A mathematical overview of mass-energy conservation in modern physics

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Abstract

The idea of energy, matter, and motion has perplexed many philosophers and physicists from antiquity to modern physics, from Plato to Einstein. New and developing physical theories raise different interpretations of energy and matter but no complete theory of everything exists at present. However, there is a law we can almost take for granted: the law of conservation of energy, which states that energy cannot be created nor destroyed although it can be transformed from one form to another. After establishing the foundational theory and history of conservation of energy, this literature review aims to provide an overview of the concept of mass and energy conservation in two of the most fundamental physical theories - quantum mechanics and general relativity. Consequences and challenges of mass-energy conservation and equivalence - dark energy - is studied in an introductory manner.

1. Energy conservation in classical mechanics

1.1. Mechanical energy conservation in classical mechanics

1.1.1. Definition

The statement of conservation of energy stayed constant since its inception- Energy cannot be created nor destroyed although it can be transformed from one form to another. While the transformation later extended to any form of energy, the first statement of energy conservation was based on the transformation between kinetic energy and potential energy in an ideal mechanical system. The reason for this limited interpretation of energy form is that energy was defined in mechanics before other forms of energy and experiments with mechanical systems were easier to measure and reproduce.[1]

1.1.2. History

Out of the conservation laws[1] that were formulated in the era of Newtonian mechanics, conservation energy was widely accepted by physicists the last, in the late 18th century and early 19th century. Despite the early discovery of the law, our understanding of the law of conservation of energy has not changed significantly since its inception. But the history of the law dates back to one of the most controversial ideas in History of Physics - vis viva.[2] The debate of vis viva boils down to whether the conserved quantity is mv or mv^2. While it is now certain that both quantities - momentum and kinetic energy - are conserved in an elastic collision, scientists at the time were at pains in proving the generality of the fundamental law of nature.[2]

René Descartes

The dispute starts when Descartes proposed the law of conservation of motion in his Principia Philosophiæ (1644). He asserts that the sum of the product of speed

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[1] Conservation laws: Conservation of energy, momentum and angular momentum were formulated and accepted in Newton's time

[2] Vis viva: Literally translated in Latin as "living force"

and mass remains constant. Or mathematically, the quantity of $\sum m_i|v_i|$ is conserved. To him, speed and velocity beg no difference, even though this leads to contradictory prediction.

Christiaan Huygens

In 1669, Huygens published the rules for head-on collision for hard spheres that he had derived in the 1650s. The paper concludes that $\sum m_i v_i^2$ is conserved.

Isaac Newton

In his Principia, the law of conservation of momentum is the direct consequence of Newton law of motion, leading to the vectorial description of conservation of motion that Descartes proposed 60 years ago. In a mathematical sense, we could say the quantity of $\sum m_i \vec{v}_i$ is conserved.

Gottfried Wilhelm von Leibniz

Despite several contributions to the interpretation of vis viva and the refutation of Cartesian conservation of motion, in the end, Leibniz acknowledges the violation of conservation of $\sum m_i v_i^2$ in inelastic collisions, whereas conservation of momentum appeared to be true for all cases. Furthermore, by realizing the fundamental aspect of vis viva, he further divided force into living force - vis viva and dead force - forces that constantly influence on an object, such as centrifugal,[3] centripetal, and many static equilibrium forces.

1.2. Mechanical equivalent of heat

1.2.1. Definition

Years after the forerunners of classical mechanics have passed away, the predecessor of thermodynamics - caloric theory - gave rise to the revision of

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[3] Note that centrifugal is not an actual force, but physicists in the past understood it as one

conservation of energy and formulated the first law of thermodynamics. The central idea of the caloric theory was that heat is conserved, and due to the interchange of mechanical work and heat, scientists deduced that energy is also conserved.

1.2.2. History

In caloric theory of the 18th century, heat is treated as a weightless fluid with a tendency of flowing from hot to cold, and objects expand when receiving heat due to the absorption of this fluid.[3]Despite many success of the caloric theory in explaining phenomena like phase change and the heat engine, the theory was unable to explain the concept of heat from friction.

Count Rumford

In doubt of the current theory of caloric, Count Rumford conducted an experiment of boring a cannon immersed in water, which resemble sharpening a pencil underwater. With the result of boiling the water within hours, and since the bore, cannon and water started out at equilibrium, the heat generated through boring violate caloric theory.[4]

Julius von Mayer and James Prescott Joule

Mayer and Joule both contributed to the apparatus and theoretical framework of reconciling heat and mechanical energy. The two scientists independently assert the statement of the interchangeability between heat and mechanical work. The result was obtained with precise experiments by Joule and with clever reasoning by Mayer.[5]

2. The transition to formalism in physics

The paradigm shift in mathematical physics of mechanics

Almost one century after the publishing of Newtonian mechanics, the birth of many sophisticated mathematical tools supplements the qualitative predictions and move into the realm of mathematical physics. Notable physicists like Legendre and Lagrange utilized advanced mathematical concepts such as partial differential equations and Legendre transformation to broaden our breadth and complexity in physics and generalize our previous knowledge.[6] The result is our theoretical framework of mechanics: Lagrangian mechanics and Hamiltonian mechanics.

