Proof that there are no odd perfect numbers

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1. Abstract

For y to be a perfect number, if one of the prime factors is p, the exponent of p is an integer $n(n \geqq 1)$, the prime factors other than p are p_1,p_2,p_3,$\cdots p_r$ and the even exponent of p_k is q_k,

$$y/p^n = (1 + p + p^2 + \cdots + p^n)\prod_{k=1}^{r}(1 + p_k + {p_k}^2 + \cdots + {p_k}^{q_k})\,/(2p^n) = \prod_{k=1}^{r}{p_k}^{q_k}$$

must be satisfied. Let m be non negative integer and q be positive integer,

$$n = 4m + 1$$

$$p = 4q + 1$$

Letting b and c be odd integers, satisfying following expressions,

$$b = \prod_{k=1}^{r}{p_k}^{q_k}$$

$$c = \prod_{k=1}^{r}(1 + p_k + {p_k}^2 + \cdots + {p_k}^{q_k})/p^n$$

$$2b = c(p^n + \cdots + 1)$$

is established. This is a known content. By the consideration of this research paper, since it turned out that the number of odd perfect numbers is one at most when $n \geqq 5$ since there is at most one solution that satisfies this equation for p and p is unique in the range of $p \geqq 5$ and p is unique in the range of $n \geqq 5$. Then since it becomes contradiction because two solutions are satisfied when n is fixed and p is changed, we have obtained a conclusion that there are no odd perfect numbers.

2. Introduction

The perfect number is one in which the sum of the divisors other than itself is the same value as itself, and the smallest perfect number is

$$1 + 2 + 3 = 6$$

It is 6. Whether an odd perfect number exists or not is currently an unsolved problem.

3. Proof

An odd perfect number is y, one of them is an odd prime number p, an exponent of p is an integer n (n ≧ 1). Let $p_1, p_2, p_3, \cdots p_r$ be the odd prime numbers of factors other than p, q_k the index of p_k, and variable a be the sum of product combinations other than prime p.

$$a = \prod_{k=1}^{r} (1 + p_k + {p_k}^2 + \cdots + {p_k}^{q_k}) \ \ldots ①$$

The number of terms N of variable a is

$$N = \prod_{k=1}^{r} (q_k + 1) \ \ldots ②$$

When y is a perfect number,

$$y = a(1 + p + p^2 + \cdots + p^n) - y \ (n > 0)$$

is established.

$$a \sum_{k=0}^{n} p^k / 2 = y$$

$$a \sum_{k=0}^{n} p^k / (2p^n) = y/p^n \ \ldots ③$$

3.1. If q_k has at least one odd integer

Letting the number of terms where q_k is an odd integer be a positive integer u, because $y/p^n = \prod_{k=1}^{r} {p_k}^{q_k}$ is an odd integer, the denominator on the left side of the expression ③ has a prime factor 2, from the expression ② variable a has more than u prime factor 2 and variable a is an even integer. Therefore $\sum_{k=0}^{n} p^k$ must be an odd integer, n is an even integer and u is 1.

3.2. When all q_k are even integers

y/p^n is an odd integer, the denominator on the left side of the expression ③ is an even integer, and since N is and odd integer when q_k are all even integers, variable a is and odd integer. Therefore $\sum_{k=0}^{n} p^k$ is necessary to include one prime factor 2, $\sum_{k=0}^{n} p^k \equiv 0 \ (mod\ 2)$ is established, and n must be an odd integer.

From 3.1, 3.2, in order to have an odd perfect number, only one exponent of the prime factor of y must be an odd integer and variable a must be an odd integer. We consider the case of 3.2 below.

In order for y to be a perfect number, the following expression must be established.

$$y/p^n = (1 + p + p^2 + \cdots + p^n) \prod_{k=1}^{r} (1 + p_k + {p_k}^2 + \cdots + {p_k}^{q_k}) / (2p^n) = \prod_{k=1}^{r} {p_k}^{q_k}$$

However, $q_1, q_2, \ldots, q_r$ are all even integers.

Here, let b be an integer

$$b = \prod_{k=1}^{r} {p_k}^{q_k} \ \ldots④$$

A following expression is established.

$y/p^n = a(1 + p + p^2 + \cdots + p^n)/(2p^n) = b$

$a(p^{n+1} - 1)/(2(p - 1)p^n) = b$

$(a - 2b)p^{n+1} + 2bp^n - a = 0$ ...⑤

Because it is an $n + 1$ order equation of p, the solution of the odd prime p is $n + 1$ at most.

