The Free Photon Wave Function's Gauge-Invariant, Lorentz-Covariant Antisymmetric-Tensor Form

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Abstract

If a free photon's wave function is taken to be a four-vector function of its space-time coordinates that has vanishing four-divergence (the Lorentz condition), it isn't uniquely determined by the free-photon Schrödinger equation. This gauge indeterminacy can be eliminated by taking that wave function to be a three-vector function of its space-time coordinates—at the expense of its Lorentz-covariant form. These conflicts are resolved by taking a free photon's wave function to be an antisymmetric-tensor function of its space-time coordinates which has vanishing four-divergence and also satisfies the Lorentz-covariant cyclic Gauss-Faraday equation that is satisfied by all antisymmetric-tensor real-valued electromagnetic fields. It is shown that for every source-free antisymmetric-tensor real-valued electromagnetic field, there exists a corresponding free-photon antisymmetric-tensor complex-valued wave function.

A free photon's configuration-representation wave function is sometimes taken to be a four-vector function of space-time $\Upsilon^\mu(\mathbf{r},t)$ that satisfies the following free-photon Schrödinger equation and Lorentz condition,

$$\big(i\hbar\partial_t - \hbar c(-\nabla^2)^{\frac{1}{2}}\big)\Upsilon^\mu = 0 \quad \text{and} \quad \partial_\mu\Upsilon^\mu = 0, \tag{1a}$$

where the entity $\hbar c(-\nabla^2)^{\frac{1}{2}}$ is the massless free photon's *Hamiltonian operator* $\widehat{H} = (|c\widehat{\mathbf{p}}|^2)^{\frac{1}{2}} = c(\widehat{\mathbf{p}}\cdot\widehat{\mathbf{p}})^{\frac{1}{2}}$, since $\widehat{\mathbf{p}} = -i\hbar\nabla$ in configuration representation. The *four-vector form of* Υ^μ *in* Eq. (1a) *is suitably Lorentz-covariant*, but Eq. (1a) *doesn't uniquely determine* Υ^μ because, given any scalar function of space-time $\chi(\mathbf{r},t)$ which satisfies the source-free wave equation,

$$\left((1/c)^2\partial_t^2 - \nabla^2\right)\chi = \partial_\nu\partial^\nu\chi = 0, \tag{1b}$$

it is the case that if Υ^μ satisfies the two equations of Eq. (1a), then so does,

$$\Upsilon^\mu_\chi \stackrel{\text{def}}{=} \Upsilon^\mu + \big(-i\hbar\partial_t - \hbar c(-\nabla^2)^{\frac{1}{2}}\big)\partial^\mu\chi. \tag{1c}$$

That Υ^μ_χ satisfies the Lorentz condition of Eq. (1a) follows from the two facts that Υ^μ satisfies that Lorentz condition and that $\partial_\mu\partial^\mu\chi = 0$. That Υ^μ_χ satisfies the Schrödinger equation of Eq. (1a) follows from the fact that Υ^μ satisfies that Schrödinger equation and the fact that,

$$\big(i\hbar\partial_t - \hbar c(-\nabla^2)^{\frac{1}{2}}\big)\big(-i\hbar\partial_t - \hbar c(-\nabla^2)^{\frac{1}{2}}\big)\partial^\mu\chi = (\hbar c)^2\left((1/c)^2\partial_t^2 - \nabla^2\right)\partial^\mu\chi = (\hbar c)^2\partial^\mu(\partial_\nu\partial^\nu\chi) = 0, \tag{1d}$$

where the final equality of Eq. (1d) follows from Eq. (1b). This *gauge indeterminacy of* Υ^μ can be *eliminated* by *setting* Υ^0 *to zero*, which *modifies* the two equations of Eq. (1a) to,

$$\big(i\hbar\partial_t - \hbar c(-\nabla^2)^{\frac{1}{2}}\big)\boldsymbol{\Upsilon} = \mathbf{0} \quad \text{and} \quad \nabla\cdot\boldsymbol{\Upsilon} = 0, \tag{1e}$$

but the *three-vector form of* $\boldsymbol{\Upsilon}$ *isn't Lorentz-covariant*. These *conflicts* with gauge invariance or formal Lorentz covariance are *resolved* by *assigning* the free photon *the antisymmetric-tensor wave function*,

$$\Psi^{\mu\nu} = \partial^\mu\Upsilon^\nu - \partial^\nu\Upsilon^\mu, \tag{2a}$$

which of course satisfies the *free-photon Schrödinger equation*,

$$\big(i\hbar\partial_t - \hbar c(-\nabla^2)^{\frac{1}{2}}\big)\Psi^{\mu\nu} = 0, \tag{2b}$$

because Υ^ν and Υ^μ satisfy the free-photon Schrödinger equation as per Eq. (1a). Also, *crucially*,

$$\partial^\mu(\partial^\nu\chi) - \partial^\nu(\partial^\mu\chi) = 0 \quad \Rightarrow \quad \Psi^{\mu\nu}_\chi \stackrel{\text{def}}{=} \partial^\mu(\Upsilon^\nu_\chi) - \partial^\nu(\Upsilon^\mu_\chi) = \partial^\mu\Upsilon^\nu - \partial^\nu\Upsilon^\mu = \Psi^{\mu\nu}, \tag{2c}$$

so within the antisymmetric-tensor $\Psi^{\mu\nu}$ *the gauge indeterminacy of* Υ^μ *cancels out.*

