Proof that there are no odd perfect numbers

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1. Abstract

For y to be a perfect number, if one of the prime factors is p, the exponent of p is an integer $n(n \geqq 1)$, the prime factors other than p are p_1,p_2,p_3,$\cdots p_r$ and the even exponent of p_k is q_k,

$$y/p^n = (1+p+p^2+\cdots+p^n)\prod_{k=1}^{r}(1+p_k+{p_k}^2+\cdots+{p_k}^{q_k})/(2p^n) = \prod_{k=1}^{r}{p_k}^{q_k}$$

must be satisfied. Let m be non negative integer and q be positive integer,

$$n = 4m+1$$

$$p = 4q+1$$

Letting b and c be odd integers, satisfying following expressions,

$$b = \prod_{k=1}^{r}{p_k}^{q_k}$$

$$c = \prod_{k=1}^{r}(1+p_k+{p_k}^2+\cdots+{p_k}^{q_k})/p^n$$

$$2b = c(p^n+\cdots+1)$$

is established. This is a known content. If we define odd integers a_x, b_x as below,

$$a_x = \prod\nolimits_{k=1}^{r(x)}({p_k}^{q_k}+\cdots+1)$$

$$b_x = \prod\nolimits_{k=1}^{r(x)}{p_k}^{q_k}$$

It is assumed that $a_1/a_2 \neq b_1/b_2$ holds when $b_1 \neq b_2$. Here, it is assumed that the prime number p_k and its exponent q_k do not have to be the same value for b_1 and b_2. By the consideration of this research paper, if this supposition is correct, since it turns out that there is a solution at most one when a is a multiple of p^n and at this time the value of b diverges to infinity, we have obtained the conclusion that there are no odd perfect numbers.

2. Introduction

The perfect number is one in which the sum of the divisors other than itself is the same value as itself, and the smallest perfect number is

$$1+2+3=6$$

It is 6. Whether an odd perfect number exists or not is currently an unsolved problem.

3. Proof

An odd perfect number is y, one of them is an odd prime number p, an exponent of p is an integer n (n ≧ 1). Let $p_1,p_2,p_3,\cdots p_r$ be the odd prime numbers of factors other than p, q_k the index of p_k, and variable a be the sum of product combinations other than prime p.

$$a = \prod_{k=1}^{r}(1 + p_k + {p_k}^2 + \cdots + {p_k}^{q_k}) \ \ldots ①$$

The number of terms N of variable a is

$$N = \prod_{k=1}^{r}(q_k + 1) \ \ldots ②$$

When y is a perfect number,

$$y = a(1 + p + p^2 + \cdots + p^n) - y \ (n > 0)$$

is established.

$$a\sum_{k=0}^{n} p^k /2 = y$$

$$a\sum_{k=0}^{n} p^k /(2p^n) = y/p^n \ \ldots ③$$

3.1. If q_k has at least one odd integer

Letting the number of terms where q_k is an odd integer be a positive integer u, because $y/p^n = \prod_{k=1}^{r} {p_k}^{q_k}$ is an odd integer, the denominator on the left side of expression ③ has a prime factor 2, from expression ② variable a has more than u prime factor 2 and variable a is an even integer. Therefore $\sum_{k=0}^{n} p^k$ must be an odd integer, n is an even integer and u is 1.

3.2. When all q_k are even integers

y/p^n is an odd integer, the denominator on the left side of expression ③ is an even integer, and since N is and odd integer when q_k are all even integers, variable a is and odd integer. Therefore $\sum_{k=0}^{n} p^k$ is necessary to include one prime factor 2, $\sum_{k=0}^{n} p^k \equiv 0 \ (mod\ 2)$ is established, and n must be an odd integer.

From 3.1, 3.2, in order to have an odd perfect number, only one exponent of the prime factor of y must be an odd integer and variable a must be an odd integer. We consider the case of 3.2 below.

In order for y to be a perfect number, the following expression must be established.

$$y/p^n = (1 + p + p^2 + \cdots + p^n)\prod_{k=1}^{r}(1 + p_k + {p_k}^2 + \cdots + {p_k}^{q_k})\,/(2p^n) = \prod_{k=1}^{r}{p_k}^{q_k}$$

However, $q_1, q_2, \dots, q_r$ are all even integers.

Here, let b be an integer

$$b = \prod_{k=1}^{r}{p_k}^{q_k} \ \dots ④$$

A following expression is established.

$y/p^n = a(1 + p + p^2 + \cdots + p^n)/(2p^n) = b$

$a(p^{n+1} - 1)/(2(p - 1)p^n) = b$

$(a - 2b)p^{n+1} + 2bp^n - a = 0$ ...⑤

Because it is an $n + 1$ order equation of p, the solution of the odd prime p is $n + 1$ at most.

