

q->q+g

$$P_{qg}(z) = T_R\left[z^2 + (1-z)^2\right]$$

g->q+g

$$P_{gq}(z) = C_F\left[\frac{1+(1-z)^2}{z}\right]$$

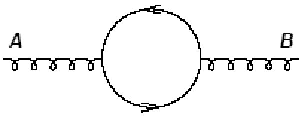
q->g+g

$$P_{gg}(z) = 2C_A\left[\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z)\right] + \delta(1-z)\frac{(11C_A - 4n_f T_R)}{6}$$

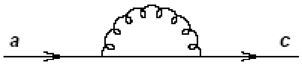
g->g+g

where color factor C_F and transmission factor T_R are defined as follows

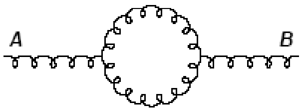
$$\mathrm{Tr}(t^A t^B) = T_R \delta^{AB}\,, \quad T_R = \frac{1}{2}$$



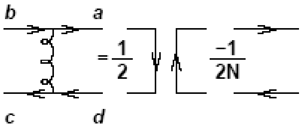
$$\sum_A t^A_{ab} t^A_{bc} = C_F \delta_{ac}\,, \quad C_F = \frac{N_C^2 - 1}{2N_C} = \frac{4}{3}$$



$$\sum_{C,D} f^{ACD} f^{BCD} = C_A \delta^{AB}\,, \quad C_A = N_C = 3$$



$$t^A_{ab} t^A_{cd} = \frac{1}{2}\delta_{bc}\delta_{ad} - \frac{1}{2N_C}\delta_{ab}\delta_{cd} \text{ (Fierz)}$$



4. QCD on lattice with Wilson-loops

QCD lattice formulation [8, 9]

Feynman path integral
$$Z = \int \mathcal{D}A_\mu \, \mathcal{D}\psi \, \mathcal{D}\overline{\psi} \, e^{-S}$$

where S is the QCD action
$$S = \int d^4x \, \left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \overline{\psi}M\psi\right)$$
, M Dirac operator $M = i\gamma^\mu D_\mu - m$

$$Z = \int \mathcal{D}A_\mu \, \det M \, e^{\int d^4x \, (-\frac{1}{4}F_{\mu\nu}F^{\mu\nu})}$$
on the gauge field A_μ

expectation value of an operator O

$$\langle \mathcal{O} \rangle = \frac{1}{Z}\int \mathcal{D}A_\mu \, \mathcal{O} \, e^{-S}$$

corresponds to the Boltzmann average

$\langle O \rangle = \mathrm{Tr} O e^{-\beta H - \mu N}$ with $\beta = \dfrac{1}{kT}$, chemical potential μ and particle number N

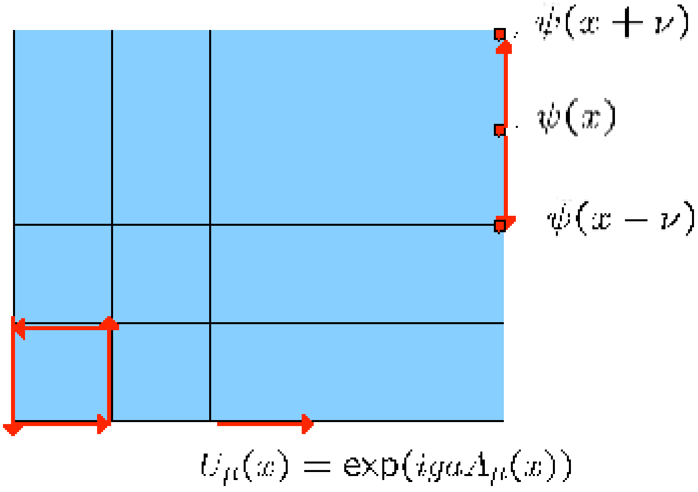
integral over 4-dimensional Euclidean lattice ($i\,t$, x_1,x_2,x_3) with complex (Wick-rot.) time on lattice

$$\langle O \rangle = Tr\int \Pi_\mu dU_\mu(x) O \det(M[U]) \exp(-S[U(x)])$$

with interaction matrix $M[U]$, under $U_\mu(x) = \exp(i\,ga\,A_\mu(x))$ the local gauge transformation with coupling constant g , lattice step size a , gluon field $A_\mu(x)$, action $S[U(x)]$

quenched approximation : we impose $\det(M) = const$ for the interaction matrix $M[U]$, quark loops are neglected

Theory on lattice with step size a [8]



momentum/energy cut-off $\mu = \pi/a$

gluon-field $A_\mu(x) \equiv A^a_\mu(x) \cdot \lambda_a/2$;

gluon-induced transformation for fermion moving from site x to y (ordered product)

$$\psi(y) = \mathcal{P}\, e^{\int_x^y igA_\mu(x)dx_\mu} \, \psi(x)$$

on-lattice $y = x + \mu a$ gluon transformation

$$U(x, x+\hat{\mu}) \equiv U_\mu(x) = e^{iagA_\mu(x+\frac{\hat{\mu}}{2})}$$

$$U(x, x-\hat{\mu}) \equiv U_{-\mu}(x) = e^{-iagA_\mu(x-\frac{\hat{\mu}}{2})} = U^\dagger(x-\hat{\mu}, x)$$

Wilson action on a plaquette (μa,νa)

$$W^{1\times1}_{\mu\nu} = U_\mu(x)U_\nu(x+\hat\mu)U^\dagger_\mu(x+\hat\nu)U^\dagger_\nu(x) \ = e^{iag[A_\mu(x+\frac{\hat\mu}{2})+A_\nu(x+\hat\mu+\frac{\hat\nu}{2})-A_\mu(x+\hat\nu+\frac{\hat\mu}{2})-A_\nu(x+\frac{\hat\nu}{2})]}$$

$$\mathrm{Re\,Tr}(1-W^{1\times1}_{\mu\nu}) = \frac{a^4g^2}{2}F_{\mu\nu}F^{\mu\nu} \qquad \mathrm{Im}(W^{1\times1}_{\mu\nu}) = a^2gF_{\mu\nu}$$ + higher terms in *a*

$$S_g = \ \beta\sum_x\sum_{\mu<\nu}\mathrm{Re\,Tr}\ \frac{1}{3}(1-W^{1\times1}_{\mu\nu})$$

where $\beta = \dfrac{6}{g^2}$

from this results the naive lattice action for fermions

$$\mathcal{S}^N = m_q\sum_x\overline{\psi}(x)\psi(x) \ + \frac{1}{2a}\sum_x\overline{\psi}(x)\gamma_\mu[U_\mu(x)\psi(x+\hat\mu)-U^\dagger_\mu(x-\hat\mu)\psi(x-\hat\mu)]$$

$$\equiv\sum_x\overline{\psi}(x)M^N_{xy}[U]\psi(y)$$

$$S_{Wilson}|_{a\to0} = \int d^4x\ tr F^2_{\mu\nu}$$ limit $a\to0$

with the interaction matrix M^N on the lattice

$$M^N_{i,j}[U] \ = \ m_q\delta_{ij} \ + \ \frac{1}{2a}\sum_\mu \ [\gamma_\mu U_{i,\mu}\delta_{i,j-\mu} \ - \ \gamma_\mu U^\dagger_{i-\mu,\mu}\delta_{i,j+\mu}]$$

Wilson fermions [9]

The naive lattice action introduces 16 for one fermion. To eliminate the copies Wilson introduced a fifth lattice dimension with step size *r* . The Wilson action becomes

$$A^W \ = \ m_q\sum_x\overline{\psi}(x)\psi(x) \ + \ \frac{1}{2a}\sum_{x,\mu}\overline{\psi}(x)\gamma_\mu[U_\mu(x)\psi(x+\hat\mu) \ - \ U^\dagger_\mu(x-\hat\mu)\psi(x-\hat\mu)]$$

$$- \ \frac{r}{2a}\sum_{x,\mu}\overline{\psi}(x)[U_\mu(x)\psi(x+\hat\mu) \ -2\psi(x)+ \ U^\dagger_\mu(x-\hat\mu)\psi(x-\hat\mu)]$$

$$= \ \frac{(m_qa \ +4r)}{a}\sum_x\overline{\psi}(x)\psi(x)$$

$$+ \ \frac{1}{2a}\sum_x\overline{\psi}[(\gamma_\mu-r)U_\mu(x)\psi(x+\hat\mu) \ -(\gamma_\mu+r)\ U^\dagger_\mu(x-\hat\mu)\psi(x-\hat\mu)]$$

$$\equiv \ \sum_{x,y}\overline{\psi}^L_x M^W_{xy}\psi^L_y$$

with the interaction matrix M^W

$$M^W_{x,y}[U]a = \delta_{xy} - \kappa\sum_\mu\left[(r-\gamma_\mu)U_{x,\mu}\delta_{x,y-\mu}+(r+\gamma_\mu)U^\dagger_{x-\mu,\mu}\delta_{x,y+\mu}\right]$$

and rescaling $\kappa = 1/(2m_qa+8r) \qquad \psi^L = \sqrt{m_qa+4r}\ \psi = \psi/\sqrt{2\kappa}$

the quark mass becomes

$$m_qa \ = \ \frac{1}{2\kappa}-4r$$

Wilson fermion propagator is

$$S_F(p) = M^{-1}_W(p) \ = \frac{a}{1 \ - \ 2\kappa\ \sum_\mu\left(r\ \cos p_\mu a - i\gamma_\mu\ \sin p_\mu a\right)}$$

Symmetries parity, time inversion, charge conjugation, chirality

$$\begin{array}{ll} \mathcal{P}: & S_F(x,y,[U]) \to \gamma_4\ S_F(x^P,y^P,[U^P])\ \gamma_4 \\ \mathcal{T}: & S_F(x,y,[U]) \to \gamma_4\gamma_5\ S_F(x^T,y^T,[U^T])\ \gamma_5\gamma_4 \\ \mathcal{C}: & S_F(x,y,[U]) \to \gamma_4\gamma_2\ S_F^{\mathrm{T}}(x,y,[U^C])\ \gamma_2\gamma_4 \\ \mathcal{H}: & S_F(x,y,[U]) \to \gamma_5\ S_F^{\dagger}(y,x,[U])\ \gamma_5 \\ \mathcal{CPH}: & S_F(x,y,[U]) \to \mathcal{C}\gamma_4\ S_F^{\dagger}(y^P,x^P,[U^C])\ \gamma_4\mathcal{C}^{-1} \end{array}$$

dispersion relation for the mass

$$m_q^{pole}a = r(\cosh Ea - 1) + \sinh Ea$$

$$e^{Ea} = \frac{(ma+r) \pm \sqrt{(1+2mar+m^2a^2)}}{1+r}$$

solution

where the second solution corresponds to the pole in the Euclidean propagator at $p_4 = \pi$ and is the temporal doubler, removed for *r=1* .

