Fermat Equation for Hypercomplex Numbers

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Abstract

We consider generalized Fermat equation for hypercomplex numbers, in order to stimulate research and development of those generalization.

Keywords: integers; Fermat equation; hypercomplex numbers

1 Introduction

In number theory Fermat's Last Theorem (sometimes called Fermat's conjecture, especially in older texts) states that no three positive integers a, b, and c satisfy the equation $a^n + b^n = c^n$ for any integer value of n greater than 2. The cases n = 1 and n = 2 have been known since antiquity to have an infinite number of solutions.

The proposition was first conjectured by Pierre de Fermat in 1637 in the margin of a copy of Arithmetica; Fermat added that he had a proof that was too large to fit in the margin. However, there were first doubts about it since the publication was done by his son without his consent, after Fermats death. After 358 years of effort by mathematicians, the first successful proof was released in 1994 by Andrew Wiles, and formally published in 1995 [6]; it was described as a "stunning advance" in the citation for Wiles' Abel Prize award in 2016.

There are several generalized Fermat equations, however all of them do not consider hypercomlex numbers.

Nevertheless, consideration of Fermat equations for hypercomplex numbers may lead to the fact of existence of the corresponding integral solutions even for n greater than 2, similar to the situation, when the equation: $x^2 + 1 = 0$ doest not have real solutions, but does have it on the complex plane.

2 Generalized Fermat equation for the Gaussian Integers

Its well-known in number theory a complex number whose real and imaginary parts are both integers: Gaussian Integer. The Gaussian integers are the set: $\mathbf{Z}[\mathbf{i}] := \{ a + b\mathbf{i} \mid a, b \in \mathbf{Z} \}$, where $\mathbf{i}^2 = -1$. Gaussian integers are closed under addition and multiplication and form commutative ring, which is a subring of the field of complex numbers. When considered within the complex plane the Gaussian integers constitute the 2-dimensional integer lattice. The Gaussian integers form unique factorization domain: it is irreducible if and only if it is a prime(Gaussian primes). The field of Gaussian rationals consists of the complex numbers whose real and imaginary part are both rational(see, e.g., [3], [4], [5]).

The norm of a Gaussian integer is its product with its conjugate:

$N(a + b\mathbf{i}) = (a + b\mathbf{i})(a - b\mathbf{i}) = a^2 + b^2$.

The norm is multiplicative, that is, one has:

$N(zw) = N(z)N(w), \; z, w \in \mathbf{Z}[\mathbf{i}]$.

Fermat problem would be considered for the commutative ring of Gaussian Integers for the equation:

$a^n + b^n = c^n, \quad a, b, c \in \mathbf{Z}[\mathbf{i}], \; n \geq 2, n \in \mathbf{Z}$.

3 Generalized Fermat equation for the Lipschitz Integers(Quaternions)

Quaternions are generally represented in the form: $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, where, $a \in \mathbf{R}$, $b \in \mathbf{R}$, $c \in \mathbf{R}$, $d \in \mathbf{R}$, and $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the fundamental quaternion units and are a number system that extends the complex numbers(see, e.g., [1], [2]).

The set of all quaternions $\mathbf{H}$ is a normed algebra, where the norm is multiplicative: $\| pq \| = \| p \| \, \| q \|$, $p \in \mathbf{H}$, $q \in \mathbf{H}$, $\| q \|^2 = a^2 + b^2 + c^2 + d^2$.

This norm makes it possible to define the distance $d(p, q) = \|p - q\|$, which makes $\mathbf{H}$ into a metric space.

Lipschitz Integer(Quaternion) is defined as:

$$L := \{ q:\ q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mid a \in \mathbf{Z},\ b \in \mathbf{Z}, c \in \mathbf{Z},\ d \in \mathbf{Z} \}.$$

Lipschitz Integer(Quaternion) is a quaternion, whose components are all integers.

Fermat problem would be considered for the normed algebra of Lipschitz Integer(Quaternions) for the equation: $a^n + b^n = c^n$, $a, b, c \in \mathbf{H}$, $n \geq 2$, $n \in \mathbf{Z}$ and similar for other hypercomplex integral numbers.

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