$(ap - 2bp + 2b)p^n = a$

Since $ap - 2bp + 2b$ is an odd integer, a/p^n is an odd integer, which is c.

$ap - 2bp + 2b = c\ (c > 0)$ ...⑥

$(2b - a)p = 2b - c$

Since variable a is an odd integer, $2b - a$ is an odd integer and $2b - a \neq 0$

$p = (2b - c)/(2b - a)$

Since $n \geqq 1$

$a - c = cp^n - c \geqq cp - c > 0$

$a > c$

is.

From equation ⑥

$2b(p-1) - (ap - c) = 0$

$2b - c(p^{n+1} - 1)/(p-1) = 0$

$(p^n + \cdots + 1)/2$ is an odd integer, $n = 4m + 1$ is required with m as an integer.

$2b(p-1) = c(p^{n+1} - 1)$

$2b = c(p^n + \cdots + 1)$

$2b = c(p+1)(p^{n-1} + p^{n-3} + \cdots + 1)$ ...⑦

b is an odd integer when $p + 1$ is not a multiple of 4. It is necessary that $p - 1$ be a multiple of 4. A positive integer is taken as q.

$p = 4q + 1$

is established.

When $p > 1$

$p^n - 1 < p^n$

$(p^n - 1)/(p - 1) < p^n/(p - 1)$

$p^{n-1} + \cdots + 1 < p^n/(p - 1)$ ...⑧

Since p is an odd prime number satisfying $p = 4q + 1$ and $p \geqq 5$

$p^{n-1} + \cdots + 1 < p^n/4$

$2b - a = c(p^n + \cdots + 1) - cp^n = c(p^{n-1} + \cdots + 1)$

$2b - a < cp^n/4 = a/4$

$2b < 5a/4$

$a > 8b/5$ ...⑨

Let a_k and b_k be integers and if

$a_k = 1 + p_k + p_k^2 + \cdots + p_k^{q_k}$, $b_k = p_k^{q_k}$,

$a_k - b_k < b_k/(p_k - 1)$

$a_k < b_k p_k/(p_k - 1)$

$$a = \prod_{k=1}^{r} a_k < \prod_{k=1}^{r} b_k p_k/(p_k - 1) = b \prod_{k=1}^{r} p_k/(p_k - 1)$$

$$a/b < \prod_{k=1}^{r} p_k/(p_k - 1)$$

When $r = 1$, since $a/b < 3/2$ is established, it becomes inappropriate contrary to inequality ⑨.

From expression ⑦,

$b = c(p + 1)/2 \times (p^{n-1} + p^{n-3} + \cdots + 1)$

holds. Since $(p + 1)/2$ is the product of only prime numbers of b, let d_k be the index,

$$(p + 1)/2 = \prod_{k=1}^{r} {p_k}^{d_k}$$

$$p = 2\prod_{k=1}^{r} {p_k}^{d_k} - 1$$

From $a = cp^n$ and expression ⑦,

$2bp^n = a(p^n + \cdots + 1)$

$a(p^n + \cdots + 1)/(2bp^n) = 1$ ... (A)

When $r = 1$,

$a = ({p_1}^{q_1+1} - 1)/(p_1 - 1)$

$b = {p_1}^{q_1}$

Equation (A) does not hold since there is no odd perfect number when $r = 1$.

Let R be a rational number,

$R = a(p^n + \cdots + 1)/(2bp^n)$

Let b' be a rational number and let A and B to be an integer,

$b' = (p_k{}^{q_k+1} - 1)/(p_k{}^{q_k}(p_k - 1)) > 1$

$A = (p_k{}^{q_k+1} - 1)/(p_k - 1)$

$B = p_k{}^{q_k}$

Multiplying R by b', there are both cases that p_k increases p or does not change. When multiplied by b', the rate of change of R is $Ap^n(p'^n + \cdots + 1)/(Bp'^n(p^n + \cdots + 1))$, if p after variation is p'. If the rate of change of R is 1,

$Ap^n(p'^n + \cdots + 1)/(Bp'^n(p^n + \cdots + 1)) = 1$

$Ap^n(p'^n + \cdots + 1) = Bp'^n(p^n + \cdots + 1)$

This expression does not hold, since the right side is not a multiple of p when $p' > p$, and $A > B$ holds when $p' = p$. Due to this operation, R may be larger or smaller than the original value, since the rate of change of R does not become 1.