The fermion density expectation value is

$$\langle\overline{\psi}\psi\rangle^{\mathrm{WI}} \equiv \langle 0|S_F(0,0)|0\rangle = \lim_{m_q\to 0} m_q \int d^4x \langle 0|P(x)P(0)|0\rangle,$$

Staggered fermions [9]

The 16-doubling of the naive action is reduced to 4 by the transformation to staggered fermions

$$\psi(x) = \Gamma_x\chi(x) \qquad \overline{\psi}(x) = \overline{\chi}(x)\Gamma_x^{\dagger} \qquad \Gamma_x = \gamma_1^{x_1}\gamma_2^{x_2}\gamma_4^{x_3}\gamma_4^{x_4}.$$

the action becomes

$$\mathcal{S}_S = m_q \sum_x \overline{\chi}(x)\chi(x) + \tfrac{1}{2}\sum_{x,\mu} \overline{\chi}_x \eta_{x,\mu}\left(U_{\mu,x}\chi_{x+\hat{\mu}} - U^{\dagger}_{\mu,x-\hat{\mu}}\chi_{x-\hat{\mu}}\right) \equiv \sum_{x,y} \overline{\chi}(x) M^S_{xy}\chi(y)$$

with the interaction matrix M^S

$$M^S[U]_{x,y} = m_q\delta_{xy} + \frac{1}{2}\sum_{\mu} \eta_{x,\mu}\left[U_{x,\mu}\delta_{x,y-\mu} - U^{\dagger}_{x-\mu,\mu}\delta_{x,y+\mu}\right]$$

with ±1-factors

$$\begin{aligned} \eta_{x,\mu} &= (-)^{\sum_{\nu<\mu} x_\nu} \\ \zeta_{x,\mu} &= (-)^{\sum_{\nu>\mu} x_\nu} \\ \epsilon_x &= (-)^{x_1+x_2+x_3+x_4} \end{aligned}$$

Improved action Lüscher-Weiss [9]

The LW action improves the naive action beyond *O(a²)* .

original leading term

$$\mathcal{L}^{(4)} = \sum_{\mu\nu} F_{\mu\nu}F_{\mu\nu}$$

corrections

$$\mathcal{L}_1^{(6)} = \sum_{\mu,\nu} \mathrm{Tr}\Big(D_\mu F_{\mu\nu} D_\mu F_{\mu\nu}\Big)$$

$$\mathcal{L}_2^{(6)} = \sum_{\mu,\nu,\rho} \mathrm{Tr}\Big(D_\mu F_{\nu\rho} D_\mu F_{\nu\rho}\Big)$$

$$\mathcal{L}_3^{(6)} = \sum_{\mu,\nu,\rho} \mathrm{Tr}\Big(D_\mu F_{\mu\rho} D_\nu F_{\nu\rho}\Big)$$

improved action

$$\mathcal{S}_g = \frac{6}{g^2}\left\{c^{(4)}(g^2)\ \mathcal{L}^{(4)} + \sum_{i=1,3} c_i^{(6)}(g^2)\ \mathcal{L}_i^{(6)}\right\}$$

with normalization condition

$$c^{(4)}(g^2) + 8c_1^{(6)}(g^2) + 8c_2^{(6)}(g^2) + 16c_3^{(6)}(g^2) = 1$$

Lüscher-Weiss:

$$c^{(4)} + 20c_1^{(6)} - 4c_2^{(6)} + 4c_3^{(6)} = 0 \quad c_2^{(6)} = 0$$

explicitly

$$c_0^{(4)}(g^2) = \frac{5}{3} + 0.2370g^2$$

$$c_1^{(6)}(g^2) = -\frac{1}{12} - 0.02521g^2$$

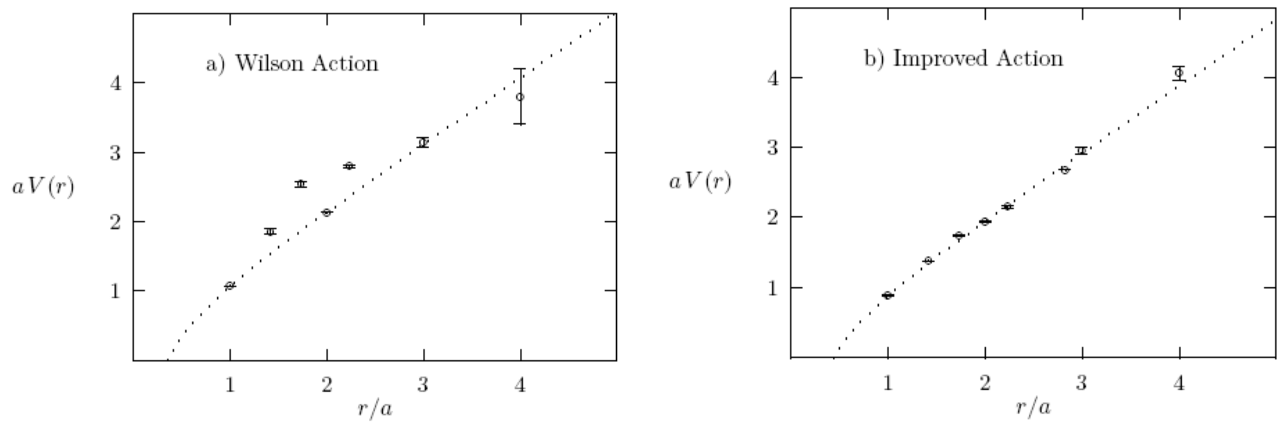
$$c_2^{(6)}(g^2) = -0.00441g^2$$

$$c_3^{(6)}(g^2) = 0\,.$$

TILW (tadpole improved Lüscher-Weiss)

$$\mathcal{S}_{TILW} = \beta_{eff}\left(\mathcal{L}^{(4)} - \frac{1}{20u_0^2}\left(1 + (0.1602 - 2X)\ g^2\right)\mathcal{L}_1^{(6)}\right.$$

with $u_0 = 1 - Xg^2$

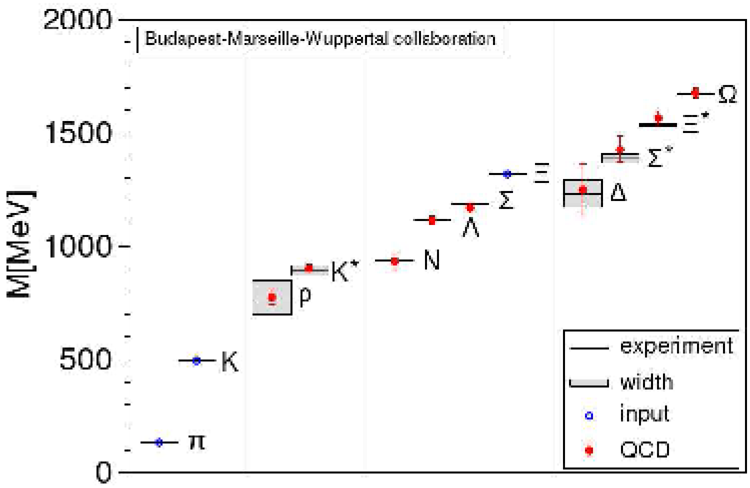


[9] Static-quark potential computed on 64 lattices with a ≈ 0.4 fm using the Wilson action and the TILW action. The dotted line is the standard infrared parameterization for the continuum potential,
V (r) = Kr − π/12r + c,
adjusted to fit the on-axis values of the potential.

Lattice calculations of hadron masses [9 18.1]
The mass (energy) eigenvalues are extracted from the decay rate

$$\Gamma(\tau) \equiv \langle O_f(\tau)O_i(0)\rangle - \langle O_f\rangle\ \langle O_i\rangle = \sum_n \frac{\langle 0|O_f|n\rangle\langle n|O_i|0\rangle}{2M_n} e^{-M_n\tau}$$

for state transition from O_i to O_f, usually $O_f = O_i^\dagger$
[4, 1.1]



The measured spectrum of hadron masses, compared to a lattice calculation [9]. The open blue circles are the hadron masses that have been used to fix the three parameters of the calculation: the value of the QCD coupling, the average of the up and down quark masses (taken equal) and the strange-quark mass. All other points are results of the calculation.