2.1. From theoretical to fundamental - Noether's Theorem

With the establishment of modern physics, German mathematician Emmy Noether proved her now eponymous theorem in 1915. The theorem states that, if a system has a continuously symmetric property or is not dependent on such property, then there is a corresponding quantity whose values is conserved.[7]For example, a system is independent of spatial-translation have its momentum conserved, and a system is independent of time-translation has its energy conserved. Notice that for systems with dissipative properties, like systems with friction, conserved quantities won't be assured.

2.2. An attempt to prove energy conservation by utilizing Lagrangian mechanics

The only tool for physicists to verify conservation laws is by experiments, as mentioned in chapter 1. Even for conservation laws that could be predicted with mathematical tools, many assumptions or idealization are required. For example, in Newton's Principia, the conclusion of energy and momentum conservation sets out from the law of motion, but with the limitation of a point mass and a perfectly elastic collision.

For the sake of formality, we will dwell into the rigorous proof of energy conservation by assuming the knowledge of Lagrangian mechanics[4] and Euler-Lagrange equation[5].[8]

Generally, mechanical energy is $E \equiv T + V$ and the Lagrangian is defined as $\mathcal{L} \equiv T - V$ where T is kinetic energy and V is generalized potential energy. The Lagrangian could be expressed in terms of a generalized coordinate q, its first derivative with respect to time $\dot{q}$ and time t is defined as $\mathcal{L} = \mathcal{L}(q, \dot{q}, t)$. To clarify,

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[4] A reformulation of Newtonian mechanics

[5] A differential equation that is equivalent to Newton's law of motion in Lagrangian mechanics

q could be any parameter, such as position, momentum, etc... Suppose that the Lagrangian does not depend on t, or $\mathcal{L} = \mathcal{L}(q, \dot{q})$ and $\frac{\partial \mathcal{L}}{\partial t} = 0$ to imply time translation symmetry for such system.

For generality, we acknowledge that q and $\dot{q}$ depend on time[6], and therefore to denote the dependence of time on $\mathcal{L}$ we have,

$$\mathcal{L}(t) = \mathcal{L}(q(t), \dot{q}(t)) \tag{2.1}$$

Differentiate $\mathcal{L}$ by time,

$$\frac{d\mathcal{L}(t)}{dt} = \frac{\partial \mathcal{L}}{\partial q}\dot{q} + \frac{\partial \mathcal{L}}{\partial \dot{q}}\ddot{q} \tag{2.2}$$

And by definition of Euler-Lagrange equation,

$$\frac{\partial \mathcal{L}}{\partial q} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}$$

Plug this into (2.2),

$$\begin{aligned} \frac{d\mathcal{L}(t)}{dt} &= (\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}})\dot{q} + \frac{\partial \mathcal{L}}{\partial \dot{q}}\ddot{q} \\ &= \frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{q}}\dot{q}) \end{aligned}$$

Move the expression $\frac{d\mathcal{L}}{dt}$ to the right side yields,

$$\frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{q}}\dot{q} - \mathcal{L}) = 0 \tag{2.3}$$

Evidently, the expression inside the parentheses is invariant with time translation and therefore conserved. Let's rephrase the Lagrangian in terms of Cartesian coordinates where q is equivalent to position x and m is mass,

$$\mathcal{L} = \frac{m}{2}\sum_{i=1}^{3}\dot{x}_i^2 - V(x(t)) \tag{2.4}$$

[6] Note that the Lagrangian or difference of T-V does not change with time but T or/and V could be dependent from time

We also could see differentiating the Lagrangian with respect to $\dot{x}$ and multiplying with $\dot{x}$ yields,

$$\frac{\partial \mathcal{L}}{\partial \dot{x}}\dot{x} = m\sum_{i=1}^{3}\dot{x}_i^2 \tag{2.5}$$

Here notice that V(x(t)) is independent of $\dot{x}$.

Plugging (2.4) and (2.5) into the back expression inside parentheses of (2.3),

$$m\sum_{i=1}^{3}\dot{x}_i^2 - \frac{m}{2}\sum_{i=1}^{3}\dot{x}_i^2 + V(x(t))$$

Applying algebraic operation, we notice that,

$$\frac{\partial \mathcal{L}}{\partial \dot{x}}\dot{x} - \mathcal{L} = \frac{m}{2}\sum_{i=1}^{3}\dot{x}^2 + V(x(t)) = T + V = E$$

What we get when replacing the expression inside the parentheses in (2.3) with total energy is,

$$\frac{d}{dt}[E(x(t), \dot{x}(t)] = 0$$

Now the expression of conserved energy is unveiled with the assumption of time translation symmetry of the Lagrangian. This matches with our prediction in section 2.3[7].

