The Burnside $\mathbb{Q}$-algebras of a monoid

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In this text "$\mathbb{Q}$-algebra" means "associative commutative $\mathbb{Q}$-algebra with one".

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Let us define A.

Say that an M-set X is *indecomposable* if $X \neq \varnothing$ and if X is not a disjoint union of two nonempty sub-M-sets.

Let Ξ be a set of finite indecomposable M-sets such that any finite indecomposable M-set is isomorphic to a unique $X \in \Xi$.

If X, Y are in Ξ, then their product $X \times Y$ is a disjoint union $Z_1 \sqcup \cdots \sqcup Z_n$ of finite indecomposable M-sets. Moreover, if $Z \in \Xi$, then the number of i such that $Z_i \simeq Z$ is a nonnegative integer $m(X, Y, Z)$ which depends only on the isomorphism classes of X, Y and Z.

We define A as the $\mathbb{Q}$-vector space with basis Ξ and multiplication given by

$$XY := \sum_{Z \in \Xi} m(X, Y, Z)\ Z.$$

In particular A is a $\mathbb{Q}$-algebra.

We temporarily denote A by $A(M)$ and Ξ by $\Xi(M)$ to emphasize the dependence on M.

Theorem 1. *The $\mathbb{Q}$-algebra $A(G)$ of a group G is von Neumann regular.*

Proof. If b is in $A(G)$, then there is a largest finite index normal subgroup N of G such that $b \in A(G/N)$. Let $\phi_{G/N} : A(G/N) \to \mathbb{Q}^{\Xi(G/N)}$ be the $\mathbb{Q}$-algebra isomorphism defined in Section 3.3 of [1], and define $b' \in A(G/N) \subset A(G)$ by

$$b' = (\phi_{G/N})^{-1}\Big(w \circ (\phi_{G/N}(b)\Big),$$

where $w : \mathbb{Q} \to \mathbb{Q}$ is defined by $w(\lambda) = \frac{1}{\lambda}$ if $\lambda \neq 0$ and $w(0) = 0$ (that is, w is a witness to the von Neumann regularity of $\mathbb{Q}$), so that we have $b^2 b' = b$ in $A(G)$, which shows that $A(G)$ is von Neumann regular. (Here $X \subset Y$ means "X is a (not necessarily proper) subset of Y".) □

We denote again by Ξ and A (instead of $\Xi(M)$ and $A(M)$) the set and the $\mathbb{Q}$-algebra defined above.

Let us define B.

Proposition 2. *For any $Z \in \Xi$ there are only finitely $(X, Y) \in \Xi^2$ such that $m(X, Y, Z)$ is nonzero.*

Proof. It suffices to show that, for $X, Y \in \Xi$ and Z an indecomposable component of $X \times Y$, the projection $p : X \times Y \to X$ maps Z *onto* X. (Indeed, up to isomorphism, there are only finitely many quotients of Z.)

Let us fix an element a of M. Say that a point of an M-set is *periodic* if it is a fixed point of a^n for some $n \geq 1$.

The following facts are clear:

(a) If v is a periodic point of an M-set U and n is a nonnegative integer, then $v = a^n u$ for some $u \in U$.

(b) If u is a point of a finite M-set, then $a^n u$ is periodic for n large enough.

Let $p : X \times Y \to X$ be the projection, and assume by contradiction that $p(Z)$ is a *proper* subset of X. Then there is a tuple (a, x_1, x_2, y_2) with

$$a \in M;\ x_1, x_2 \in X;\ x_1 \notin p(Z);\ ax_1 = x_2;\ y_2 \in Y;\ (x_2, y_2) \in Z.$$

It suffices to show $x_1 \in p(Z)$. By (b) we can pick an $n \in \mathbb{N}$ such that $a^n(x_2, y_2) \in Z$ is periodic. Set

$$x_3 := a^n x_2 = a^{n+1} x_1,\ y_3 := a^n y_2.$$

By (a) there is a $y_1 \in Y$ such that $a^{n+1} y_1 = y_3$, and we get

$$a^{n+1}(x_1, y_1) = (x_3, y_3) \in Z,$$

which implies $(x_1, y_1) \in Z$ and thus $x_1 \in p(Z)$, contradiction. This completes the proof. $\square$

Proposition 2 implies that the multiplication we defined above on A extends to the $\mathbb{Q}$-vector space of **all** expressions of the form

$$\sum_{X \in \Xi} a_X \ X$$

with $a_X \in \mathbb{Q}$. We denote by B the $\mathbb{Q}$-algebra obtained by this process.