Part B Minimization of QCD-QED-action on lattice and its results

1. Solution methods in lattice-QCD [8,9,12,13,14]

Basically, there are four solution methods in lattice-QCD (LQCD):

Perturbative analytic Feynman solution

Here one calculates the reaction cross-sections from Feynman diagrams evaluating the corresponding Feynman-integrals in analogy to the QED. As the QCD is renormalizable, all Feynman integrals can be made finite. However, this works only for convergent Feynman series, i.e. if the interaction constant $g_c<1$. This is the case for large energies $E>E_\Lambda=220MeV$.

Non-perturbative on-lattice Wilson-loop method

Here the expectation value of an operator (e.g. energy=Hamilton operator) is calculated using path integrals

$$\langle O\rangle = Tr\int \Pi_\mu dU_\mu(x)\, O\, \det(M[U])\exp(-S[U(x)])$$

with interaction matrix $M[U]$, under $U_\mu(x)=\exp(i\, ga\, A_\mu(x))$ the local gauge transformation, with coupling constant g , lattice step size a , gluon field $A_\mu(x)$, action $S[U(x)]$, on closed loops on the lattice.

Non-perturbative on-lattice eom solution

The QCD equations-of-motion (eom) are derived from the minimal-action-principle as the Euler-Lagrange-equations corresponding to the QCD Lagrangian. They are

$\partial^\mu F^a_{\mu\nu} + g f^{abc} A^{\mu b} F^c_{\mu\nu} = 0,$ the Yang-Mills-equations for the gluon wavefunction $A^a{}_\mu(x)$ and the color-field-tensor $F^a{}_{\mu\nu}(x)$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$$

and the Dirac equation

$$(i\hbar D_\mu \gamma^\mu - mc)\psi^a = 0$$

with the color-covariant-derivative $D_\mu = I\partial_\mu - igT^a A^a_\mu$ and the quark-wavefunction $\psi^a{}_\mu(x)$

These are n_q+n_g partial differential equations (pdeq) first order in x^μ , for the $n_q=2$ or $n_q=3$ quarks and $n_g=8$ gluons, adding a gauge condition and a boundary condition for $A^a{}_\mu(x)$.

They must be solved numerically on a lattice as an eigenvalue problem of the Dirac equation, which is very difficult and time-consuming for a one-dimensional lattice of, say, $n_1=100$ points (total number of points $n=n_1^4=10^8$) .

Non-perturbative on-lattice minimization of action

The starting point is the minimum-action-principle for QCD:

$S=\int L_{QCD}(x^\mu, q_i, Ag_i)\, dx = \min$ with a gauge condition and a boundary condition for $Ag_i(x)$.

It can be extended to QCD+QED

$$S=\int \left(L_{QCD}(x^\mu, q_i, Ag_i) + L_{QED}(x^\mu, q_i, Ae_i)\right)dx = \min$$

for the quarks q_i, QCD-gluons Ag_i , QED-photons Ae_i ,

In order to carry out the minimization numerically, we introduce an equidistant 4-dimensional lattice $L(t_k, r_k, \theta_k, \varphi_k) = (t_k)\times(r_k)\times(\theta_k)\times(\varphi_k)$, extract a small random sub-lattice L_{sub} and approximate the integral by a sum over L_{sub} :

$\tilde{S} = \sum_{x\in L_{sub}} L_{QCD}(x, q_i, Ag_i)\Delta V$, where $\Delta V = \Delta t\, \Delta r\, \Delta\theta\, \Delta\varphi$ is the elementary integration volume in spherical coordinates, and model the quark wavefunctions as parameterized Gauss functions $q = q(x, par(q))$ and the gluon-wavefunctions as Ritz-Galerkin series on a function system $f_k(x)$ with coefficients α_k :

$Ag = Ag\left(\sum \alpha_k f_k(x), par(Ag)\right)$, according the photon-wavefunctions $Ae = Ae(x, \{\alpha_k\}, par(Ae))$.

We impose the gauge condition for Ag_i : $\partial_\mu Ag_i^\mu = 0$ and a boundary condition : $Ag_i(r=r_0)=0$, the quark-wavefunctions are normalized $\int q_i(x)\, d^3x = 1$.

The minimization is carried out in dependence on $par = (par(q), \{\alpha_k\}, par(Ag))$

$par_0 = \min\left(\widetilde{S}, par\right)$, where par_0 yields information about the energy (=mass) , the sizes and the inner structure of the considered hadron.

Non-perturbative on-lattice minimization of action

lattice
$L(t_k, r_k, \theta_k, \varphi_k) = (t_k) \times (r_k) \times (\theta_k) \times (\varphi_k)$

quark wf: gaussian
$q = q(x, par(q))$
normalized $\int q_i(x)\, d^3x = 1$

gluon wf: RG expansion on f_k
$Ag = Ag\left(\sum \alpha_k f_k(x), par(Ag)\right)$

action minimization
$$S = \int L_{QCD}(x^\mu, q_i, Ag_i)\, dx = \min$$

gauge cond.
$\partial_\mu Ag_i{}^\mu = 0$
boundary cond.
$Ag_i(r = r_0) = 0$

discretized action
$\tilde{S} = \sum_{x \in L_{sub}} L_{QCD}(x, q_i, Ag_i) \Delta V = \min$
$par = (par(q), \{\alpha_k\}, par(Ag))$
$\rightarrow \quad par_0 = \min(\tilde{S}, par)$

2. The ansatz for the quark and gluon wavefunctions [19]

Gluon wavefunction

For the gluon wavefunction we apply here the full Ritz-Galerkin series on the function system $f_k(r,\theta)=\{bfunc(r,r_0,dr_0)\,r^{k_1}, k_1=0,...,n_r\}\times\{(\cos^{k_2}\theta,\cos^{k_2}\theta\sin\theta), k_2=0,...,n_\theta\}$ with coefficients α_k ,

where $bfunc(r,r_0,dr_0)=\dfrac{1}{1+\exp\left(\dfrac{r-r_0}{dr_0}\right)}$ is a Fermi-step-function which limits the region $r\le r_0$ of the hadron

with „smearing width“ dr_0 .

$Ag(t,r,\theta)=\{\begin{pmatrix} Ag_i(t,r,\theta)\cos aA_i \\ Ag_i(t,r,\theta)\sin aA_i \end{pmatrix}, i=1,...,8\}$, where aA_i is the phase angle between the particle and the anti-particle part of the gluon, and with the Ritz-Galerkin-expansion

$Ag_k(t,r,\theta)=\sum_j \alpha[k,j]\,f_j(r,\theta)\exp(-i\,t\,EA_k)$ with energies EA_k

Because of color-symmetry, the active (non-zero) gluons are

$Ag=\{Ag_1,...,Ag_8\}$ all gluons for nucleons

$Ag=\{Ag_1,Ag_2,Ag_4,Ag_5,Ag_6,Ag_7\}$ 6 non-diagonal gluons for vector-mesons

$Ag=\{Ag_2,Ag_5,Ag_7\}$ 3 quark-antiquark gluons for for pseudo-scalar mesons

Quark wavefunction

The first-generation (u,d)-hadrons consist of three quarks (nucleons) or three color-symmetric quark-antiquark-combinations (vector-mesons) or two quark-antiquark-combinations (pseudo-scalar mesons)

$q=\{\begin{pmatrix} q_1 \\ 0 \end{pmatrix},\begin{pmatrix} q_2 \\ 0 \end{pmatrix},\begin{pmatrix} q_3 \\ 0 \end{pmatrix}\}$ for nucleons

$q=\{\left(\dfrac{\begin{pmatrix} q_1 \\ \bar q_1 \end{pmatrix}\pm\begin{pmatrix} q_2 \\ \bar q_2 \end{pmatrix}}{\sqrt{2}}\right),\left(\dfrac{\begin{pmatrix} q_1 \\ \bar q_1 \end{pmatrix}\pm\begin{pmatrix} q_2 \\ \bar q_2 \end{pmatrix}}{\sqrt{2}}\right),\left(\dfrac{\begin{pmatrix} q_1 \\ \bar q_1 \end{pmatrix}\pm\begin{pmatrix} q_2 \\ \bar q_2 \end{pmatrix}}{\sqrt{2}}\right)\}$ or $q=\{\begin{pmatrix} q_1 \\ \bar q_2 \end{pmatrix},\begin{pmatrix} q_1 \\ \bar q_2 \end{pmatrix},\begin{pmatrix} q_1 \\ \bar q_2 \end{pmatrix}\}$ for vector-mesons

(omega0, rho0, rho+)

$q=\{\begin{pmatrix} q_1 \\ 0 \end{pmatrix},\begin{pmatrix} 0 \\ \bar q_2 \end{pmatrix},0\}$ or $q=\{\left(\dfrac{\begin{pmatrix} q_1 \\ 0 \end{pmatrix}+\begin{pmatrix} 0 \\ \bar q_2 \end{pmatrix}}{\sqrt{2}}\right),\left(\dfrac{\begin{pmatrix} q_1 \\ 0 \end{pmatrix}-\begin{pmatrix} 0 \\ \bar q_2 \end{pmatrix}}{\sqrt{2}}\right),0\}$ for pseudo-scalar mesons (pi+, pi0)

A Ritz-Galerkin series for quarks would blow up the complexity of calculation, therefore we use here a simpler model, based on the asymptotic-freedom property of quarks: gaussian “blobs”

$q_k(t,r,\theta)=\exp(-i\,t\,Eu_k)\exp\left(-\dfrac{(\vec r-\vec r_{u,k})^2}{2\,dr_{u,k}}\right)\cos a_k$,where Eu_k is the energy, $\vec r_{u,k}=(ru_k,\theta u_k)$ and $dr_{u,k}$ is the position(r,θ) and its width, a_k is the quark-antiquark phase and the antiquark is

$\bar q_k(t,r)=\exp(-i\,t\,Eu_k)\exp\left(-\dfrac{(\vec r-\vec r_{u,k})^2}{2\,dr_{u,k}}\right)\sin a_k$

The ansatz and the color symmetry

The form of the quark color-wavefunction and the corresponding set of active gluons are *enforced* by the color-symmetry and the number of particles equal to the number of combinations.