3. Treatment of mass-energy in modern physics

Redefine the intrinsic property of nature - mass

Amidst the never-ending bloodbath in Europe and geopolitics shifts all around the world at the change of the 20th-century, physicists were still able to make unprecedented developments in our understanding of the universe. The result was two of the most prominent and well-established field in all physics - quantum

[7] Note that a similar analysis could be applied for different conservation laws

mechanics and general relativity. The birth of new physics also leads to the refinement of fundamental concepts like mass, time and energy. But problems arise when our definition of those properties don't agree in different branches of physics. And more than 100 years later, general relativity and quantum mechanics have not yet been reconciled.[9]

3.1. A recall of mass in classical mechanics

In classical mechanics, the concept of mass appears in Newton's law of motion and his law of universal gravitation.

Inertial mass

Inertial mass is defined as the resistance of an object to acceleration. The mathematical expression is retained in Newton's second law,

$$\mathbf{F} = m_i \mathbf{a} \tag{3.1}$$

Where $\mathbf{F}$ is the force exerted on the object with inertial mass m_i and accelerate with the amount $\mathbf{a}$.

Inertial mass can be defined by Newton's third law. The third law states that if one object exerts a force on a second object, it will experience an equal and opposite force, or $\mathbf{F}_{12} = -\mathbf{F}_{21}$. Where we define $\mathbf{F}_{12} = m_{i1}\mathbf{a}_1$ (3.1) is the force exerted on m_{i2} by m_{i1} and correspondingly, $\mathbf{F}_{21} = m_{i2}\mathbf{a}_2$ (3.2) is the force exerted on m_{i1} by m_{i2}. Note that both m_{i1} and m_{i2} are constant inertial masses. From that, we could derive the definition of inertial mass 1 to be:

$$m_{i1} = m_{i2} \frac{|\mathbf{a}_2|}{|\mathbf{a}_1|}$$

Gravitational mass

Gravitational mass is defined by Newton's law of universal gravitation, which states that the attractive force between two masses is proportional to the product of their masses and inversely proportional to the square of the distance between their centers. Or mathematically,

$$\mathbf{F}_{12} = -G \frac{m_{g1} m_{g2}}{|\mathbf{r}_{12}|^2} \hat{\mathbf{r}}_{12} \tag{3.4}$$

Where

$\mathbf{F}_{12}$ is the force exerted on m_{g2} by m_{g1}

G is the gravitational constant ($G = 6.67408(31)\times10^{-11}\ m^3 . kg^{-1}. s^{-2}$) [10]

m_{g2} and m_{g1} are gravitational masses of objects.

$|\mathbf{r}_{12}|$is the distance between two objects.

$\hat{\mathbf{r}}_{12}$ is the unit vector point from object one to object two.

So far experiments verified the equivalence between inertial mass and gravitational mass or $m_{i1} = m_{g1}$, to extraordinary precision.[11] If we apply Newton's 2nd Law equation for m_{i1}(3.2) and plug it into (3.4), it becomes,

$$m_{i1}\mathbf{a}_1 = -G\frac{m_{g1}m_{g2}}{|\mathbf{r}_{12}|^2}\hat{\mathbf{r}}_{12}$$

Cancel out same quantities,

$$\mathbf{a}_1 = -G\frac{m_{g2}}{|\mathbf{r}_{12}|^2}\hat{\mathbf{r}}_{12}$$

The quantity $\mathbf{a}_1$ is also called the gravitational acceleration due to mass m_{g2} on m_{g1}. On the surface of the Earth, $\mathbf{a}_1$ is roughly the constant $g = 9.8m.s^-2$ with the assumption of uniform density and perfect spherical shape of the Earth,[12] m_{g2} is the mass of the Earth and $|\mathbf{r}_{12}|^2$ is the square of the radius of the Earth.

3.2. The concept of mass in special relativity

3.2.1. Mass in special relativity

The concept of mass had gone through radical changes since the development of Lagrangian and Hamiltonian mechanics and mathematical formality in Physics in the late 19th-century. Physicists had different approaches to the problem of fast-moving objects or specifically charged particles that would violate our classical predictions. Many attempted solutions had been proposed, including electromagnetic mass[8] and radiation pressure[9]. All were discarded upon the development of the mass-energy equivalence. In 1946, Einstein stated, “We might

[8] Mass is attributed to the electrostatic field

[9] The pressure exerted due to movement in the electromagnetic field by a fictitious fluid

say that the principle of the conservation of energy, having previously swallowed up that of the conservation of heat, now proceeded to swallow that of the conservation of mass—and holds the field alone”.[13]

The statement of mass-energy equivalence is usually stated in pop culture as, $E = mc^2$ (3.4) and there is some easily-overlooked nuance in the meaning of the mass m in this equation.

Rest mass

Rest mass, or m_0 is defined as the mass of an object when measured by an observer moving along with the object, or the object is not moving in the observer inertial frame of reference. This is crucial because the mass-energy equivalence for rest mass is not $E = m_0c^2$ (3.5) but instead,

$$E^2 \equiv m_0^2c^4 + p^2c^2 \tag{3.6}$$

Where E is energy, m_0 is the rest mass and c is the speed of light with p is the regular Newtonian momentum in 3D space.

We could see that our former equation (3.5) is only valid when $p^2 = 0 \iff (mv)^2 = 0$ or $v = 0$.