$(ap - 2bp + 2b)p^n = a$

Since $ap - 2bp + 2b$ is an odd integer, a/p^n is an odd integer, which is c.

$ap - 2bp + 2b = c\ (c > 0)$ ...⑥

$(2b - a)p = 2b - c$

Since variable a is an odd integer, $2b - a$ is an odd integer and $2b - a \neq 0$

$p = (2b - c)/(2b - a)$

Since $n \geqq 1$

$a - c = cp^n - c \geqq cp - c > 0$

$a > c$

is.

From the equation ⑥

$2b(p - 1) - (ap - c) = 0$

$2b - c(p^{n+1} - 1)/(p - 1) = 0$

$(p^n + \cdots + 1)/2$ is an odd integer, $n = 4m + 1$ is required with m as an integer.

$2b(p - 1) = c(p^{n+1} - 1)$

$2b = c(p^n + \cdots + 1)$

$2b = c(p + 1)(p^{n-1} + p^{n-3} + \cdots + 1)$ ...⑦

b is an odd integer when $p + 1$ is not a multiple of 4. It is necessary that $p - 1$ be a multiple of 4. A positive integer is taken as q.

$p = 4q + 1$

is established.

When $p > 1$

$p^n - 1 < p^n$

$(p^n - 1)/(p - 1) < p^n/(p - 1)$

$p^{n-1} + \cdots + 1 < p^n/(p - 1)$ ...⑧

Since p is an odd prime number satisfying $p = 4q + 1$ and $p \geqq 5$

$p^{n-1} + \cdots + 1 < p^n/4$

$2b - a = c(p^n + \cdots + 1) - cp^n = c(p^{n-1} + \cdots + 1)$

$2b - a < cp^n/4 = a/4$

$2b < 5a/4$

$a > 8b/5$ ...⑨

Let a_k and b_k be integers and if

$a_k = 1 + p_k + p_k^2 + \cdots + p_k^{q_k},\ b_k = p_k^{q_k},$

$a_k - b_k < b_k/(p_k - 1)$

$a_k < b_k p_k/(p_k - 1)$

$$a = \prod_{k=1}^{r} a_k < \prod_{k=1}^{r} b_k p_k/(p_k - 1) = b \prod_{k=1}^{r} p_k/(p_k - 1)$$

$$a/b < \prod_{k=1}^{r} p_k/(p_k - 1)$$

When $r = 1$, since $a/b < 3/2$ is established, it becomes inappropriate contrary to inequality ⑨.

From the expression ⑦,

$b = c(p + 1)/2 \times (p^{n-1} + p^{n-3} + \cdots + 1)$

holds. Since $(p + 1)/2$ is the product of only prime numbers of b, let d_k be the index,

$$(p + 1)/2 = \prod_{k=1}^{r} {p_k}^{d_k}$$

$$p = 2\prod_{k=1}^{r} {p_k}^{d_k} - 1$$

From $a = cp^n$ and the expression ⑦,

$2bp^n = a(p^n + \cdots + 1)$

$a(p^n + \cdots + 1)/(2bp^n) = 1$ ... (A)

When $r = 1$,

$a = ({p_1}^{q_1+1} - 1)/(p_1 - 1)$

$b = {p_1}^{q_1}$

The equation (A) does not hold since there is no odd perfect number when $r = 1$.

Let R be a rational number,

$R = a(p^n + \cdots + 1)/(2bp^n)$

Let b' be a rational number and let A and B to be an integer,

$b' = (p_k{}^{q_k+1} - 1)/(p_k{}^{q_k}(p_k - 1)) > 1$

$A_k = (p_k{}^{q_k+1} - 1)/(p_k - 1)$

$B_k = p_k{}^{q_k}$

Multiplying R by b', there are both cases that p_k increases p or does not change. When multiplied by b', the rate of change of R is $A_{r+1}p^n(p'^n + \cdots + 1)/(B_{r+1}p'^n(p^n + \cdots + 1))$, if p after variation is p'. If the rate of change of R is 1,

$A_{r+1}p^n(p'^n + \cdots + 1)/(B_{r+1}p'^n(p^n + \cdots + 1)) = 1$

$A_{r+1}p^n(p'^n + \cdots + 1) = B_{r+1}p'^n(p^n + \cdots + 1)$

This expression does not hold since the right side is not a multiple of p when $p' > p$, and $A_{r+1} > B_{r+1}$ holds when $p' = p$. Due to this operation, R may be larger or smaller than the original value since the rate of change of R does not become 1.