The 8 **gluons** of the SU(5) form 3 families: the diagonal $\{Ag_3,Ag_8\}$, which *map color indices into itself*,

the non-diagonal $\{Ag_1,Ag_4,Ag_6\}$, which exchange *color-index with a different color* index, and

the non-diagonal $\{Ag_2,Ag_5,Ag_7\}$, which exchange *color-index with a different anti-color* index.

The **nucleons** consist of three quarks with color *(r,g,b)*, and the color wavefunction *q* is mapped into itself under color-permutations, therefore the full set of 8 gluons Ag_i is required, and there are only two possibilities for first-generation hadrons: *p=uud* and *n=ddu* .
The **vector mesons** consist of quark-antiquark pairs, where the color wavefunction *q* has three identical components.
q is mapped into itself under the corresponding set of 6 non-diagonal gluons
$Ag = \{Ag_1, Ag_2, Ag_4, Ag_5, Ag_6, Ag_7\}$ (each *flips two color indices*).
It is seen immediately that the three combinations listed above are the only possible ones, which is confirmed by the existence of the three v-mesons omega0, rho0, rho+ .
The **pseudo-scalar** mesons consist of quark-antiquark pairs, where the color wavefunction *q* has two non-zero components. The corresponding gluon set are the 3 non-diagonal color-anticolor gluons $Ag = \{Ag_2, Ag_5, Ag_7\}$, which exchange a *color-quark with a different anti-color-quark*. For example, Ag_2 flips color-indices (3,1) and transforms $q_{12} = \{\begin{pmatrix} q_{1c} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \overline{q}_{2\overline{c}} \end{pmatrix}, 0\}$ into $q_{23} = \{0, \begin{pmatrix} q_{1\overline{c}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \overline{q}_{2c} \end{pmatrix}\}$. So in reality, the wavefunction is a superposition of the three (q_{12}, q_{23}, q_{31}) and is mapped by the gluon set into itself.
Again, one can see immediately that there are only two possible combinations, which correspond to the two known ps-mesons pi+ and pi0 .

3. The numerical algorithm [19]

The energy, length, and time are made dimensionless by using the units: E($E_0 = \frac{\hbar c}{1\,fm} = 0.196 GeV$), r(*fm*) , t(*fm/c*) $fm=10^{-15}m$. The hadrons have axial symmetry, so we can set $\varphi=0$ and use the spherical coordinates (t, r, θ) .

We choose the equidistant lattice for the intervals $(t,r,\theta) \in [0,1]\times[0,1]\times[0,\pi]$ with 21x21x11 points and, for the minimization 8x in parallel, 8 random sublattices :

$l[ix, j] = \{\{(t_{i1}, r_{i2}, t_{i3}) \,|\, (i1,i2,i3) = random(lattice, j=1...100)\} \,|\, ix = 1,...,8\}$.

For the Ritz-Galerkin expansion we use the 12 functions

$f_k(r,\theta) = \{bfunc(r, r_0, dr_0)\, r^{k_1}, k_1 = 0,...,n_r\} \times \{(\cos^{k_2}\theta, \cos^{k_2}\theta \sin\theta), k_2 = 0,...,n_\theta\}$

The action $S = \int L_{QCD}(x^\mu, q_i, Ag_i)\, r^2 \sin\theta \, dt\, dr\, d\theta\, d\varphi$ becomes a mean-value on the sublattice $l[ix]$

$\tilde{S}[ix] = \frac{1}{N(l[ix])} \sum_{x \in l[ix]_{sub}} L_{QCD}(x, q_i, Ag_i) 2\pi V_{tr\theta}$,where $V_{tr\theta} = \pi$ the (t,r,θ)-volume and $N(l[ix])$ number of points is. We impose the gauge condition and the boundary condition for Ag_i via penalty-function (imposing exact conditions is possible, but slows down the minimization process enormously).

$\tilde{S}$ is minimized 8x in parallel with the Mathematica-minimzation method “simulated annealing” , the execution time on a 2.7GHz Xeon E5 is 9100s for the proton *p=uud* , the complexity $K(\tilde{S}[ix])$ =8.4 million terms.

The minimization is performed in the parameters $par = (par(q), \{\alpha_k\}, par(Ag))$, for the proton is the number of parameters $N(\{\alpha_k\})=16*12=164$, $N(par(q))=3*5=15$, $N(par(Ag))=8*2=16$.

The proper parameters of the quarks and the gluons are:

$par(q_i) = \{Eu_i, a_i, ru_i, \theta u_i, dru_i\}$, $par(Ag_i) = \{EA_i, aA_i\}$

Criteria for correctness of the ansatz

1.Convergence of minimization

As we found out during the computation, a wrong ansatz, e.g. lacking color symmetry, leads to a non-convergent minimization. We chose a high goal precision of *prec=10-4*, so there was a high probability that a convergent minimizations hits a real (global) minimum.

2.High relative deviation between solutions

Strongly differing solutions indicate a non-correct ansatz, as we found out e.g. for the nucleons with too many degrees –of-freedom for the gluons: the relative deviation for crucial variables, like energy, should be no more than 2% for the nucleons and 6% for the ps-mesons.

3.Vanishing parameter-derivatives

A true minimum must satisfy the derivative-condition $\frac{\partial S}{\partial p_i} = 0$, where p_i is one of the minimization parameters, Normally, the parameter-derivatives are close to zero, otherwise the minimum is not genuine, or the ansatz is wrong.

4.Boundary condition and gauge condition

The boundary and gauge condition must have values close to zero, otherwise the weight for the penalty function is too low.

5.Minimum value

The minimum value should be -30,...,30 for the considered parameter range. Very large positive values result in the case of too high penalty weights. Very large negative values may come out, if the Ritz-Galerkin parameters α_i are not bounded appropriately.

6.Correct energy scale and number of particles

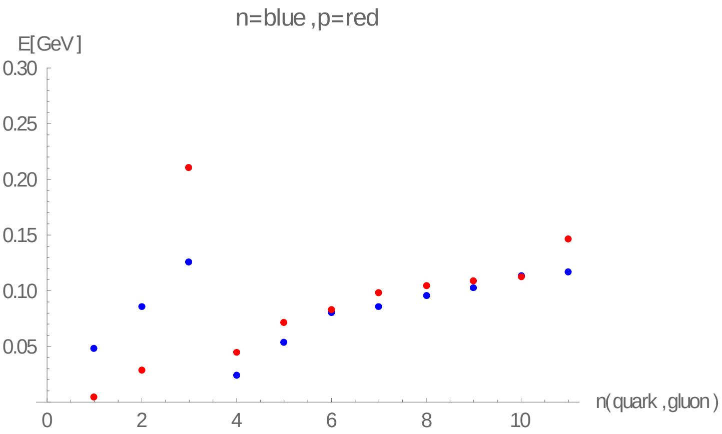
The three types of first-generation hadrons have energy scales: E(nucleon)≈0.98GeV, E(v-meson)≈0.78GeV, E(ps-meson)≈0.14GeV , and these values emerge automatically with 8, 6 and 3 gluons respectively.

Furthermore, with the above ansatz, the number of possible particles is 2, 3, 2 respectively.

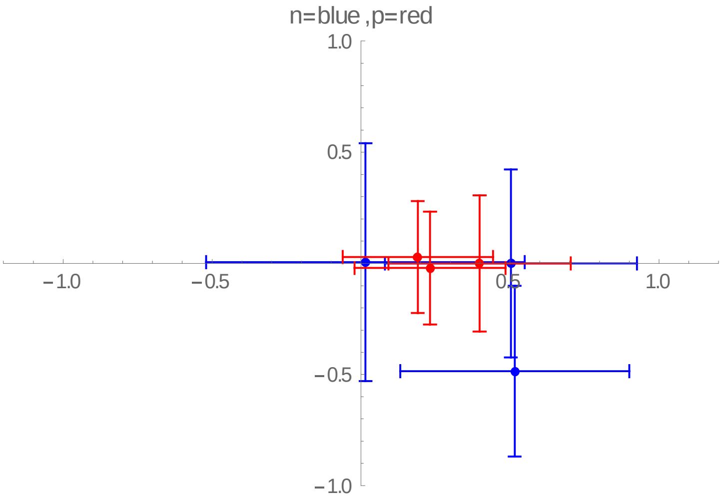
4. The results for first-generation hadrons [19]

nucleons n, p quarks(3),gluons(8),spin=1/2

energy quarks, gluons



distribution quarks (r[fm])



The quark distribution differs largely between the nucleons: the proton has only one orbital orthogonal to the z-axis, the neutron has two orbitals at an with an angle of $\alpha=\pi/4$. The small mass difference is probably due to the electromagnetic contribution, which is about 1% of the total mass.
The mass of the nucleons, as is the case for all first-generation-hadrons is generated almost exclusively by the energy of the gluons and the quarks, the rest masses of *u* and *d* ($m_u=2.3MeV$, $m_d=4.8MeV$) contribute very little to the total mass.
The gluon distribution is practically the same for both nucleons, which is to be expected, since the two particles are identical for the color interaction.

