Proof that there are no odd perfect numbers

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July 09th, 2019

1. Abstract

For y to be a perfect number, if one of the prime factors is p, the exponent of p is an integer $n(n \geqq 1)$, the prime factors other than p are $p_1, p_2, p_3, \cdots p_r$ and the even exponent of p_k is q_k,

$$y/p^n = (1 + p + p^2 + \cdots + p^n) \prod_{k=1}^{r} (1 + p_k + {p_k}^2 + \cdots + {p_k}^{q_k}) / (2p^n) = \prod_{k=1}^{r} {p_k}^{q_k}$$

must be satisfied. Let m be non negative integer and q be positive integer,

$$n = 4m + 1$$

$$p = 4q + 1$$

Letting b and c be odd integers, satisfying following expressions,

$$b = \prod_{k=1}^{r} {p_k}^{q_k}$$

$$c = \prod_{k=1}^{r} (1 + p_k + {p_k}^2 + \cdots + {p_k}^{q_k}) / p^n$$

$$2b = c(p^n + \cdots + 1)$$

is established. This is a known content. By the consideration of this research paper, since it turned out that the solution (a, b, p, n) that satisfies this equation is at most one and that one is an inappropriate solution, we have obtained the conclusion that there are no odd perfect numbers.

2. Introduction

The perfect number is one in which the sum of the divisors other than itself is the same value as itself, and the smallest perfect number is

$$1 + 2 + 3 = 6$$

It is 6. Whether an odd perfect number exists or not is currently an unsolved problem.

3. Proof

An odd perfect number is y, one of them is an odd prime number p, an exponent of p is an integer n (n ≧ 1). Let $p_1,p_2,p_3,\cdots p_r$ be the odd prime numbers of factors other than p, q_k the index of p_k, and variable a be the sum of product combinations other than prime p.

$$a=\prod_{k=1}^{r}(1+p_k+{p_k}^2+\cdots+{p_k}^{q_k})\ \ldots①$$

The number of terms N of variable a is

$$N=\prod_{k=1}^{r}(q_k+1)\ \ldots②$$

When y is a perfect number,

$$y=a(1+p+p^2+\cdots+p^n)-y\ (n>0)$$

is established.

$$a\sum_{k=0}^{n}p^k/2=y$$

$$a\sum_{k=0}^{n}p^k/(2p^n)=y/p^n\ \ldots③$$

3.1. If q_k has at least one odd integer

Letting the number of terms where q_k is an odd integer be a positive integer u, because $y/p^n=\prod_{k=1}^{r}{p_k}^{q_k}$ is an odd integer, the denominator on the left side of expression ③ has a prime factor 2, from expression ② variable a has more than u prime factor 2 and variable a is an even integer. Therefore $\sum_{k=0}^{n}p^k$ must be an odd integer, n is an even integer and u is 1.

3.2. When all q_k are even integers

y/p^n is an odd integer, the denominator on the left side of expression ③ is an even integer, and since N is and odd integer when q_k are all even integers, variable a is and odd integer. Therefore $\sum_{k=0}^{n}p^k$ is necessary to include one prime factor 2, $\sum_{k=0}^{n}p^k\equiv 0\ (mod\ 2)$ is established, and n must be an odd integer.

From 3.1, 3.2, in order to have an odd perfect number, only one exponent of the prime factor of y must be an odd integer and variable a must be an odd integer. We consider the case of 3.2 below.

In order for y to be a perfect number, the following expression must be established.

$$y/p^n = (1+p+p^2+\cdots+p^n)\prod_{k=1}^{r}(1+p_k+{p_k}^2+\cdots+{p_k}^{q_k})/(2p^n) = \prod_{k=1}^{r}{p_k}^{q_k}$$

However, $q_1, q_2, \dots, q_r$ are all even integers.

Here, let b be an integer

$$b = \prod_{k=1}^{r}{p_k}^{q_k} \ \dots④$$

A following expression is established.

$y/p^n = a(1+p+p^2+\cdots+p^n)/(2p^n) = b$

$a(p^{n+1}-1)/(2(p-1)p^n) = b$

$(a-2b)p^{n+1} + 2bp^n - a = 0$ ...⑤

Because it is an $n+1$ order equation of p, the solution of the odd prime p is $n+1$ at most.