Assuming that $R = 1$ in some r, letting x be an integer and by multiplying fractions $b' = A_{r+1}/B_{r+1}$, $b'' = A_{r+2}/B_{r+2}$, $\cdots b''^{\cdots\prime} = A_x/B_x$ to R. Furthermore, assuming that $A_{s+1}A_{s+2} \ldots A_r$ is not a multiple of p, R is divided by A_{s+1}/B_{s+1}, $A_{s+2}/B_{s+2}, \cdots A_r/B_r$ and it is assumed that finally $R = 1$. At this time, assuming that n changes, the change rate of R by this operation when multiplying by A_{r+1}/B_{r+1} is

$A_{r+1}p^n(p^{n_{r+1}} + \cdots + 1)/(B_{r+1}p^{n_{r+1}}(p^n + \cdots + 1))$

$$
\begin{aligned}
1 \times B_{s+1}p^n(p^{n_{s+1}} + \cdots + 1)/(A_{s+1}p^{n_{s+1}}(p^n + \cdots + 1)) \times \ldots \times B_r p^{n_{r-1}}(p^{n_r} + \cdots \\
+ 1)/(A_r p^{n_r}(p^{n_{r-1}} + \cdots + 1)) \times A_{r+1}p^{n_r}(p^{n_{r+1}} + \cdots + 1)/(B_{r+1}p^{n_{r+1}}(p^{n_r} \\
+ \cdots + 1)) \times A_{r+2}p^{n_{r+1}}(p^{n_{r+2}} + \cdots + 1)/(B_{r+2}p^{n_{r+2}}(p^{n_{r+1}} + \cdots + 1)) \times \ldots \\
\times A_x p^{n_{x-1}}(p^{n_x} + \cdots + 1)/(B_x p^{n_x}(p^{n_{x-1}} + \cdots + 1)) = 1
\end{aligned}
$$

$$
\begin{aligned}
B_{s+1}B_{s+2} \ldots B_r A_{r+1}A_{r+2} \ldots A_x p^{n-n_x}(p^{n_x} + \cdots + 1) \\
= A_{s+1}A_{s+2} \ldots A_r B_{r+1}B_{r+2} \ldots B_x(p^n + \cdots + 1) \quad \ldots \text{(B)}
\end{aligned}
$$

When $n_x < n$, it becomes contradiction since the right side of above expression does not include factor p.

When $n_x = n$,

$B_{s+1}B_{s+2} \ldots B_r A_{r+1}A_{r+2} \ldots A_x = A_{s+1}A_{s+2} \ldots A_r B_{r+1}B_{r+2} \ldots B_x \quad \ldots \text{(C)}$

Let e_r, f_r be odd integers and g_r be a rational number,

$$e_r = \prod_{k=1}^{r} (p_k{}^{q_k} + \cdots + 1)$$

$$f_r = \prod_{k=1}^{r} p_k{}^{q_k}$$

$$g_r = e_r/f_r$$

holds.

$g_{r+1} = e_{r+1}/f_{r+1} = e_r/f_r \times (p_{r+1}{}^{q_{r+1}} + \cdots + 1)/p_{r+1}{}^{q_{r+1}} > e_r/f_r = g_r$

Let $q_1{}'$ be even integer and $q_1' > q_1$ holds. Let g_r be $g_r{}'$ when q_1 becomes q_1',

$g_r' = (p_1{}^{q_1}(p_1{}^{q_1{}'} + \cdots + 1)/p_1{}^{q_1{}'}(p_1{}^{q_1} + \cdots + 1))g_r > g_r$

is established.