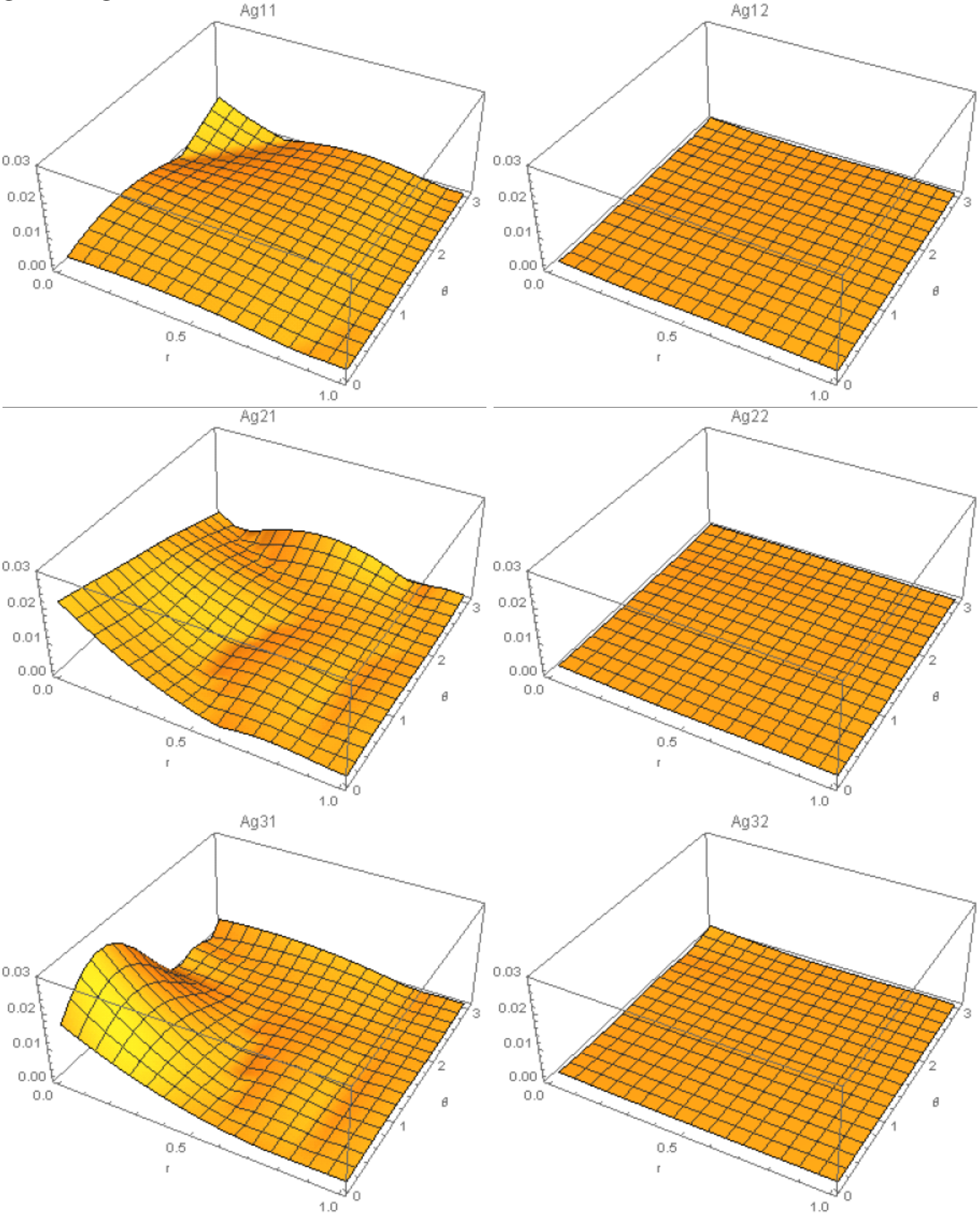
proton p=uud

m=0.938GeV, r_0=0.84fm

E_{tot}=0.945GeV, ΔE_{tot}=0.032, dE_{em}=-0.013

Eu_i	EA_i	a_i	aA_i	dru_i	ru_i	θu_i
0.0047,0.028,0.211	0.044,0.071,0.083,0.098,0.105, 0.108,0.113,0.146	-.99,-.99,.99	0,...,0	.16,.27,.75	.16,.15,.41	-.12,.08,0
ΔEu_i	ΔEA_i	Δa_i	ΔaA_i	Δdru_i	Δru_i	$\Delta\theta u_i$
0.004,0.007,0.014	0.018,0.006,0.005,0.006,0.004, 0.002,0.001,0.062	.0041,.0037,.0014	0,...,0	.29,.26,.25	.20,.050,.016	.50,.42,0

gluons Agi



The proton *p* has one rotation plane (orbital), the two quarks (u,d) are close at $r=0.15$ low energy $E\leq 0.03$, the second u-quark further outside $r=0.4$, and high energy $E=0.2$. The "smearing" width is comparable, $\delta r\approx 0.3$. The electromagnetic correction is negative and much larger than with the neutron, $dE_{em}=-0.013GeV$, which is probably the reason for the proton's smaller mass.

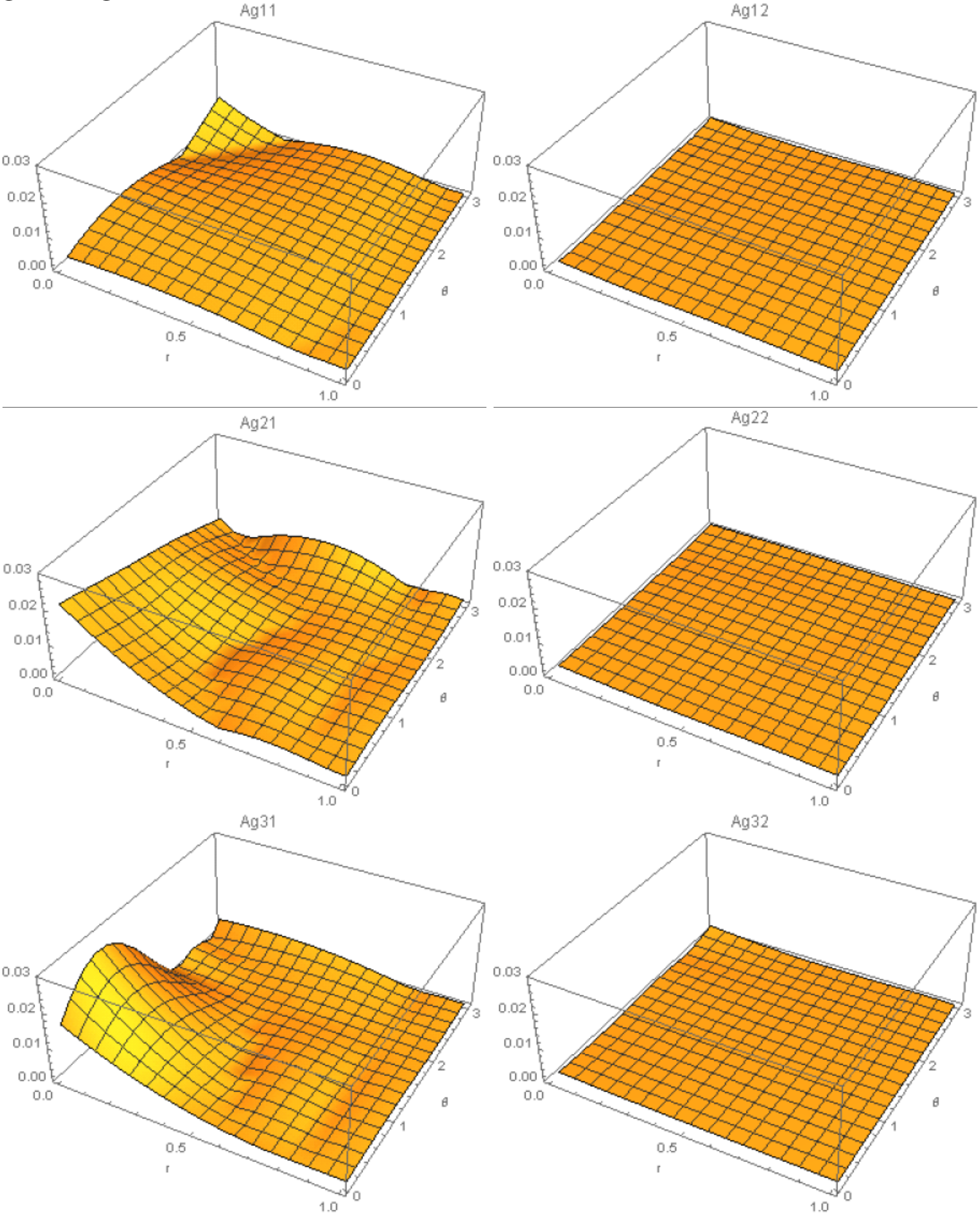
neutron n=ddu

m=0.939GeV, r_0=0.84fm

E_{tot}=0.945GeV, ΔE_{tot}=0.018, dE_{em}=+0.0017

Eu_i	EA_i	a_i	aA_i	dru_i	ru_i	θu_i
0.048,0.086,0.126	0.024,0.054,0.08,0.086,0.096, 0.103,0.113,0.117	-.92,-.95,.93	0,...,0	.72,1.05,.82	.71,.016,.50	-.68,.35,0
ΔEu_i	ΔEA_i	Δa_i	ΔaA_i	Δdru_i	Δru_i	$\Delta\theta u_i$
0.011,0.012,0.002	0.0005,0.005,0.0009,0.004, 0.00001,0.0005,0.0004,0.003	.017,.021,.041	0,...,0	.031,.052,.034	.042,.021,.021	.008,.007,0

gluons Agi



The neutron *n* has two orbitals with an angle of $\alpha=\pi/4$, the u-quark is at the center with low energy $E=0.05$, the two d-quarks sit in the orbitals with higher energies $E=0.09,0.013$. The “smearing” width is comparable, $\delta r\approx 0.4$ and higher than with the proton.