$(ap - 2bp + 2b)p^n = a$

Since $ap - 2bp + 2b$ is an odd integer, a/p^n is an odd integer, which is c.

$ap - 2bp + 2b = c \ (c > 0)$ ...⑥

$(2b - a)p = 2b - c$

Since variable a is an odd integer, $2b - a$ is an odd integer and $2b - a \neq 0$

$p = (2b - c)/(2b - a)$

Since $n \geqq 1$

$a - c = cp^n - c \geqq cp - c > 0$

$a > c$

is.

From equation ⑥

$2b(p - 1) - (ap - c) = 0$

$2b - c(p^{n+1} - 1)/(p - 1) = 0$

$(p^n + \cdots + 1)/2$ is an odd integer, $n = 4m + 1$ is required with m as an integer.

$2b(p - 1) = c(p^{n+1} - 1)$

$2b = c(p^n + \cdots + 1)$

$2b = c(p + 1)(p^{n-1} + p^{n-3} + \cdots + 1)$ ...⑦

b is an odd integer when $p + 1$ is not a multiple of 4. It is necessary that $p - 1$ be a multiple of 4. A positive integer is taken as q.

$p = 4q + 1$

is established.

When $p > 1$

$p^n - 1 < p^n$

$(p^n - 1)/(p - 1) < p^n/(p - 1)$

$p^{n-1} + \cdots + 1 < p^n/(p - 1)$ ...⑧

Since p is an odd prime number satisfying $p = 4q + 1$ and $p \geqq 5$

$p^{n-1} + \cdots + 1 < p^n/4$

$2b - a = c(p^n + \cdots + 1) - cp^n = c(p^{n-1} + \cdots + 1)$

$2b - a < cp^n/4 = a/4$

$2b < 5a/4$

$a > 8b/5$ ...⑨

Let a_k and b_k be integers and if

$a_k = 1 + p_k + p_k^2 + \cdots + p_k^{q_k}$, $b_k = p_k^{q_k}$,

$a_k - b_k < b_k/(p_k - 1)$

$a_k < b_k p_k/(p_k - 1)$

$$a = \prod_{k=1}^{r} a_k < \prod_{k=1}^{r} b_k p_k/(p_k - 1) = b \prod_{k=1}^{r} p_k/(p_k - 1)$$

$$a/b < \prod_{k=1}^{r} p_k/(p_k - 1)$$

When $r = 1$, since $a/b < 3/2$ is established, it becomes inappropriate contrary to inequality ⑨.

From expression ⑦,

$b = c(p + 1)/2 \times (p^{n-1} + p^{n-3} + \cdots + 1)$

holds. Since $(p + 1)/2$ is the product of only prime numbers of b, let d_k be the index,

$$(p + 1)/2 = \prod_{k=1}^{r} {p_k}^{d_k}$$

$$p = 2\prod_{k=1}^{r} {p_k}^{d_k} - 1$$

From $a = cp^n$ and expression ⑦,

$2bp^n = a(p^n + \cdots + 1)$

$a(p^n + \cdots + 1)/(2bp^n) = 1$ ... (A)

When $r = 1$,

$a = ({p_1}^{q_1+1} - 1)/(p_1 - 1)$

$b = {p_1}^{q_1}$

Equation (A) does not hold since there is no odd perfect number when $r = 1$.

Let R be a rational number,

$R = a(p^n + \cdots + 1)/(2bp^n)$

Let b' be a rational number and let A and B to be an integer,

$b' = (p_k{}^{q_k+1} - 1)/(p_k{}^{q_k}(p_k - 1)) > 1$

$A_k = (p_k{}^{q_k+1} - 1)/(p_k - 1)$

$B_k = p_k{}^{q_k}$

Multiplying R by b', there are both cases that p_k increases p or does not change. When multiplied by b', the rate of change of R is $A_{r+1}p^n(p'^n + \cdots + 1)/(B_{r+1}p'^n(p^n + \cdots + 1))$, if p after variation is p'. If the rate of change of R is 1,

$A_{r+1}p^n(p'^n + \cdots + 1)/(B_{r+1}p'^n(p^n + \cdots + 1)) = 1$

$A_{r+1}p^n(p'^n + \cdots + 1) = B_{r+1}p'^n(p^n + \cdots + 1)$

This expression does not hold since the right side is not a multiple of p when $p' > p$, and $A_{r+1} > B_{r+1}$ holds when $p' = p$. Due to this operation, R may be larger or smaller than the original value since the rate of change of R does not become 1.