Relativistic mass

So if m_0 is not the mass in $E = mc^2$ (3.4), what is? Here we introduce the relativistic mass or m_{rel}, where it is the correct type of mass in the equation (3.4), which is also the mass we defined for $p \equiv m_{rel}v$. To find the relation between rest mass and relativistic mass let’s redefine (3.4) as $E = m_{rel}c^2$(3.7) and plug it into (3.6),

$$(m_{rel}c^2)^2 = m_0^2c^4 + p^2c^2$$

Use the definition of $p \equiv m_{rel}v$,

$$m_{rel}^2c^4 = m_0^2c^4 + m_{rel}^2v^2c^2$$

Divide everything with c^4,

$$m_{rel}^2 = m_0^2 + m_{rel}^2\frac{v^2}{c^2} \tag{3.8}$$

Move the term with m_{rel} to the left side,

$$m_{rel}^2 - m_{rel}^2\frac{v^2}{c^2} = m_0^2$$

Equivalently,

$$m_{rel}^2(1 - \frac{v^2}{c^2}) = m_0^2$$

Solve for m_{rel}, we have

$$m_{rel} = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

The factor $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ is also called γ or the Lorentz factor, so that the relation between relativistic mass and rest mass is defined as, (3.9)

$$m_{rel} \equiv \gamma m_0$$

Note that this factor increases unboundedly when v approaches c. So that for an object that has $m_0 \neq 0$, and is approaching the speed of light, its m_{rel} becomes tremendously large that due to the proportionality of energy and relativistic mass in (3.7). When v=c, γ will be infinite, which corresponds to infinite energy input in order for a massive object to reach the speed of light. By that argument, the speed of light is the limit of the universe, and it is impossible for massive objects to achieve.

Note that for photons, where v=c, (3.8) becomes,

$$m_{rel}^2 = m_0^2 + m_{rel}^2\frac{c^2}{c^2} \iff m_0 = 0$$

So that for photons, their rest mass equals to zero. That also explains why for massive objects($m_0 \neq 0$), traveling at the speed of light is impossible.

To avoid confusion between m_0 and m_{rel}, m_{rel} in (3.7) is often defined as $\frac{E}{c^2}$, or the total energy scaled with a constant. And rest mass, m_0 is m, the mass-energy equivalence in (3.6) would be $E^2 = m^2c^4 + p^2c^2$. Similarly, using (3.9) and the definition of momentum, $p \equiv \gamma mv$.[14]

3.2.2. An iteration of conservation of energy utilizing relativistic mechanics and inner product of two 4-vectors

We will delve into the transition between Newtonian mechanics to relativistic mechanics to prove the energy conservation in every inertial frame.[15]

In Newtonian mechanics, we take the Euclidean space or most commonly expressed in Cartesian coordinates (x,y,z) for granted. The Euclidean norm or the ℓ^2 norm in $\mathbb{R}^3$ is defined as the square root of the sum of the squares of the components or $\sqrt{x^2+y^2+z^2}$. Our usual treatment of space and time in Newtonian Mechanics are invariant, or there is no disagreement between different observers on the length of time between two events or the distance between the two events. However, the experience of time and space becomes subjective in the realm of special relativity due to length contraction and time dilation. Because of the relativistic property of time, in order to accurately describe an object inside spacetime, we need to define for the dimension of length. In Minkowski coordinates, position is defined as (ct,x,y,z). Interestingly, the analog for distance - interval - is defined as

$$ds^2 \equiv -(cdt)^2 + dx^2 + dy^2 + dz^2 \tag{3.10}$$

Where ds is the spacetime interval, dx, dy and dz are spatial distances, dt is temporal distance and c is the speed of light. The nuance of this metric is the notion of constant speed of light for all observers. Let's express the distance light traveled at time t in Euclidean metric,[16]

$$c^2dt^2 = dx^2 + dy^2 + dx^2$$

or,

$$0 = -c^2dt^2 + dx^2 + dy^2 + dx^2$$

Since the expression on the right-hand side doesn't change for lights regardless of the choice of coordinates, we could imply that the expression is invariant. In other words, the spacetime interval is an analog of Euclidean distance, where that quantity is unchanged regardless of your choice of coordinates.

To best represent the relation between position, velocity, and acceleration, we would need to employ the concept of 4-vector. For convenience, we use (x^0, x^1, x^2, x^3) and (ct, x, y, z) interchangeably as well as express 4-vectors with a bold face and 3-vector with an arrow above it. Note that the superscripts are not powers but rather to number (ct,x,y,z) components.
Here we introduce the indices for the convenience of computing inner products, where

$$\mathbf{A} = (a^0, a^1, a^2, a^3) \equiv a^\mu$$

μ takes a value of 0,1,2 and 3 to indicate component.