The gluon distribution is practically the same as for the proton, which is to be expected, since the two particles are identical for the color interaction.
The electromagnetic correction is positive and much smaller than with the proton, dE_{em}=+0.0017GeV , which is probably the reason for the proton's smaller mass.

magnetic moment of nucleons

The magnetic moment is $\mu = \frac{q}{2m}L = \frac{q}{2m}m\omega r^2$, for a rotating charge distribution: $\mu = \frac{\omega}{2}\sum_i q_i r_i^2 = \frac{\omega}{2}I_q$,
where $I_q = \sum_i q_i r_i^2 \to \int r^2 dq$ is the momentum of charge, in analogy to the momentum of inertia $I_m = \int r^2 dm$.
For a rotating solid sphere with radius r_0 with constant charge density $I_q = \frac{2}{5}qr_0^2$.
The magnetic moment of the nucleons is measured in nuclear magnetons $\mu_N = \frac{e\hbar}{2m}$, which is the magnetic moment of a rotating solid sphere with constant charge density $\mu_N = \frac{\omega}{2}I_q(sphere) = \frac{\omega}{2}\frac{2}{5}er_0^2$.
The actual momentum of charge is therefore:

$$I_q = \sum_i q(q_i)r(q_i)^2$$

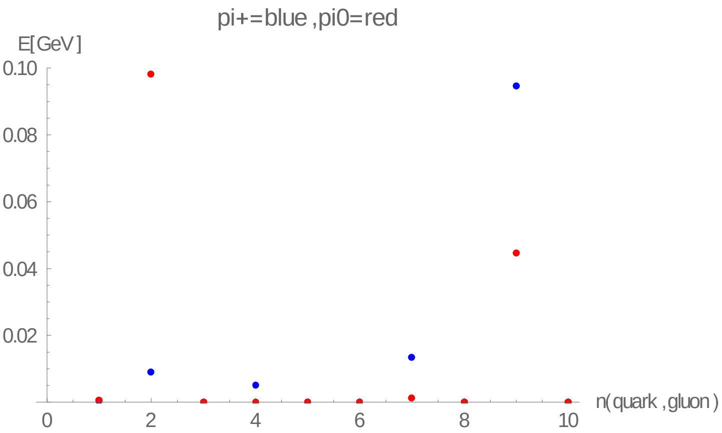
We have to take into account the "smearing" Δr_i of radius r_i $\left\langle r_i^2 \right\rangle = \frac{\int r^2 \exp(-\frac{(r-r_i)^2}{2\Delta r})dr}{\int \exp(-\frac{(r-r_i)^2}{2\Delta r})dr}$, so it becomes

$$I_q = \sum_i q(q_i)\left\langle r(q_i)^2 \right\rangle$$

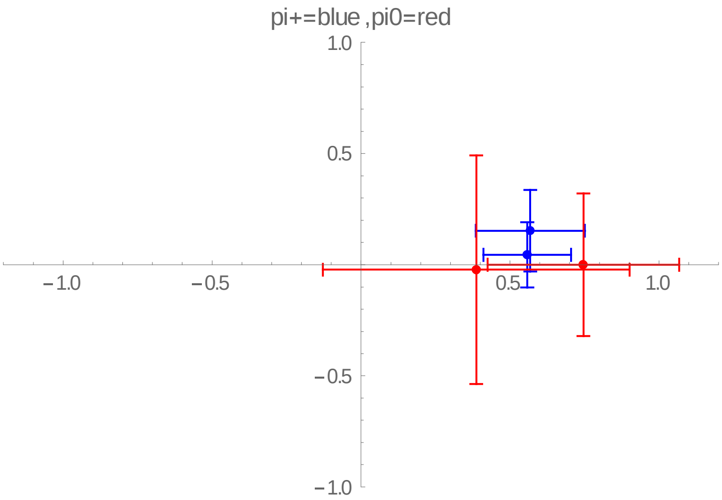
We get for the neutron
$I_{qn}=-0.1766\ e$, $I_{qNn}= 0.106\ e$, so $\frac{I_{qn}}{I_{qNn}} = -1.766$, measured $\frac{\mu}{\mu_N} = -1.91$
and for the proton
$I_{qp}=+\ 0.2226\ e$, $I_{qNp}=\ 0.0909\ e$, so $\frac{I_{qp}}{I_{qNp}} = +\ 2.448$, measured $\frac{\mu}{\mu_N} = +2.793$
The calculation does not take into account the orbitals, and there is also the statistical uncertainty of the order 7% , so the results are satisfactory.

pseudo-scalar mesons pi+, pi0 quarks(2),gluons(3),spin=0-

energy quarks, gluons



distribution quarks (r[fm]): independent(θ)=spherical



The pseudo-scalar mesons are spherically-symmetric, there is no θ-dependence: θ≈0 in the quark-distribution, the gluon-wavefunctions show little θ-dependence, and the gluon amplitudes are much smaller (factor 30) for pi0 than for pi+ .

For the pi0, $u\bar{u}$ and $d\bar{d}$ sit at *r=0.4 E≈0* , and at *r=0.75 E≈0.1* .

For the pi+, the *u* and $\bar{d}$ have practically equal radii, but different energies: *r=0.6 E≈0.001* , and *r=0.6 E≈0.01* .

The measured masses of the ps-mesons (0.135, 0.139) are reproduced by the calculation (0.155±0.025, 0.129±0.026), but only roughly within the error bounds.

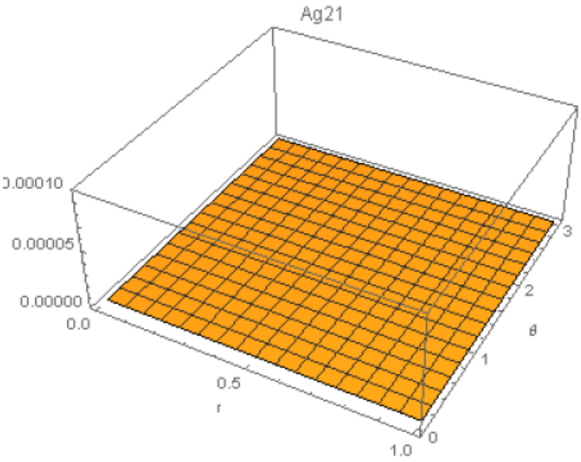
ps-meson pi0= $\left(u\bar{u} - d\bar{d}\right)/\sqrt{2}$

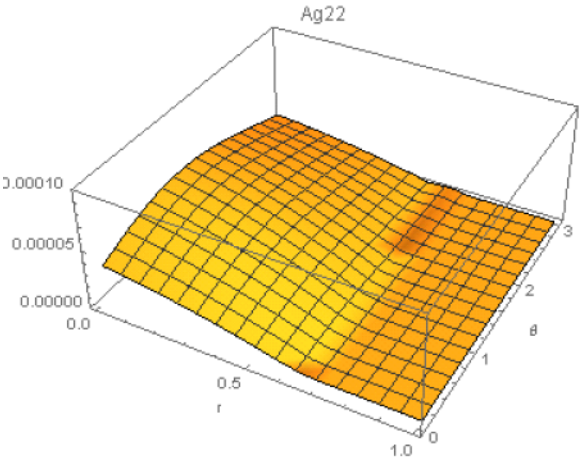
m=0.135GeV, r_0=0.66fm

E_{tot}=0.155GeV, ΔE_{tot}=0.025, dE_{em}=+0.007

Eu_i	EA_i	a_i	aA_i	dru_i	ru_i	θu_i
0.0007,0.098	0,0,0,0,0.0012,0,0.045,0	.073,-.650	0,-.77,0,0,-.131,0,-.634,0	.985,.631	.387,.746	-.058,0
ΔEu_i	ΔEA_i	Δa_i	ΔaA_i	Δdru_i	Δru_i	$\Delta\theta u_i$
0.001,0.013	0,0,0,0,0.002,0,0.022,0	.028,.018	0,.40,0,0,.38,0,.25,0	.040,.031	.039,.011	.010,0

gluons Agi



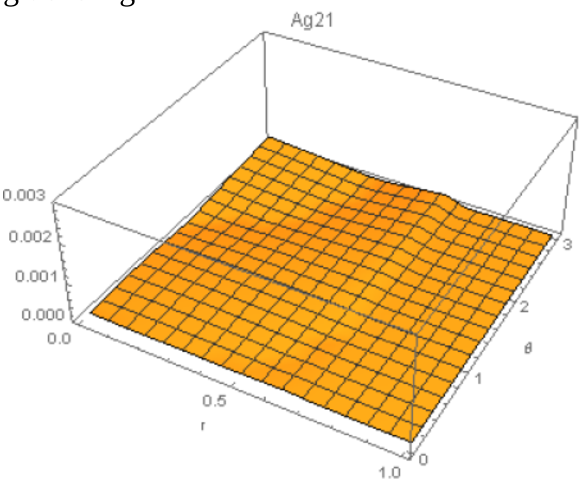


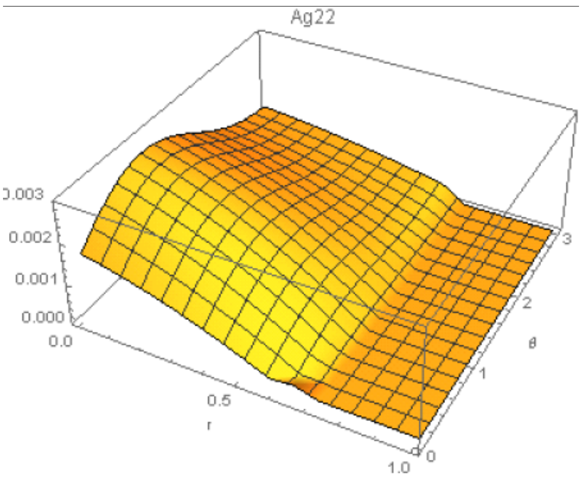
ps-meson pi+= $u\bar{d}$
m=0.139GeV, r_0=0.66fm