Assuming that $R = 1$ in some r, letting x be an integer and by multiplying fractions $b' = A_{r+1}/B_{r+1}$, $b'' = A_{r+2}/B_{r+2}$, $\cdots b''^{\cdots\prime} = A_x/B_x$ to R. Furthermore, assuming that $A_{s+1}A_{s+2} \ldots A_r$ is not a multiple of p, R is divided by A_{s+1}/B_{s+1}, $A_{s+2}/B_{s+2}, \cdots A_r/B_r$ and it is assumed that finally $R = 1$. At this time, assuming that n changes, the change rate of R by this operation when multiplying by A_{r+1}/B_{r+1} is

$A_{r+1}p^n(p^{n_{r+1}} + \cdots + 1)/(B_{r+1}p^{n_{r+1}}(p^n + \cdots + 1))$

$$\begin{aligned}
1 \times B_{s+1}p^n(p^{n_{s+1}} + \cdots + 1)/(A_{s+1}p^{n_{s+1}}(p^n + \cdots + 1)) \times \ldots \times B_r p^{n_{r-1}}(p^{n_r} + \cdots \\
+ 1)/(A_r p^{n_r}(p^{n_{r-1}} + \cdots + 1)) \times A_{r+1}p^{n_r}(p^{n_{r+1}} + \cdots + 1)/(B_{r+1}p^{n_{r+1}}(p^{n_r} \\
+ \cdots + 1)) \times A_{r+2}p^{n_{r+1}}(p^{n_{r+2}} + \cdots + 1)/(B_{r+2}p^{n_{r+2}}(p^{n_{r+1}} + \cdots + 1)) \times \ldots \\
\times A_x p^{n_{x-1}}(p^{n_x} + \cdots + 1)/(B_x p^{n_x}(p^{n_{x-1}} + \cdots + 1)) = 1
\end{aligned}$$

$$\begin{aligned}
B_{s+1}B_{s+2} \ldots B_r A_{r+1}A_{r+2} \ldots A_x p^{n-n_x}(p^{n_x} + \cdots + 1) \\
= A_{s+1}A_{s+2} \ldots A_r B_{r+1}B_{r+2} \ldots B_x(p^n + \cdots + 1) \quad \ldots \text{(B)}
\end{aligned}$$

When $n_x < n$, it becomes contradiction since the right side of above expression does not include factor p.

When $n_x = n$,

$B_{s+1}B_{s+2} \ldots B_r A_{r+1}A_{r+2} \ldots A_x = A_{s+1}A_{s+2} \ldots A_r B_{r+1}B_{r+2} \ldots B_x \quad \ldots \text{(C)}$

Let e_r, f_r be odd integers and g_r be a rational number,

$$e_r = \prod_{k=1}^{r} (p_k{}^{q_k} + \cdots + 1)$$

$$f_r = \prod_{k=1}^{r} p_k{}^{q_k}$$

$$g_r = e_r/f_r$$

holds.

$$g_{r+1} = e_{r+1}/f_{r+1} = e_r/f_r \times (p_{r+1}{}^{q_{r+1}} + \cdots + 1)/p_{r+1}{}^{q_{r+1}} > e_r/f_r = g_r$$

Let $q_1{}'$ be even integer and $q_1' > q_1$ holds. Let g_r be $g_r{}'$ when q_1 becomes q_1',

$$g_r' = (p_1{}^{q_1}(p_1{}^{q_1'} + \cdots + 1)/p_1{}^{q_1'}(p_1{}^{q_1} + \cdots + 1))g_r > g_r$$

is established.