Additionally, 4-vector differs from four-dimensional Euclidean vectors in how the metric (3.10) is computed. Therefore, the inner product of two 4-vectors a^μ and b_μ is,

$$a^\mu b_\mu = -a^0 b_0 + a^1 b_1 + a^2 b_2 + a^3 b_3$$

Let's define proper time (τ) as the time taken in the rest frame in which the Minkowski coordinate is $(c\tau, \vec{0})$. t is the time in an arbitrary frame other than the rest frame of the particle. Take the spacetime interval (3.10) of the particle in the reference frame of t.

$$ds^2 = (cdt)^2 - d\vec{x}^2 = cd\tau^2 - d\vec{0}^2 = (cd\tau)^2$$

So that

$$(cdt)^2 - d\vec{x}^2 = (cd\tau)^2$$

Differentiate both sides by dt,

$$(\frac{cdt}{dt})^2 - (\frac{d\vec{x}}{dt})^2 = (\frac{cd\tau}{dt})^2$$

Equivalently,

$$c^2 - \vec{v}^2 = c^2(\frac{d\tau}{dt})^2$$

Divide both sides by c^2 and take the square root,

$$\sqrt{1 - \frac{\vec{v}^2}{c^2}} = \frac{d\tau}{dt}$$

take the reciprocal,

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}}$$

Notice that the expression on the right-hand side is the Lorentz factor that we defined in section 3.3.1. So that, $\frac{dt}{d\tau} = \gamma$.

This relation also emerges as an interesting property - time dilation. We could obtain the relation $dt = \gamma d\tau$ from above, we deduce that dt would get tremendously large when the velocity approaches the speed of light. For example, say you are on a spaceship traveling with 0.95c or 95% speed of light to a planet that is 9.5 light years away. A person on Earth will measure 10 years for the trip. While your clock will differ, we could calculate the disagreement by using the relation above. Your clock will now calculate the proper time since the clock is at rest relative to you, and a person on Earth would be moving relative to you. So that when we plug in numbers in the relation, it yields,

$$10(years) = \frac{1}{\sqrt{1-\frac{(0.95c)^2}{c^2}}}d\tau \iff d\tau = 3.12(years)$$

So, during the trip, a person on Earth would age 10 years while you on the spaceship would be about 3 years older.

With sufficient background, we now can define 4-position,

$$\mathbf{X} \equiv (ct, \vec{x}) = x^\mu$$

Where $\vec{x} = (x_1, x_2, x_3)$

Similar to Newtonian mechanics, the 4-velocity is defined as,

$$\mathbf{V} \equiv \frac{d\mathbf{X}}{d\tau} = (c\frac{dt}{d\tau}, \frac{dt}{d\tau}\frac{d\vec{x}}{dt}) = (\gamma c, \gamma\vec{v}) = \gamma(c, \vec{v}) = v^\mu$$

Let's take the inner product of a 4-velocity and itself,

$$v^\mu v_\mu = \gamma^2(-c^2 + v^2) = \frac{1}{\frac{c^2-v^2}{c^2}}(-c^2 + v^2) = -c^2 \tag{3.12}$$

Now we take the inner product between a 4-acceleration and a 4-velocity vector,

$$\mathbf{A} = a^\mu \equiv \frac{dv^\mu}{d\tau}$$

$$a_\mu v^\mu = \frac{dv_\mu}{d\tau}v^\mu = \frac{1}{2}\frac{d}{d\tau}(v_\mu v^\mu) = \frac{1}{2}\frac{d}{d\tau}(-c^2) = 0 \tag{3.13}$$

And we know from Newton's 2nd Law,

$$m\mathbf{A} = \mathbf{F} \tag{3.14}$$

Where m is the rest mass of an object and $\mathbf{F}$ is the 4-force.
Since(3.13) shows that $\mathbf{A} \cdot \mathbf{V} = 0$, with (3.14), we conclude,

$$\mathbf{F} \cdot \mathbf{V} = 0$$

Apply the work-energy theorem for a constant force, $W = \Delta E = \mathbf{F} \cdot \mathbf{X}$
we could imply

$$\mathbf{F} \cdot \mathbf{V} = \mathbf{F} \cdot \frac{d\mathbf{X}}{d\tau} = \frac{d}{d\tau}(\mathbf{F} \cdot \mathbf{X}) = \frac{dE}{d\tau}$$

therefore,

$$\frac{dE}{d\tau} = 0$$

Or mechanical energy is conserved.

The physical significance of the magnitude of a 4-velocity

Let's consider the physical significance of the dot product between two 4-velocities (3.12).

$$v^\mu v_\mu = ||\mathbf{V}||^2 = -c^2$$

Or

$$||\mathbf{V}|| = \pm c \quad \text{(Depends on the sign convention)}$$

The magnitude of a 4-velocity is equivalent to the speed of light. It tells us that the magnitude of 4-velocity is invariant for all observers. Interestingly, when we shift to the frame where the object is subject is at rest, the 4-velocity becomes,

$$v^\mu = (c, \vec{0})$$

Since $\vec{v} = \vec{0}$ and $\gamma = 1$ when $v^2 = 0$.
This draws an important conclusion that any inertial observers move through time at the same rate or proper time τ is experienced the same for each observer.[17]

3.3. The concept of mass in quantum mechanics

3.3.1. Mass in quantum mechanics

The quantization of observables like energy or momentum is the defining feature of Quantum Mechanics. A quantized observable, by definition, must adhere to certain discrete values. We could assert that for every observable one could deduce a corresponding a linear, real eigenvalue operator. For example, the operator corresponding to energy is the Hamiltonian.[18]If we apply the Hamiltonian $\hat{\mathcal{H}}$ on a wave function $\psi(t)$, it is equivalent as multiplying the associated eigenvalues E by $\psi(t)$, or in mathematical terms,

$$\hat{\mathcal{H}}\psi(t) = E\psi(t) \tag{3.15}$$

The time-dependent Hamiltonian is defined as

$$\hat{\mathcal{H}}\psi(t) = i\hbar\frac{\partial\psi}{\partial t}$$

Interestingly, mass is not quantized: there is no known operator whose eigenvalues correspond to the mass of a particle. Without the quantization of mass, the definition of inertial mass in quantum mechanics and classical mechanics is identical.