E_{tot}=0.129GeV, ΔE_{tot}=0.026, dE_{em}=+0.0014

Eu_i	EA_i	a_i	aA_i	dru_i	ru_i	θu_i
0.0004,0.009	0,0.005,0,0,0.014,0,0.0945,0	-.136,-.319	0,-.868,0,0,-.011,0,-.556,0	.020,.025	.588,.560	.180,0
ΔEu_i	ΔEA_i	Δa_i	ΔaA_i	Δdru_i	Δru_i	$\Delta\theta u_i$
0.001,0.012	0,0.003,0,0,0.016,0,0.017,0	.68,.67	0,.294,0,0,.100,0,.223,0	.0,.008	.190,.171	.243,0

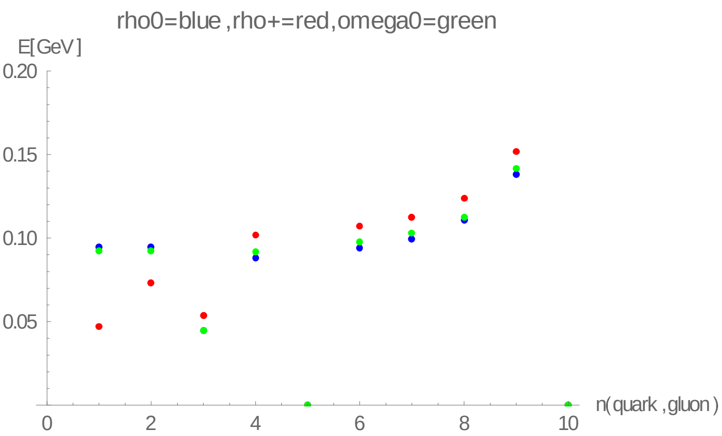
gluons Agi



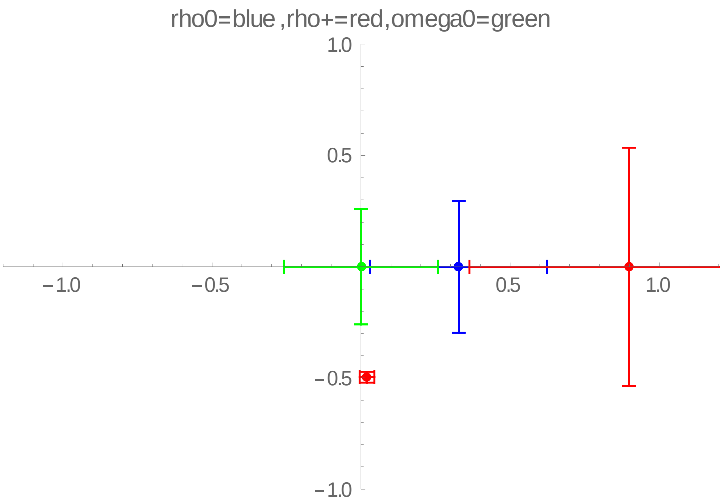


vector mesons rho0, rho+, omega0 quarks(2),gluons(6),spin=1

energy quarks, gluons



distribution quarks (r[fm])



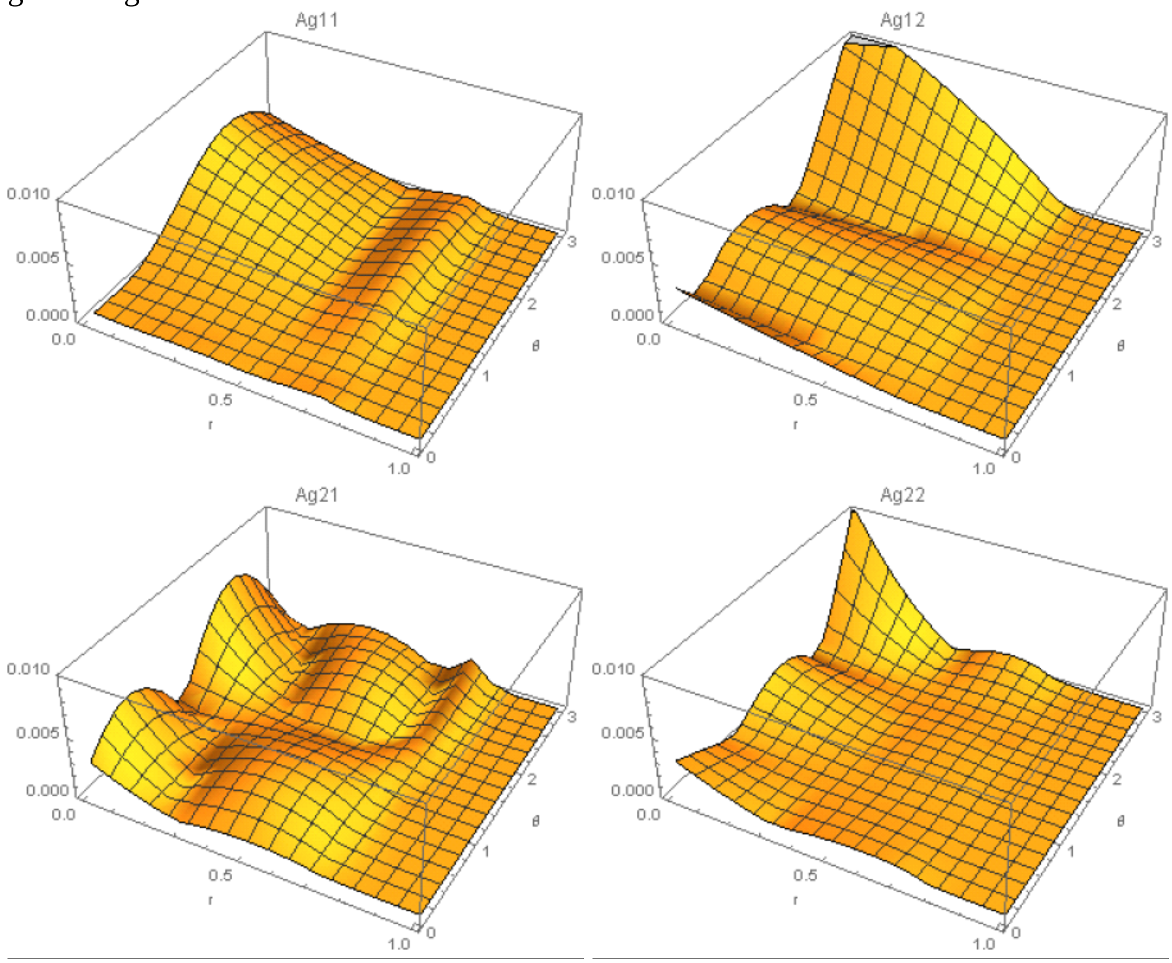
The vector mesons are spin-1 bosons but only rho+ shows an explicit θ-dependence of quark-distribution: it has two orbitals. The gluons show explicit θ-dependence and are, as for the nucleons, practically equal for all three particles.
For rho0: the quarks $u\bar{u}$ and $d\bar{d}$ have identical parameters $r=0.5,\ \delta r=0.3,\ E=0.1$
For omega0: the quarks $u\bar{u}$ and $d\bar{d}$ again have identical parameters, are at center, $\delta r=0.25,\ E=0.1$
For rho+: the heavier quark $\bar{d}$ has $r=0.5$, $\delta r=0.05,\ E=0.05$, the light quark u has $r=0.9$, $\delta r=0.5,\ E=0.07$, rho+ has two orthogonal orbitals. Its two quarks have completely different width; the $\bar{d}$ quark closer to the center has a small bandwidth, the light u quark is strongly "smeared" like all the other quarks in the 3 particles.
The measured masses of the v-mesons (0.775, 0.775, 0.782) are reproduced correctly by the calculation (0.771±0.0052, 0.779±0.012, 0.782±0.007).