Here, it is assumed that q_k becomes $q_k - h_k$ by making q_k smaller than before for g_r. h_k is an even non-negative integer. Then it is assume that r becomes $s (s > r)$, $g_s = g_r$ and g_s is not changed.

$$g_s/g_r = p_1{}^{q_1} \times \ldots \times p_r{}^{q_r}\left(p_1{}^{q_1-h_1} + \cdots + 1\right) \ldots (p_r{}^{q_r-h_r} + \cdots + 1)/(p_1{}^{q_1-h_1} \times \ldots \times p_r{}^{q_r-h_r}(p_1{}^{q_1} + \cdots + 1) \ldots (p_r{}^{q_r} + \cdots + 1)) = 1$$

$$p_1{}^{h_1} \times \ldots \times p_r{}^{h_r}\left(p_1{}^{q_1-h_1} + \cdots + 1\right) \ldots (p_r{}^{q_r-h_r} + \cdots + 1)/((p_1{}^{q_1} + \cdots + 1) \ldots (p_r{}^{q_r} + \cdots + 1)) \times p_{r+1}{}^{q_{r+1}} \times \ldots \times p_s{}^{q_s} = 1$$

$$p_{r+1}{}^{q_{r+1}} \times \ldots \times p_s{}^{q_s} \times p_1{}^{h_1} \times \ldots \times p_r{}^{h_r}\left(p_1{}^{q_1-h_1} + \cdots + 1\right) \ldots \left(p_r{}^{q_r-h_r} + \cdots + 1\right) = (p_1{}^{q_1} + \cdots + 1) \ldots (p_r{}^{q_r} + \cdots + 1)$$

$$p_{r+1}{}^{q_{r+1}} \times \ldots \times p_s{}^{q_s}\left(p_1{}^{q_1} + \cdots + p_1{}^{h_1}\right) \ldots \left(p_r{}^{q_r} + \cdots + p_r{}^{h_r}\right) = (p_1{}^{q_1} + \cdots + 1) \ldots (p_r{}^{q_r} + \cdots + 1)$$

$a = (p_1{}^{q_1} + \cdots + 1) \ldots (p_r{}^{q_r} + \cdots + 1) = cp^n$ holds and from the expression ⑦, c must be a product of primes from p_1 to p_r. Thereby the above equation does not hold since it is inappropriate when there is even one prime number other than p_1 to p_r. When changing the value of p_k, it is equivalent to dividing by $p_k{}^{q_k}$ and then multiplying by new $p_k{}^{q_k}$, so it is sufficient to consider only the changes of q_k and r. From above, since g_r does not chord the original value when q_k or r is increased or decreased, it takes unique values for the variables p_k, q_k, r.

From the above proof,

$g_r = A_1A_2 \ldots A_s/B_1B_2 \ldots B_s \times A_{r+1}A_{r+2} \ldots A_x/B_{r+1}B_{r+2} \ldots B_x$

g_r must be represented uniquely, and the expression (C) does not satisfied. When dividing by the prime number in the expression of p, a contradiction arises since the prime number not included in b is in the expression of p. Therefore when p holds $p \equiv 1 \pmod 4$ and $p \geqq 5$, the number of the solution (a, b, p, n) satisfying $R = 1$ is at most one.

Since $(a, b, p, n) = (1,1,1,1)$ is inappropriate solution and the expression (C) becomes contradiction, there is one solution when $n_x = n = 1$. Therefore there are no odd perfect numbers when $n = 1$.

Define the operation [multiplication] and the operation [division] as follows.
Assuming that p in the equation of R is replaced by p' by multiplying A_i/B_i, define operation [multiplication] to R as follows.

$$p' = 2\prod_{k=1}^{r} p_k^{d_k} \times p_i^{d_i} - 1$$

$$0 \leqq d_i \leqq q_i$$

Here, let i be $i > r$. Suppose operation [division] is division by A_j/B_j for R, and if p_j is included in p in the expression R, p_j is deleted as $d_j = 0$. Here, assuming that j satisfies $1 \leqq j \leqq r$.

In the proof of the expression (B), it is assumed that p changes on the way, and finally p becomes p_x.

$$A_1 \dots A_r = cp^n$$

$$2B_1 \dots B_r = c(p^n + \cdots + 1)$$

$$A_1 \dots A_x = c' {p_x}^{n_x}$$

$$2B_1 \dots B_x = c'({p_x}^{n_x} + \cdots + 1)$$

It is assumed that the above expressions are satisfied.