3.3.2. A verification of energy conservation in quantum mechanics with time-independent Hamiltonian and Ehrenfest theorem

The equation (3.15) immediately implies that when the Hamiltonian is time-independent, energy is conserved.

To verify our conserved property, we can utilize Ehrenfest theorem. The theorem state that, for an arbitrary observable $\hat{O}$, the time derivative of the expectation value of $\hat{O}$ satisfies,

$$\frac{d}{dt}\langle\psi(t)|\hat{O}|\psi(t)\rangle = \frac{i}{\hbar}\langle\psi(t)|[\hat{\mathcal{H}},\hat{O}]|\psi(t)\rangle + \langle\psi(t)|\frac{\partial\hat{O}}{\partial t}|\psi(t)\rangle$$

Where:
$i^2 = -1$
$\hbar$ is the Planck constant

t is time

$\langle\psi|\hat{X}|\psi\rangle$ is the average value or expected value of an arbitrary operator $\hat{X}$.
Where the ket $|\psi\rangle$ represents a column vector
The bra $\langle\psi|$ represents a row vector
The combination of operator, bra, and ket represents matrix multiplication such as $\langle\psi|\hat{X}|\psi\rangle$.

The Poisson bracket of two operators $\hat{X}$ and $\hat{Y}$ is defined as $[\hat{X},\hat{Y}] \equiv \hat{X}\hat{Y} - \hat{Y}\hat{X}$.

The proof of this theorem is could be deduced by using the product rule,[19]

$$\frac{d}{dt}\langle\psi(t)|\hat{O}|\psi(t)\rangle = \langle\frac{\partial\psi(t)}{\partial t}|\hat{O}|\psi(t)\rangle + \langle\psi(t)|\frac{\partial\hat{O}}{\partial t}|\psi(t)\rangle + \langle\psi(t)|\hat{O}|\frac{\partial\psi(t)}{\partial t}\rangle$$

Let's simplify our notation where $\langle\psi(t)|\hat{X}|\psi(t)\rangle \equiv \langle\hat{X}\rangle$ for any operator $\hat{X}$. Making use of time-dependent Hamiltonian in (3.16) and its complex counterpart, we could imply,

$$\begin{aligned}\frac{d}{dt}\langle\hat{O}\rangle &= -\frac{1}{i\hbar}\langle\hat{\mathcal{H}}^{\dagger}\hat{O}\rangle + \left\langle\frac{\partial\hat{O}}{\partial t}\right\rangle + \frac{1}{i\hbar}\langle\hat{O}\hat{\mathcal{H}}\rangle \\ &= \frac{1}{i\hbar}\left\langle[\hat{\mathcal{H}},\hat{O}]\right\rangle + \left\langle\frac{\partial\hat{O}}{\partial t}\right\rangle\end{aligned}$$

Which matches with our definition of Ehrenfest theorem stated above.

In this case, we want to study the time derivative of the expectation value of $\hat{\mathcal{H}}$, which yields,

$$\frac{d}{dt}\langle\hat{\mathcal{H}}\rangle = \frac{1}{i\hbar}\left\langle[\hat{\mathcal{H}},\hat{\mathcal{H}}]\right\rangle + \left\langle\frac{\partial\hat{\mathcal{H}}}{\partial t}\right\rangle$$

It's obvious that $[\hat{\mathcal{H}},\hat{\mathcal{H}}] = \hat{\mathcal{H}}\hat{\mathcal{H}} - \hat{\mathcal{H}}\hat{\mathcal{H}} = 0$ and the Hamiltonian does not explicitly change with time, therefore $\frac{\partial\hat{\mathcal{H}}}{\partial t} = 0$. We could conclude that $\frac{d}{dt}\langle\hat{\mathcal{H}}\rangle = 0$. In other terms, the expectation value of the Hamiltonian or total energy is conserved.