Question 3. *Is B flat over A?*

Beside the case of groups, there is only one case where I know that the answer is Yes. It is the case of the monoid $M := \{0, 1\}$ with the obvious multiplication. In the post

https://math.stackexchange.com/a/3154240/660

Eric Wofsey shows the isomorphism $A \simeq \mathbb{Q}[x_1, x_2, \dots]$, where the x_i are indeterminates, and it is clear that we have $B \simeq \mathbb{Q}[[x_1, x_2, \dots]]$.

Proposition 4. *The ring* $\mathbb{Q}[[x_1, x_2, \dots]]$ *is flat over* $\mathbb{Q}[x_1, x_2, \dots]$.

The poof of Proposition 4 will use two lemmas:

Lemma 5. *If A is a commutative ring with one, if $(M_i)_{i \in I}$ is a filtered inductive system of A-modules, and if $N \to P$ is a morphism of A-modules, then the natural morphisms*

$$\begin{aligned}
&\operatorname{colim} \operatorname{Ker}(M_i \otimes_A N \to M_i \otimes_A P) \\
&\qquad \to \operatorname{Ker}\left(\operatorname{colim}(M_i \otimes_A N) \to \operatorname{colim}(M_i \otimes_A N)\right) \\
&\qquad \to \operatorname{Ker}\left((\operatorname{colim} M_i) \otimes_A N \to (\operatorname{colim} M_i) \otimes_A N\right)
\end{aligned}$$

are bijective.

Proof. This follows respectively from Lemmas 4.19.2

https://stacks.math.columbia.edu/tag/002W

and 10.11.9

https://stacks.math.columbia.edu/tag/00DD

of [2]. $\square$

Lemma 6. *Filtered colimits preserve flatness. More precisely, if A and $(M_i)_{i \in I}$ are as above, and if in addition M_i is flat for all i, then* $\operatorname{colim} M_i$ *is flat.*

Proof. This follows immediately from Lemma 5. □

Proof of Proposition 4. We claim:

(a) $\mathbb{Q}[[x_1, x_2, \dots]]$ is flat over $\mathbb{Q}[x_1, \dots, x_n]$.

(b) Claim (a) implies the proposition.

Proof of (b). Set

$$A_n := \mathbb{Q}[[x_1, x_2, \dots]] \otimes_{\mathbb{Q}[x_1,\dots,x_n]} \mathbb{Q}[x_1, x_2, \dots].$$

The ring A_n being flat over $\mathbb{Q}[x_1, x_2, \dots]$ and $\mathbb{Q}[[x_1, x_2, \dots]]$ being the colimit of the A_n, Claim (b) follows from Lemma 6.

Proof of (a). The ring $\mathbb{Q}[[x_1, \dots, x_n]]$ being noetherian by Lemma 10.30.2

https://stacks.math.columbia.edu/tag/0306

of [2], and flat over $\mathbb{Q}[x_1, \dots, x_n]$ by Lemma 10.96.2(1)

https://stacks.math.columbia.edu/tag/00MB

of [2], it is enough to verify that $\mathbb{Q}[[x_1, x_2, \dots]]$ is flat over $\mathbb{Q}[[x_1, \dots, x_n]]$.

But, since $\mathbb{Q}[[x_1, x_2, \dots]]$, viewed as an $\mathbb{Q}[[x_1, \dots, x_n]]$-module, is just a product of copies of $\mathbb{Q}[[x_1, \dots, x_n]]$, it is flat over $\mathbb{Q}[[x_1, \dots, x_n]]$ by Lemma 10.89.5

https://stacks.math.columbia.edu/tag/05CY

and Proposition 10.89.6

https://stacks.math.columbia.edu/tag/05CZ

of [2], we are done. □

References.

[1] Serge Bouc, Burnside rings, Chapter 1, pages 739-804, in **Handbook of Algebra**, Volume 2, 2000, doi 0.1037/a0028240

https://tinyurl.com/y6trypqv.

[2] The Stacks Project https://stacks.math.columbia.edu/.

Tex file available at

https://tinyurl.com/y5ska9jm and https://tinyurl.com/y5jfbv5r

May 5, 2019