$$B_{s+1}B_{s+2} \dots B_r A_{r+1} A_{r+2} \dots A_x p^n ({p_x}^{n_x} + \cdots + 1)$$
$$= A_{s+1}A_{s+2} \dots A_r B_{r+1} B_{r+2} \dots B_x {p_x}^{n_x} (p^n + \cdots + 1)$$

$$B_{s+1}B_{s+2} \dots B_r A_1 \dots A_r A_{r+1} A_{r+2} \dots A_x p^n ({p_x}^{n_x} + \cdots + 1)$$
$$= A_1 \dots A_r A_{s+1} A_{s+2} \dots A_r B_{r+1} B_{r+2} \dots B_x {p_x}^{n_x} (p^n + \cdots + 1)$$

$$B_{s+1}B_{s+2} \dots B_r c' {p_x}^{n_x} p^n ({p_x}^{n_x} + \cdots + 1)$$
$$= A_1 \dots A_r A_{s+1} A_{s+2} \dots A_r B_{r+1} B_{r+2} \dots B_x {p_x}^{n_x} (p^n + \cdots + 1)$$

$$B_{s+1}B_{s+2} \dots B_r c' p^n ({p_x}^{n_x} + \cdots + 1) = A_1 \dots A_r A_{s+1} A_{s+2} \dots A_r B_{r+1} B_{r+2} \dots B_x (p^n + \cdots + 1)$$

$$B_1 \dots B_r B_{s+1} B_{s+2} \dots B_r c' p^n ({p_x}^{n_x} + \cdots + 1)$$
$$= A_1 \dots A_r A_{s+1} A_{s+2} \dots A_r B_1 \dots B_r B_{r+1} B_{r+2} \dots B_x (p^n + \cdots + 1)$$

$$B_1 \dots B_r B_{s+1} B_{s+2} \dots B_r c' p^n ({p_x}^{n_x} + \cdots + 1)$$
$$= A_1 \dots A_r A_{s+1} A_{s+2} \dots A_r c'({p_x}^{n_x} + \cdots + 1)/2 \times (p^n + \cdots + 1)$$

$$B_1 \dots B_r B_{s+1} B_{s+2} \dots B_r p^n = A_1 \dots A_r A_{s+1} A_{s+2} \dots A_r/2 \times (p^n + \cdots + 1)$$

$c(p^n + \cdots + 1)/2 \times B_{s+1}B_{s+2} \dots B_r p^n = cp^n A_{s+1}A_{s+2} \dots A_r/2 \times (p^n + \cdots + 1)$

$B_{s+1}B_{s+2} \dots B_r = A_{s+1}A_{s+2} \dots A_r$

is established. It becomes contradiction since $A_k > B_k$ holds when the operation [division] is performed.

We consider in the case of $n \geqq 5$ as follows. Consider a tree whose vertex is $(a, b, p, n) = (1,1,1,1)$, and it becomes a child node when the operation [multiplication] is performed. For example, consider a child node connected to a vertex as follows.

$(a, b, p, n) = (13,9,5,5)$ as $p_1 = 3$, $q_1 = 2$ and $d_1 = 1$

$(a, b, p, n) = (13,9,17,9)$ as $p_1 = 3$, $q_1 = 2$ and $d_1 = 2$

$(a, b, p, n) = (57,49,97,13)$ as $p_1 = 7$, $q_1 = 2$ and $d_1 = 2$

In the above proof, since by considering $x = s$ it becomes inconsistent, the two points connected without returning to the bifurcation except for the top of the tree will not be odd perfect numbers. …(D)

The following lemma holds as a corollary of Zsigmondy's theorem.

[lemma Z]

For odd prime p and odd $n \geqq 5$, where $p \equiv 1, n \equiv 1 \pmod 4$, $p^{n+1} - 1$ has at least one prime factor different from any prime factor of $p^2 - 1$.

By using this lemma Z, the following theorem can be proved.

[theorem]

For odd prime p and odd $n \geqq 5$, where $p \equiv 1, n \equiv 1 \pmod 4$, $p^{n-1} + p^{n-3} + \cdots + p^2 + 1$ has a prime factor different from at least one prime factor of $(p + 1)/2$.

[proof]

From lemma Z, $p^{n+1} - 1$ has at least one prime factor different from any prime factor of $p^2 - 1$. Let this be q.

$p^{n+1} - 1 = (p^{n-1} + p^{n-3} + \cdots + p^2 + 1) \times (p^2 - 1)$ and since q is not a prime factor of $p^2 - 1$, $p^{n-1} + p^{n-3} + \cdots + p^2 + 1$ always has q as a prime factor.

Since $p^2 - 1$ is a multiple of $(p + 1)/2$, this q is different from any prime factor of $(p + 1)/2$. □

From the above, for odd prime number p and odd number $n \geqq 5$ where $p \equiv 1, n \equiv 1 \pmod 4$ $p^{n-1} + p^{n-3} + \cdots + p^2 + 1$ can not be the product of only $(p + 1)/2$ prime factors.