3.4. An approach in quantifying mass-energy conservation in general relativity

Understanding the problem of mass-energy in general relativity

Despite the success of general relativity explaining the dynamics of spacetime, the theory complicates the definition of mass-energy. To deduce the law conservation of mass-energy, we need time-translation symmetry according to Noether's theorem. But for general relativity, time is dependent on reference frame [10], and since mass-energy curves spacetime, the metric also fluctuates, making even harder to define the direction of time.[20]

3.4.1. Landau–Lifshitz pseudotensor

The analog of mass-energy in general relativity is the stress-energy tensor[11],[21]

$$T^{\mu\nu} = \frac{1}{8\pi G}G^{\mu\nu} \tag{3.16}$$

Where $T^{\mu\nu}$ is the stress-energy tensor that describes matter, G is the gravitational constant and $G^{\mu\nu}$ is the Einstein tensor that gives you the curvature of spacetime. The equation is best represented by words of John Archibald Wheeler: "Spacetime tells matter how to move; matter tells spacetime how to curve"[22]

Unfortunately, the stress-energy tensor is not conserved in general relativity due to nonlinearity of the metric that lives inside $G^{\mu\nu}$. However, physicists weren't satisfied with breaking one of our most fundamental principles in physics. In 1951, Lev Davidovich Landau and Evgeny Mikhailovich Lifshitz developed Landau–Lifshitz pseudotensor $\tau^{\mu\nu}$ that includes the energy of gravitational potential in addition to the curvature of spacetime. $\tau^{\mu\nu}$ is defined so that the sum of this pseudotensor and the stress-energy tensor is divergenceless,

$$\partial_\mu(T^{\mu\nu} + \tau^{\mu\nu}) = 0 \tag{3.17}$$

Or energy is conserved. One way to express Landau-Lifshitz pseudotensor is,[23]

$$\tau^{\mu\nu} = -\frac{1}{8\pi G}G^{\mu\nu} + \frac{1}{16\pi G(-g)}((-g)(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta}))_{,\alpha\beta}$$

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[10] This is also the case in special relativity, but we have a canonical definition of time by the inertial observer

[11] For convenience, we set c=1 for future mathematical expressions

The negative portion $-\frac{1}{8\pi G}G^{\mu\nu}$ is purely a geometric object that is designed to cancel out stress-energy tensor in (3.16). The positive portion $\frac{1}{16\pi G(-g)}((-g)(g^{\mu\nu}g^{\alpha\beta}-g^{\mu\alpha}g^{\nu\beta}))_{,\alpha\beta}$ is the combination of second order derivatives of the metrics, where the notation $_{,\alpha\beta} \equiv \frac{\partial^2}{\partial x^\alpha \partial x^\beta}$ .The expression is set up so that the first derivative of the metric vanishes, leading to the relation (3.17). The construction of this pseudotensor assures mass-energy conservation, however, the pseudotensor only works properly in certain coordinates, in which we can not find a general expression.

3.4.2. ADM formalism

In 1959, Richard Arnowitt, Stanley Deser, and Charles W. Misner created a model where one could quantify the mass of the universe, with a caveat that observer would be at spatial infinity.[24] The logic behind this is that for an observer at spatial infinity, the dynamical of spacetime appears flat and we could avoid the nonlinearity of spacetime. With ADM formalism, the inertial observer at spatial infinity with respect to an object could have a notion of time and energy of the object, granting time-translation symmetry and energy conservation. The ADM mass-energy could be expressed as,[25]

$$E_{\mathrm{ADM}} \equiv \frac{1}{16\pi} \lim_{r\to\infty} \int_{S_r^2} \mathrm{d}A\, n_i(\partial_j h_{ij} - \partial_i h_{jj}) \tag{3.18}$$

Where the limit of the radius of a two-sphere S_r^2 [12] goes to infinity signifies the observer at spatial infinity, A is the surface area of S_r^2, n_i is the unit normal outward of this sphere, h_{ij} is the 3-metric on a spatial surface.

A drawback of this formalism is that for an observer at spatial infinity, this expression calculates the total energy of the entire spacetime. As a consequence, one could not differentiate mass-energy due to gravitational radiation and inertial mass-energy of the object.

[12] A ball

3.4.3. Bondi-Sach formalism

The treatment in conceptualizing mass-energy in Bondi-Sach formalism implies that the observer would be at light-like infinity[13].[26] As it takes time for information to reach the observer traveling at the speed of light, we need to take account for the time for the signal to travel. Here we define retarded time [14] and advanced time [15], as $u \equiv t - \frac{r}{c}$ and $v \equiv t + \frac{r}{c}$ or if $c \equiv 1$, $u \equiv t - r$ and $v \equiv t + r$. The construction leads to similar consequence as ADM formalism - flat spacetime and energy conservation and the equation for Bondi-Sach mass-energy,[27]

$$E_{\mathrm{BS}} \equiv \frac{1}{16\pi} \lim_{u,v \to \infty} \int_{S_r^2} \mathrm{d}A\, n_i(\partial_j h_{ij} - \partial_i h_{jj})$$

Which is analogous with the definition of ADM mass in (3.18), with the exception of u and v go to infinity instead of r. For spacetime appears to be flat, the time it takes for an object to the observer should go to infinity to diminish the curvature in the neighborhood of the source.