Here, it is assumed that q_k becomes $q_k - h_k$ by making q_k smaller than before for g_r. h_k is an even non-negative integer. Then it is assume that r becomes $s(s > r)$, $g_s = g_r$ and g_s is not changed.

$$g_s/g_r = p_1{}^{q_1} \times \ldots \times p_r{}^{q_r}(p_1{}^{q_1-h_1} + \cdots + 1) \ldots (p_r{}^{q_r-h_r} + \cdots + 1)/(p_1{}^{q_1-h_1} \times \ldots \times p_r{}^{q_r-h_r}(p_1{}^{q_1} + \cdots + 1) \ldots (p_r{}^{q_r} + \cdots + 1)) = 1$$

$$p_1{}^{h_1} \times \ldots \times p_r{}^{h_r}(p_1{}^{q_1-h_1} + \cdots + 1) \ldots (p_r{}^{q_r-h_r} + \cdots + 1)/((p_1{}^{q_1} + \cdots + 1) \ldots (p_r{}^{q_r} + \cdots + 1)) \times p_{r+1}{}^{q_{r+1}} \times \ldots \times p_s{}^{q_s} = 1$$

$$p_{r+1}{}^{q_{r+1}} \times \ldots \times p_s{}^{q_s} \times p_1{}^{h_1} \times \ldots \times p_r{}^{h_r}(p_1{}^{q_1-h_1} + \cdots + 1) \ldots (p_r{}^{q_r-h_r} + \cdots + 1) = (p_1{}^{q_1} + \cdots + 1) \ldots (p_r{}^{q_r} + \cdots + 1)$$

$$p_{r+1}{}^{q_{r+1}} \times \ldots \times p_s{}^{q_s}(p_1{}^{q_1} + \cdots + p_1{}^{h_1}) \ldots (p_r{}^{q_r} + \cdots + p_r{}^{h_r}) = (p_1{}^{q_1} + \cdots + 1) \ldots (p_r{}^{q_r} + \cdots + 1)$$

$a = (p_1{}^{q_1} + \cdots + 1) \ldots (p_r{}^{q_r} + \cdots + 1) = cp^n$ holds and from expression ⑦, c must be a product of primes from p_1 to p_r. Thereby, the above equation does not hold since it is inappropriate when there is even one prime number other than p_1 to p_r. When changing the value of p_k, it is equivalent to dividing by $p_k{}^{q_k}$ and then multiplying by new $p_k{}^{q_k}$, so it is sufficient to consider only the changes of q_k and r. From above, since g_r does not chord the original value when q_k or r is increased or decreased, it takes unique values for the variables p_k, q_k, r.

From above proof,

$$g_r = A_1A_2 \ldots A_{r-1}/B_1B_2 \ldots B_{r-1} \times A_{s+1}A_{s+2} \ldots A_r/B_{s+1}B_{s+2} \ldots B_r$$

is represented uniquely, and expression (C) does not satisfied. When dividing by the prime number in the expression of p, a contradiction arises since the prime number not included in b is in the expression of p. Therefore, when p holds $p \equiv 1 \pmod 4$ and $p \geqq 5$, the number of the solution (a, b, p, n) satisfying $R = 1$ is at most one.

In the proof of expression (B), it is assumed that p changes on the way, and finally p becomes p_x.

$A_1 \dots A_r = cp^n$

$2B_1 \dots B_r = c(p^n + \cdots + 1)$

$A_1 \dots A_x = c' {p_x}^n$

$2B_1 \dots B_x = c'({p_x}^n + \cdots + 1)$

It is assumed that the above expressions are satisfied.

$$B_{s+1}B_{s+2} \dots B_r A_{r+1}A_{r+2} \dots A_x p^n({p_x}^{n_x} + \cdots + 1)$$
$$= A_{s+1}A_{s+2} \dots A_r B_{r+1}B_{r+2} \dots B_x {p_x}^{n_x}(p^n + \cdots + 1)$$

$$B_{s+1}B_{s+2} \dots B_r A_1 \dots A_r A_{r+1}A_{r+2} \dots A_x p^n({p_x}^{n_x} + \cdots + 1)$$
$$= A_1 \dots A_r A_{s+1}A_{s+2} \dots A_r B_{r+1}B_{r+2} \dots B_x {p_x}^{n_x}(p^n + \cdots + 1)$$