We quoted the above lemma Z, theorem and proof from as below.
Proof for the existence of an odd complete number 3
https://rio2016.5ch.net/test/read.cgi/math/1544361065/498

From above theorem, when $n \geqq 5$, if b is only a prime number of $(p+1)/2$, it does not become an odd perfect number. …(E)

It is assumed that a set of nodes is branched when p is changed by an operation [multiplication] in nodes in two or more layers. Here, when there is a solution in a certain p, if there is a solution even in the other values p', by the proposition (D) and the proposition (E), the operation [division] must be performed to return to the bifurcation. At this time from the above proof, it becomes inconsistent. Thereby p must be unique. Therefore since for p satisfying $p \geqq 5$ there is at most one solution with $R = 1$, the number of odd perfect number is one at most where $n \geqq 5$.

Assuming that $R = 1$ holds in some r when $n \geqq 5$ and by fixing n and performing operation [multiplication], finally p becomes p_x and $R = 1$ holds again.

$A_1 \dots A_r = cp^n$

$2B_1 \dots B_r = c(p^n + \cdots + 1)$

$A_1 \dots A_x = c' {p_x}^n$

$2B_1 \dots B_x = c'({p_x}^n + \cdots + 1)$

$2B_1 \dots B_r p^n = A_1 \dots A_r(p^n + \cdots + 1)$

$2B_1 \dots B_x {p_x}^n = A_1 \dots A_x({p_x}^n + \cdots + 1)$

$B_{r+1} \dots B_x \times {p_x}^n/p^n = A_{r+1} \dots A_x \times ({p_x}^n + \cdots + 1)/(p^n + \cdots + 1)$

$B_{r+1} \dots B_x {p_x}^n(p^n + \cdots + 1) = A_{r+1} \dots A_x p^n({p_x}^n + \cdots + 1)$

Let t be an odd integer,

$A_{r+1} \dots A_x = t{p_x}^n \dots (E)$

$B_{r+1} \dots B_x(p^n + \cdots + 1) = tp^n({p_x}^n + \cdots + 1)$

If $B_x = p^{q_x}$,

$B_{r+1} \dots B_{x-1} p^{q_x - n}(p^n + \cdots + 1) = t({p_x}^n + \cdots + 1)$

From the expression (E),

$A_1 \ldots A_x = ctp^n {p_x}^n = c' {p_x}^n$

$c' = ctp^n$

$B_{r+1} \ldots B_{x-1} p^{q_x - n}(p^n + \cdots + 1) = c'/(cp^n) \times ({p_x}^n + \cdots + 1)$

$cB_{r+1} \ldots B_{x-1} p^{q_x}(p^n + \cdots + 1) = c'({p_x}^n + \cdots + 1)$

$B_1 \ldots B_r B_{r+1} \ldots B_{x-1} p^{q_x} = B_1 \ldots B_x$

When $B_x = p^{q_x}$, above expression holds. Since the number of odd perfect numbers is one at most when $n \geqq 5$, an assumption that there are two solutions with the same n is false. It becomes contradiction because the false assumption holds. From above there are no odd perfect numbers.

4. Complement

From the equation ⑤,

$2bp^n(p-1) = a(p^{n+1}-1)$

$2 = a(p^{n+1}-1)/(bp^n(p-1))$

$2 = (p_1{}^{q_1+1}-1)(p_2{}^{q_2+1}-1)\ldots(p_r{}^{q_r+1}-1)(p^{n+1}-1)$

$\qquad /(p_1{}^{q_1}p_2{}^{q_2}\ldots p_r{}^{q_r}p^n(p_1-1)(p_2-1)\ldots(p_r-1)(p-1))$

$2(p_1{}^{q_1+1}-p_1{}^{q_1})(p_2{}^{q_2+1}-p_2{}^{q_2})\ldots(p_r{}^{q_r+1}-p_r{}^{q_r})(p^{n+1}-p^n)$

$\qquad = (p_1{}^{q_1+1}-1)(p_2{}^{q_2+1}-1)\ldots(p_r{}^{q_r+1}-1)(p^{n+1}-1)$

We consider when $r = 2$.