Assuming that $R = 1$ in some r, letting x be an integer and by multiplying fractions $b' = A_{r+1}/B_{r+1}$, $b'' = A_{r+2}/B_{r+2}$, $\cdots b''^{\cdots}{}' = A_x/B_x$ to R, if $R = 1$ holds finally. At this time, assuming that n changes, the change rate of R by this operation when multiplying by A_{r+1}/B_{r+1} is

$A_{r+1}p^n(p^{n_{r+1}} + \cdots + 1)/(B_{r+1}p^{n_{r+1}}(p^n + \cdots + 1))$

$$1 \times A_{r+1}p^n(p^{n_{r+1}} + \cdots + 1)/(B_{r+1}p^{n_{r+1}}(p^n + \cdots + 1)) \times A_{r+2}p^{n_{r+1}}(p^{n_{r+2}} + \cdots + 1)/(B_{r+2}p^{n_{r+2}}(p^{n_{r+1}} + \cdots + 1)) \times \ldots \times A_x p^{n_{x-1}}(p^{n_x} + \cdots + 1)/(B_x p^{n_x}(p^{n_{x-1}} + \cdots + 1)) = 1$$

$A_{r+1}A_{r+2} \ldots A_x p^n(p^{n_x} + \cdots + 1) = B_{r+1}B_{r+2} \ldots B_x p^{n_x}(p^n + \cdots + 1) \ldots (B)$

When $n = n_x$

$A_{r+1}A_{r+2} \ldots A_x = B_{r+1}B_{r+2} \ldots B_x$

holds. It becomes contradiction. Therefore, there is one solution when p and n are fixed.

The odd integers a_x and b_x are defined as follows, and it is assumed that the following proposition holds.

$$a_x = \prod_{k=1}^{r(x)} (p_k{}^{q_k} + \cdots + 1)$$

$$b_x = \prod_{k=1}^{r(x)} p_k{}^{q_k}$$

When $b_1 \neq b_2$, $a_1/a_2 \neq b_1/b_2$ holds. $\cdots$(C)

Here, it is assumed that the prime number p_k and its exponent q_k do not have to be the same value for b_1 and b_2.

Assuming that $R = 1$ in some r, letting x be an integer and fixing p and n and by multiplying fractions $b' = A_{r+1}/B_{r+1}$, $b'' = A_{r+2}/B_{r+2}$, $\cdots b''^{\cdots\prime} = A_x/B_x$ to R, and assuming that $R = 1$ holds finally. Dividing by $A_s A_{s+1} \dots A_r / B_s B_{s+1} \dots B_r$, from expression (B),

$$B_s B_{s+1} \dots B_r A_{r+1} A_{r+2} \dots A_x = A_s A_{s+1} \dots A_r B_{r+1} B_{r+2} \dots B_x$$

holds. If proposition (C) is correct, it becomes contradiction in this expression. When dividing by a prime number in the expression of p, contradiction arises since the prime number not included in b is in the expression of p. Therefore, when a is divided by p^n, the number of solutions is one.

When A_1 is divided by p, let t be an odd integer,

$$p_1{}^{q_1} + \cdots + 1 = tp$$

$$p_1{}^{q_1+1} - 1 = t(p_1 - 1)p$$

$$p_1{}^{q_1+1} \equiv 1 \pmod{p}$$

Let u be a rational number. From Fermat's little theorem,

$$(q_1 + 1)u = p - 1$$

is established. Thereby, q_1 can be changed as large as possible.

When $A_1 A_2 \dots A_{s-1}$ can be divided by p^n, the combinations of primes are infinite, and there is at most one solution for one of the combinations. Let a set having infinite number of elements which are odd prime multiples of the values of $B_1 B_2 \dots B_r$ be a set P, and consider a set Q having as an element the value of b when a is an odd multiple of p^n. When b is included in the set P or Q, the number of solutions is one for each set. Since set Q is a proper subset of the sum of all the sets considered as set P, there is at most one solution for all product sets of the set P. Therefore, even if an odd perfect number exists, since its value diverges to infinity, if proposition (C) is correct, there are no odd perfect numbers.

4. Complement

From equation ⑤,

$$2bp^n(p-1) = a(p^{n+1}-1)$$

$$2 = a(p^{n+1}-1)/(bp^n(p-1))$$

$$2 = (p_1{}^{q_1+1}-1)(p_2{}^{q_2+1}-1)\ldots(p_r{}^{q_r+1}-1)(p^{n+1}-1)$$
$$/(p_1{}^{q_1}p_2{}^{q_2}\ldots p_r{}^{q_r}p^n(p_1-1)(p_2-1)\ldots(p_r-1)(p-1))$$

$$2(p_1{}^{q_1+1}-p_1{}^{q_1})(p_2{}^{q_2+1}-p_2{}^{q_2})\ldots(p_r{}^{q_r+1}-p_r{}^{q_r})(p^{n+1}-p^n)$$
$$=(p_1{}^{q_1+1}-1)(p_2{}^{q_2+1}-1)\ldots(p_r{}^{q_r+1}-1)(p^{n+1}-1)$$