$$B_{s+1}B_{s+2} \dots B_r c' {p_x}^{n_x} p^n({p_x}^{n_x} + \cdots + 1)$$
$$= A_1 \dots A_r A_{s+1}A_{s+2} \dots A_r B_{r+1}B_{r+2} \dots B_x {p_x}^{n_x}(p^n + \cdots + 1)$$

$$B_{s+1}B_{s+2} \dots B_r c' p^n({p_x}^{n_x} + \cdots + 1) = A_1 \dots A_r A_{s+1}A_{s+2} \dots A_r B_{r+1}B_{r+2} \dots B_x(p^n + \cdots + 1)$$

$$B_1 \dots B_r B_{s+1}B_{s+2} \dots B_r c' p^n({p_x}^{n_x} + \cdots + 1)$$
$$= A_1 \dots A_r A_{s+1}A_{s+2} \dots A_r B_1 \dots B_r B_{r+1}B_{r+2} \dots B_x(p^n + \cdots + 1)$$

$$B_1 \dots B_r B_{s+1}B_{s+2} \dots B_r c' p^n({p_x}^{n_x} + \cdots + 1)$$
$$= A_1 \dots A_r A_{s+1}A_{s+2} \dots A_r c'({p_x}^{n_x} + \cdots + 1)/2 \times (p^n + \cdots + 1)$$

$$B_1 \dots B_r B_{s+1}B_{s+2} \dots B_r p^n = A_1 \dots A_r A_{s+1}A_{s+2} \dots A_r/2 \times (p^n + \cdots + 1)$$

$$c(p^n + \cdots + 1)/2 \times B_{s+1}B_{s+2} \dots B_r p^n = cp^n A_{s+1}A_{s+2} \dots A_r/2 \times (p^n + \cdots + 1)$$

$$B_{s+1}B_{s+2} \dots B_r = A_{s+1}A_{s+2} \dots A_r \dots \text{(D)}$$

is established. It becomes contradiction since $A_k > B_k$ holds. Thus, the number of solutions (a, b, p, n) for which $R = 1$ does not depend on the values of p and n is one at most. However, since $(a, b, p, n) = (1,1,1,1)$ becomes a solution, there is not any solution other than this combination. When the division is not performed, the above expression holds. From expression (B),

$$A_{r+1}A_{r+2} \dots A_x p^{n-n_x}(p^{n_x} + \cdots + 1) = B_{r+1}B_{r+2} \dots B_x(p^n + \cdots + 1)$$

When $n_x < n$, it becomes contradiction for $p \geqq 5$ since the right side of above expression does not include factor p.

When $n_x = n$,

$$A_{r+1}A_{r+2} \dots A_x = B_{r+1}B_{r+2} \dots B_x$$

It becomes contradiction. Thereby, when expression (D) is satisfied, it becomes inappropriate since p must be $p = p_x = 1$. Therefore, there are no odd perfect numbers.

4. Complement

From equation ⑤,

$2bp^n(p-1) = a(p^{n+1}-1)$

$2 = a(p^{n+1}-1)/(bp^n(p-1))$

$2 = (p_1^{q_1+1}-1)(p_2^{q_2+1}-1)\ldots(p_r^{q_r+1}-1)(p^{n+1}-1)$

$\quad /(p_1^{q_1}p_2^{q_2}\ldots p_r^{q_r}p^n(p_1-1)(p_2-1)\ldots(p_r-1)(p-1))$

$2(p_1^{q_1+1}-p_1^{q_1})(p_2^{q_2+1}-p_2^{q_2})\ldots(p_r^{q_r+1}-p_r^{q_r})(p^{n+1}-p^n)$

$\quad = (p_1^{q_1+1}-1)(p_2^{q_2+1}-1)\ldots(p_r^{q_r+1}-1)(p^{n+1}-1)$

We consider when $r = 2$.

$(p_1^{q_1+1}-1)(p_2^{q_2+1}-1)(p^{n+1}-1) = 2(p_1^{q_1+1}-p_1^{q_1})(p_2^{q_2+1}-p_2^{q_2})(p^{n+1}-p^n)$

Let s, t, u be integers,

$s = p_1^{q_1+1}-1$

$t = p_2^{q_2+1}-1$

$u = p^{n+1}-1$

are.