v-meson rho0= $\left(u\bar{u}-d\bar{d}\right)/\sqrt{2}$
m=0.775GeV, r_0=0.75fm

E_{tot}=0.771GeV, ΔE_{tot}=0.0052, dE_{em}=+0.002

Eu_i	EA_i	a_i	aA_i	dru_i	ru_i	θu_i
0.094,0.094	0.045,0.088,0,0.094,0.099, 0.111,0.138,0	-.0057,-.0057	.018,-,003,0,.250,-.809,.227,-.533,0	.56,.56	.2327,.327	0,0
ΔEu_i	ΔEA_i	Δa_i	ΔaA_i	Δdru_i	Δru_i	$\Delta\theta u_i$
0.0003,0.0003	0.005,0.0005,0,0.0005,0.0005 ,0.002,0.0005,0	.0005,.0006	.015,.002,0,.008,.002,.006,.003,0	.071,.071	.033,.033	0,0

gluons Agi

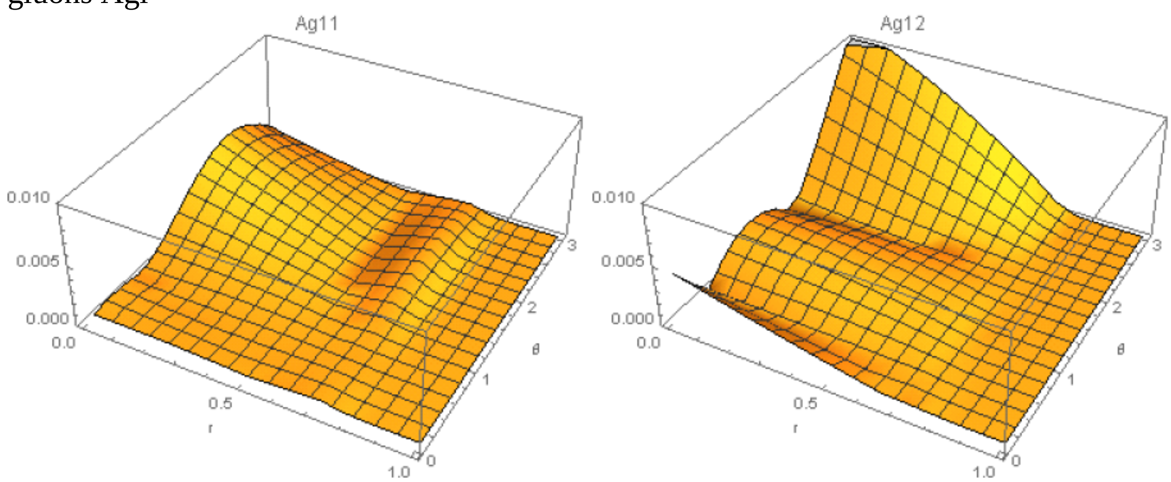


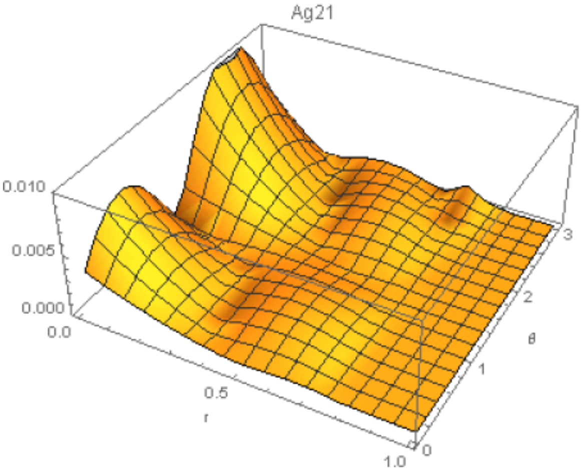
v-meson rho+= $u\bar{d}$
m=0.775GeV, r_0=0.75fm

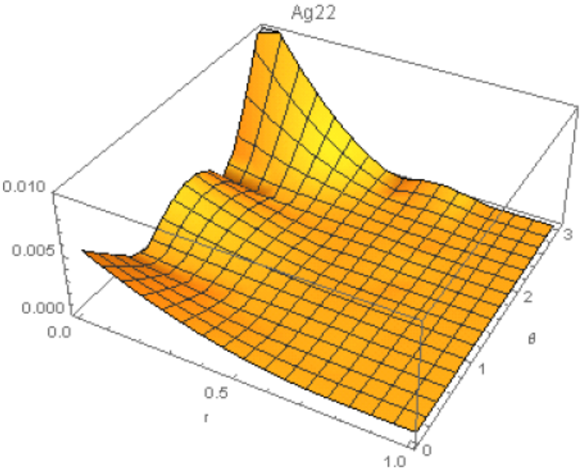
E_{tot}=0.779GeV, ΔE_{tot}=0.012, dE_{em}=+0.002

Eu_i	EA_i	a_i	aA_i	dru_i	ru_i	θu_i
0.047,0.073	0.054,0.102,0,0.107,0.113, 0.124,0.152,0	-.628,.620	.011,-,003,0,.250,-.810,.229,-.534,0	1.05,.02	.89,.48	0,-1.0
ΔEu_i	ΔEA_i	Δa_i	ΔaA_i	Δdru_i	Δru_i	$\Delta\theta u_i$
0.004,0.009	0.006,0.001,0,0.001,0.001, 0.003,0.001,0	.018,.0	.012,.003,0,.011,.001,.003,.001,0	.019,.012	.018,.011	0,.001

gluons Agi





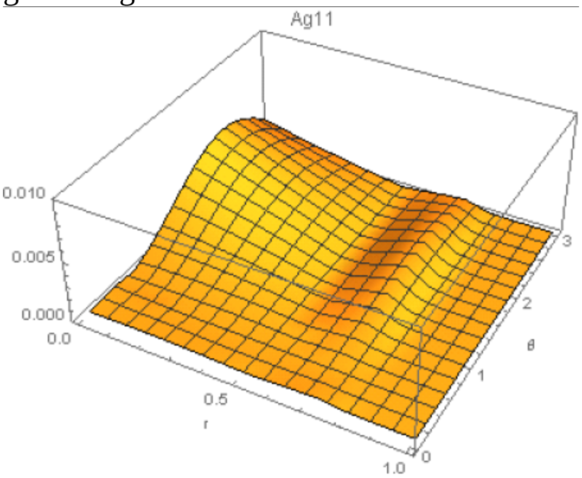


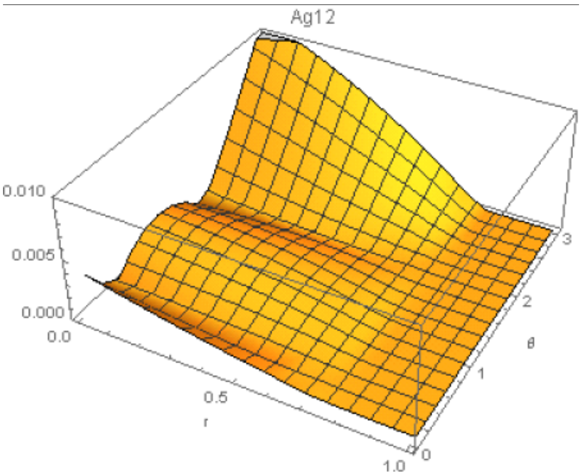
v-meson omega0= $\left(u\bar{u}+d\bar{d}\right)/\sqrt{2}$
m=0.782GeV, r_0=0.75fm

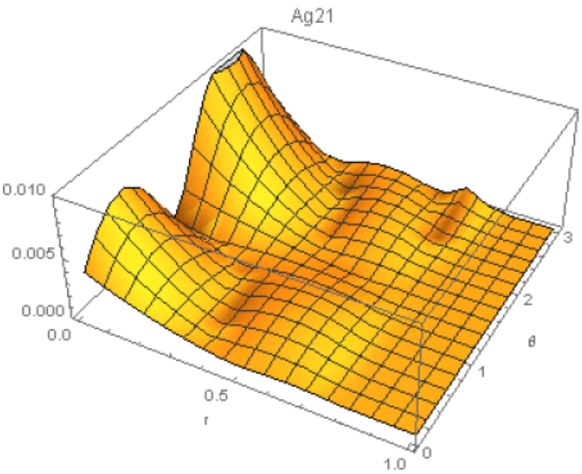
E_{tot}=0.782GeV, ΔE_{tot}=0.007, dE_{em}=+0.002

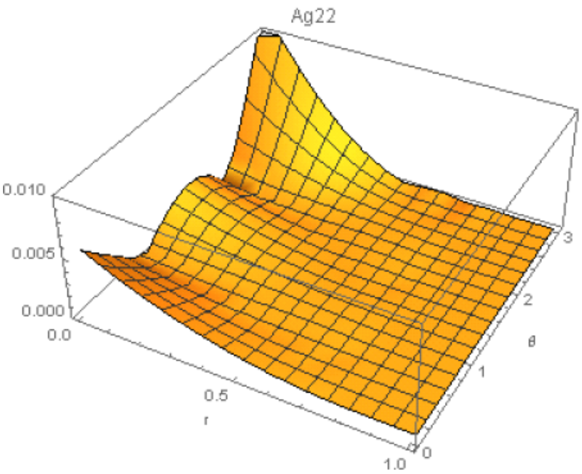
Eu_i	EA_i	a_i	aA_i	dru_i	ru_i	θu_i
0.092,0.092	0.045,0.092,0,0.097,0.103, 0.113,0.142,0	.750,-.750	.012,-,003,0,.241,-.810,.228,- .534,0	.517,.517	0,0	-.45,-.07
ΔEu_i	ΔEA_i	Δa_i	ΔaA_i	Δdru_i	Δru_i	$\Delta\theta u_i$
0.002,0.002	0.006,0.0008,0,0.0008,0.0007, 0.002,0.0008,0	.707,.707	.007,.002,0,.012,.001,.005,.003,0	.118,.118	.0,.0	.207,.200

gluons Agi









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