We consider when $r = 2$.

$$(p_1{}^{q_1+1}-1)(p_2{}^{q_2+1}-1)(p^{n+1}-1) = 2(p_1{}^{q_1+1}-p_1{}^{q_1})(p_2{}^{q_2+1}-p_2{}^{q_2})(p^{n+1}-p^n)$$

Let s, t, u be integers,

$s = p_1{}^{q_1+1}-1$

$t = p_2{}^{q_2+1}-1$

$u = p^{n+1}-1$

are.

$$stu = 2(p_1{}^{q_1+1}-1-(p_1{}^{q_1}-1))(p_2{}^{q_2+1}-1-(p_2{}^{q_2}-1))(p^{n+1}-1-(p^n-1))$$

$$stu = 2(s-(s+1)/p_1+1)(t-(t+1)/p_2+1)(u-(u+1)/p+1)$$

$$pp_1p_2stu = 2((s+1)p_1-(s+1))((t+1)p_2+(t+1))\big((u+1)p+(u+1)\big)$$

$$pp_1p_2stu = 2(s+1)(p_1-1)(t+1)(p_2-1)(u+1)(p-1)$$

$$stu/((s+1)(t+1)(u+1)) = 2(p_1-1)(p_2-1)(p-1)/(p_1p_2p)$$

Since $stu/((s+1)(t+1)(u+1))$ is a monotonically increasing function for variables s, t and u, if

$s \geqq 3^{2+1}-1 = 26,\ p_1 = 3,\ q_1 = 2$

$t \geqq 7^{2+1}-1 = 342,\ p_2 = 7,\ q_2 = 2$

$u \geqq 5^2-1 = 24,\ p = 5,\ n = 1$

holds,

$$stu/((s+1)(t+1)(u+1)) \geqq 26\times342\times24/(27\times343\times25) = 7904/8575$$

$$2(p_1-1)(p_2-1)(p-1)/(p_1p_2p) = 2\times2\times6\times4/(3\times7\times5) = 32/35$$

Since $stu/((s+1)(t+1)(u+1))$ is limited to 1 when s, t and u are infinite,
$stu/((s+1)(t+1)(u+1)) < 1$

If $f(p_1, p_2, p) = 2(p_1-1)(p_2-1)(p-1)/(p_1p_2p)$ holds, it is sufficient to consider a combination where $f(p_1, p_2, p) < 1$.
$f(3,7,5) = 2 \times 2 \times 6 \times 4/(3 \times 7 \times 5) = 32/35$
$f(3,11,5) = 2 \times 2 \times 10 \times 4/(3 \times 11 \times 5) = 32/33$
$f(3,13,5) = 2 \times 2 \times 12 \times 4/(3 \times 13 \times 5) = 64/65$
$f(3,17,5) = 2 \times 2 \times 16 \times 4/(3 \times 17 \times 5) = 256/255$
$f(3,7,13) = 2 \times 2 \times 6 \times 12/(3 \times 7 \times 13) = 96/91$
$f(3,5,17) = 2 \times 2 \times 4 \times 16/(3 \times 5 \times 17) = 256/255$
From the above, when $r = 2$, a combination $(p_1, p_2, p) = (3,7,5), (3,11,5), (3,13,5)$ can be considered.

Let q_k be 2 and $n = 1$, if $g(p_1, p_2, p) = ({p_1}^3-1)({p_2}^3-1)(p^2-1)/({p_1}^3{p_2}^3p^2)$,
$g(3,7,5) = 26 \times 342 \times 24/(3^3 7^3 5^2) = 7904/8575 > 32/35$
$g(3,11,5) = 26 \times 1330 \times 24/(3^3 11^3 5^2) = 55328/59895$
$g(3,13,5) = 26 \times 2196 \times 24/(3^3 13^3 5^2) = 3904/4225$
Since the function g is the minimum in the case of $q_k = 2$ and $n = 1$, there is no solution q_k and n when $g > f$, so the case of $(p_1, p_2, p) = (3,7,5)$ becomes unsuitable.

$stu/((s+1)(t+1)(u+1)) = 2(p_1-1)(p_2-1)(p-1)/(p_1p_2p)$
$({p_1}^{q_1+1}-1)({p_2}^{q_2+1}-1)(p^{n+1}-1)/({p_1}^{q_1+1}{p_2}^{q_2+1}p^{n+1})$
$= 2(p_1-1)(p_2-1)(p-1)/(p_1p_2p)$

If $F(p_1, p_2, p) = (p_1-1)(p_2-1)(p-1)/(p_1p_2p)$,
$F({p_1}^{q_1+1}, {p_2}^{q_2+1}, p^{n+1}) = 2F(p_1, p_2, p)$

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6. References

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