$stu = 2(p_1^{q_1+1}-1-(p_1^{q_1}-1))(p_2^{q_2+1}-1-(p_2^{q_2}-1))(p^{n+1}-1-(p^n-1))$

$stu = 2(s-(s+1)/p_1+1)(t-(t+1)/p_2+1)(u-(u+1)/p+1)$

$pp_1p_2stu = 2((s+1)p_1-(s+1))((t+1)p_2+(t+1))\big((u+1)p+(u+1)\big)$

$pp_1p_2stu = 2(s+1)(p_1-1)(t+1)(p_2-1)(u+1)(p-1)$

$stu/((s+1)(t+1)(u+1)) = 2(p_1-1)(p_2-1)(p-1)/(p_1p_2p)$

Since $stu/((s+1)(t+1)(u+1))$ is a monotonically increasing function for variables s, t and u, if

$s \geqq 3^{2+1}-1 = 26,\ p_1 = 3,\ q_1 = 2$

$t \geqq 7^{2+1}-1 = 342,\ p_2 = 7,\ q_2 = 2$

$u \geqq 5^2-1 = 24,\ p = 5,\ n = 1$

holds,

$stu/((s+1)(t+1)(u+1)) \geqq 26\times 342\times 24/(27\times 343\times 25) = 7904/8575$

$2(p_1-1)(p_2-1)(p-1)/(p_1p_2p) = 2\times 2\times 6\times 4/(3\times 7\times 5) = 32/35$

Since $stu/((s+1)(t+1)(u+1))$ is limited to 1 when s, t and u are infinite,
$stu/((s+1)(t+1)(u+1)) < 1$

If $f(p_1, p_2, p) = 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p)$ holds, it is sufficient to consider a combination where $f(p_1, p_2, p) < 1$.

$f(3,7,5) = 2 \times 2 \times 6 \times 4/(3 \times 7 \times 5) = 32/35$

$f(3,11,5) = 2 \times 2 \times 10 \times 4/(3 \times 11 \times 5) = 32/33$

$f(3,13,5) = 2 \times 2 \times 12 \times 4/(3 \times 13 \times 5) = 64/65$

$f(3,17,5) = 2 \times 2 \times 16 \times 4/(3 \times 17 \times 5) = 256/255$

$f(3,7,13) = 2 \times 2 \times 6 \times 12/(3 \times 7 \times 13) = 96/91$

$f(3,5,17) = 2 \times 2 \times 4 \times 16/(3 \times 5 \times 17) = 256/255$

From the above, when $r = 2$, a combination $(p_1, p_2, p) = (3,7,5), (3,11,5), (3,13,5)$ can be considered.

Let q_k be 2 and $n = 1$, if $g(p_1, p_2, p) = (p_1{}^3 - 1)(p_2{}^3 - 1)(p^2 - 1)/(p_1{}^3 p_2{}^3 p^2)$,

$g(3,7,5) = 26 \times 342 \times 24/(3^3 7^3 5^2) = 7904/8575 > 32/35$

$g(3,11,5) = 26 \times 1330 \times 24/(3^3 11^3 5^2) = 55328/59895$

$g(3,13,5) = 26 \times 2196 \times 24/(3^3 13^3 5^2) = 3904/4225$

Since the function g is the minimum in the case of $q_k = 2$ and $n = 1$, there is no solution q_k and n when $g > f$, so the case of $(p_1, p_2, p) = (3,7,5)$ becomes unsuitable.

$$stu/((s+1)(t+1)(u+1)) = 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p)$$

$$(p_1{}^{q_1+1} - 1)(p_2{}^{q_2+1} - 1)(p^{n+1} - 1)/(p_1{}^{q_1+1} p_2{}^{q_2+1} p^{n+1})$$
$$= 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p)$$

If $F(p_1, p_2, p) = (p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p)$,

$F(p_1{}^{q_1+1}, p_2{}^{q_2+1}, p^{n+1}) = 2F(p_1, p_2, p)$

5. Acknowledgement

In writing this research document, we asked anonymous reviewers to point out several tens of mistakes. We would like to thank you for giving appropriate guidance and counter-arguments.

6. References

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