The lightlike observer is dynamical and time dependent due to the nature of being in motion at the speed of light, as opposed to the stationary observer at spatial infinity in ADM formalism. Therefore a lightlike observer would not be able to capture the gravitational radiation traverses to spatial infinity. As a result, Bondi-Sach formalism could take account for the mass-energy loss due to gravitational energy,[28][29]

$$\frac{dE_{BS}}{du} = -\frac{1}{A} \oint |N|^2 dA \tag{3.19}$$

Where A is the surface area of a two-sphere mentioned in (3.18) with constant u. N determines the energy flux of gravitational radiation. Equation (3.19) implies that if there is an energy flux in gravitational energy, Bondi-Sach mass-energy would decrease, and if N=0 then Bondi-Sach mass-energy will stay constant as expected.

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[13] Travel at the speed of light

[14] Where the object is **behind** the observer

[15] Where the object is **ahead of** the observer

4. An unprecedented factor of violating conservation of energy - Dark energy

4.1. An expanding universe

Before the science revolution in the 20th century, scientists and philosophers have asked among themselves on the shape and the dynamic of the universe. All came to the conclusion of a static and never-changing cosmos that encompasses the dynamical celestial bodies.[30] But over the course from 1927 to 1929, Georges Lemaître and Edwin Hubble, observed and measured the expansion of the universe, changing our previous understanding of the universe.[31][32] Surprisingly, the speed as the universe expands outruns the speed of light. The reason for this irregularity is that the maximum speed of light that was governed by the Minkowski metric only applies to flat spacetime. In the cosmological scale, curvature herein cannot be neglected. Due to that nature, we could only observe the universe so far, or roughly 46.508 billion light years.[33] Therefore, the expansion of the universe will not expand “into” anything, because an object that passes our border of the observable universe will be lost to our eyesight, we can only measure the distance between galaxies gradually drift apart in an accelerated manner.

4.2. Einstein’s cosmological constant

How could the expansion happen? Wouldn’t gravity should attract all object into a huge blob of mass in a matter of time, or at least static due to or sensation of the universe? In 1917, Einstein introduced the constant in his field equations to balance the gravitational field induced by mass-energy, creating a static universe.[34] Ten years later, when Hubble and Lemaître proposed the expanding model of the universe, Einstein abandoned his notion of cosmological constant.[35] Surprisingly, the constant made a comeback in 1998 when the accelerating expansion of the universe was discovered by observing distant supernovae to measure the acceleration.[36] The equation (3.16) now becomes,

$$T^{\mu\nu} = \frac{1}{8\pi G}(G^{\mu\nu} + \Lambda g_{\mu\nu}) \tag{4.1}$$

Where $g_{\mu\nu}$ is the metric of spacetime, and Λ represents the energy density of vacuum, or in another term, dark energy.

4.3. Friedmann equation

In order to differentiate dark energy to ordinary matter and radiation, we would need to dwell into the technical definition of Friedmann equation. Let's define the critical density ρ_c as the density of matter so that the universe is flat, and density parameter,

$$\Omega_x \equiv \frac{\rho_x}{\rho c}$$

where ρ_x represents the density of some type x of energy-matter. Now we define Ω_M, Ω_R, Ω_D and Ω_{total}as the density parameter of ordinary matter[16], radiation[17], dark energy and the sum of all parameter, respectively. One of the definitions of the Friedmann equation could be expressed as follow,[37]

$$\frac{\dot{a}(t)}{a(t)} = \frac{\dot{a}(t_0)}{a(t_0)}\sqrt{\Omega_M a^{-3} + \Omega_R a^{-4} + (1-\Omega_{total})a^{-2} + \Omega_D a^{-3(1+w)}} \tag{4.2}$$

Where a is the scale factor and represents the ratio between the proper distance at time t and time t_0 , $a(t) = \frac{d(t)}{d(t_0)}$. And $w \equiv \frac{p}{\rho}$ where p is the pressure induced by the energy-matter and ρ is the energy density.

Because experiments and theoretical calculations predict a flat, homogeneous and isotropic[18] universe[38] or in mathematical terms, $\Omega_{total} = \frac{\rho_{total}}{\rho_c} = 1$ and $w = -1$, (4.2) becomes

$$\frac{\dot{a}(t)}{a(t)} = \frac{\dot{a}(t_0)}{a(t_0)}\sqrt{\Omega_M a^{-3} + \Omega_R a^{-4} + \Omega_D a} \tag{4.3}$$

(4.3) expresses the proportion between energy density to the spatial unit. For example, the energy density of ordinary matter is inversely proportional to its volume, as expected. While radiation and relativistic particles scale inversely to the 4th order - by volume and another order to take account for redshifting. And surprisingly, dark energy has a constant energy density, or in other terms, dark energy is constantly created as the universe expands. This poses a serious problem in the foundation of physics, as the law of conservation of energy forbid this energy creation. Or maybe the constraint of $\Omega_{total} = 1$ is a form of energy conservation, because the energy density of all matter must sum to unity.[37] Nevertheless, our

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[16] Which includes regular matter and dark matter
[17] Which includes photons and relativistic particles (neutrinos,...)
[18] Appears the same in all directions

understanding of the universe is still lacking in the cosmological scale, but one should remind oneself that problems don't lead to failure but rather an ingenious interpretation of the cosmos.

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We could verify the law conservation of energy for classical mechanics, in classical mechanics, quantum mechanics and special relativity. Here we show proofs for the law in each branch of physics.