$(p_1{}^{q_1+1}-1)(p_2{}^{q_2+1}-1)(p^{n+1}-1) = 2(p_1{}^{q_1+1}-p_1{}^{q_1})(p_2{}^{q_2+1}-p_2{}^{q_2})(p^{n+1}-p^n)$

Let s, t, u be integers,

$s = p_1{}^{q_1+1}-1$

$t = p_2{}^{q_2+1}-1$

$u = p^{n+1}-1$

are.

$stu = 2(p_1{}^{q_1+1}-1-(p_1{}^{q_1}-1))(p_2{}^{q_2+1}-1-(p_2{}^{q_2}-1))(p^{n+1}-1-(p^n-1))$

$stu = 2(s-(s+1)/p_1+1)(t-(t+1)/p_2+1)(u-(u+1)/p+1)$

$pp_1p_2stu = 2((s+1)p_1-(s+1))((t+1)p_2+(t+1))\big((u+1)p+(u+1)\big)$

$pp_1p_2stu = 2(s+1)(p_1-1)(t+1)(p_2-1)(u+1)(p-1)$

$stu/((s+1)(t+1)(u+1)) = 2(p_1-1)(p_2-1)(p-1)/(p_1p_2p)$

Since $stu/((s+1)(t+1)(u+1))$ is a monotonically increasing function for variables s, t and u, if

$s \geqq 3^{2+1}-1 = 26,\ p_1 = 3,\ q_1 = 2$

$t \geqq 7^{2+1}-1 = 342,\ p_2 = 7,\ q_2 = 2$

$u \geqq 5^2-1 = 24,\ p = 5,\ n = 1$

holds,

$stu/((s+1)(t+1)(u+1)) \geqq 26 \times 342 \times 24/(27 \times 343 \times 25) = 7904/8575$

$2(p_1-1)(p_2-1)(p-1)/(p_1p_2p) = 2 \times 2 \times 6 \times 4/(3 \times 7 \times 5) = 32/35$

Since $stu/((s+1)(t+1)(u+1))$ is limited to 1 when s, t and u are infinite,
$stu/((s+1)(t+1)(u+1)) < 1$

If $f(p_1, p_2, p) = 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p)$ holds, it is sufficient to consider a combination where $f(p_1, p_2, p) < 1$.
$f(3,7,5) = 2 \times 2 \times 6 \times 4/(3 \times 7 \times 5) = 32/35$
$f(3,11,5) = 2 \times 2 \times 10 \times 4/(3 \times 11 \times 5) = 32/33$
$f(3,13,5) = 2 \times 2 \times 12 \times 4/(3 \times 13 \times 5) = 64/65$
$f(3,17,5) = 2 \times 2 \times 16 \times 4/(3 \times 17 \times 5) = 256/255$
$f(3,7,13) = 2 \times 2 \times 6 \times 12/(3 \times 7 \times 13) = 96/91$
$f(3,5,17) = 2 \times 2 \times 4 \times 16/(3 \times 5 \times 17) = 256/255$
From the above, when $r = 2$, a combination $(p_1, p_2, p) = (3,7,5), (3,11,5), (3,13,5)$ can be considered.

Let q_k be 2 and $n = 1$, if $g(p_1, p_2, p) = (p_1{}^3 - 1)(p_2{}^3 - 1)(p^2 - 1)/(p_1{}^3 p_2{}^3 p^2)$,
$g(3,7,5) = 26 \times 342 \times 24/(3^3 7^3 5^2) = 7904/8575 > 32/35$
$g(3,11,5) = 26 \times 1330 \times 24/(3^3 11^3 5^2) = 55328/59895$
$g(3,13,5) = 26 \times 2196 \times 24/(3^3 13^3 5^2) = 3904/4225$
Since the function g is the minimum in the case of $q_k = 2$ and $n = 1$, there is no solution q_k and n when $g > f$, so the case of $(p_1, p_2, p) = (3,7,5)$ becomes unsuitable.

$$stu/((s+1)(t+1)(u+1)) = 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p)$$
$$(p_1{}^{q_1+1} - 1)(p_2{}^{q_2+1} - 1)(p^{n+1} - 1)/(p_1{}^{q_1+1} p_2{}^{q_2+1} p^{n+1})$$
$$= 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p)$$

If $F(p_1, p_2, p) = (p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p)$,
$F(p_1{}^{q_1+1}, p_2{}^{q_2+1}, p^{n+1}) = 2F(p_1, p_2, p)$

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6. References

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