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In *addition* to the property of $\Psi^{\mu\nu} = \partial^\mu\Upsilon^\nu - \partial^\nu\Upsilon^\mu$ of its being *antisymmetric*,

$$\Psi^{\nu\mu} = -\Psi^{\mu\nu}, \tag{2d}$$

it also satisfies the *Lorentz-covariant cyclic Gauss-Faraday equation*,

$$\partial^\lambda\Psi^{\mu\nu} + \partial^\mu\Psi^{\nu\lambda} + \partial^\nu\Psi^{\lambda\mu} = \left(\partial^\lambda\partial^\mu\Upsilon^\nu + \partial^\mu\partial^\nu\Upsilon^\lambda + \partial^\nu\partial^\lambda\Upsilon^\mu\right) - \left(\partial^\lambda\partial^\nu\Upsilon^\mu + \partial^\mu\partial^\lambda\Upsilon^\nu + \partial^\nu\partial^\mu\Upsilon^\lambda\right) = 0, \tag{2e}$$

and its *four-divergence vanishes* because,

$$\partial_\mu\Psi^{\mu\nu} = \partial_\mu\partial^\mu\Upsilon^\nu - \partial^\nu(\partial_\mu\Upsilon^\mu), \tag{2f}$$

and Eq. (1a) *imposes* $(\partial_\mu\Upsilon^\mu) = 0$, and *it also implies that* $\partial_\mu\partial^\mu\Upsilon^\nu = 0$ *via its* Υ^ν *Schrödinger equation*,

$$0 = \left[(1/(\hbar c))^2\left(-i\hbar\partial_t - \hbar c(-\nabla^2)^{\frac{1}{2}}\right)\right]\left(i\hbar\partial_t - \hbar c(-\nabla^2)^{\frac{1}{2}}\right)\Upsilon^\nu = \left((1/c)^2\partial_t^2 - \nabla^2\right)\Upsilon^\nu = \partial_\mu\partial^\mu\Upsilon^\nu. \tag{2g}$$

Although in Eq. (2a) we *synthesized* the gauge-invariant antisymmetric-tensor $\Psi^{\mu\nu} = \partial^\mu\Upsilon^\nu - \partial^\nu\Upsilon^\mu$ from the gauge-indeterminate four-vector Υ^μ, $\Psi^{\mu\nu}$ in fact *is characterized* by its Eq. (2b) *free-photon Schrödinger equation*, its Eq. (2d) *antisymmetry*, its Eq. (2e) *Lorentz-covariant cyclic Gauss-Faraday equation* and its Eq. (2f)–(2g) *vanishing four-divergence*. The Eq. (2d)–(2g) properties of $\Psi^{\mu\nu}$ *are exact analogs of the Lorentz-covariant Heaviside-Maxwell equations for source-free antisymmetric-tensor real-valued electromagnetic fields* $F^{\mu\nu}$, i.e., the Eq. (2d)–(2g) properties of $\Psi^{\mu\nu}$ *are exact analogs of*,

$$F^{\nu\mu} = -F^{\mu\nu}, \quad \partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0 \quad \text{and} \quad \partial_\mu F^{\mu\nu} = 0. \tag{3a}$$

Also, it turns out that for *every* such source-free real-valued electromagnetic field $F^{\mu\nu}$, there exists a *corresponding* free-photon complex-valued wave function $\Psi^{\mu\nu}$ that is given by,

$$\Psi^{\mu\nu}(\mathbf{r}, t) = N^{-\frac{1}{2}}\left(-i\hbar\partial_t - \hbar c(-\nabla^2)^{\frac{1}{2}}\right)F^{\mu\nu}(\mathbf{r}, t), \tag{3b}$$

where $N^{-\frac{1}{2}}$ is regarded here as an arbitrary positive constant, whose value we *further on* can *legitimately select to normalize* $\Psi^{\mu\nu}$. In light of Eq. (3a), it is apparent that the Eq. (3b) $\Psi^{\mu\nu}$ *does* satisfy the Eq. (2d)–(2g) *properties of* $\Psi^{\mu\nu}$. We next use Eq. (3a) to establish the *lemma* that,

$$0 = \partial_\lambda\left(\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu}\right) = \partial_\lambda\partial^\lambda F^{\mu\nu} - \partial^\mu\partial_\lambda F^{\lambda\nu} + \partial^\nu\partial_\lambda F^{\lambda\mu} = \partial_\lambda\partial^\lambda F^{\mu\nu}, \tag{3c}$$

which *enables* us to show that the Eq. (3b) $\Psi^{\mu\nu}$ *does* satisfy the Eq. (2b) *Schrödinger equation for* $\Psi^{\mu\nu}$,

$$\begin{gathered}\left(i\hbar\partial_t - \hbar c(-\nabla^2)^{\frac{1}{2}}\right)\Psi^{\mu\nu} = \left(i\hbar\partial_t - \hbar c(-\nabla^2)^{\frac{1}{2}}\right)N^{-\frac{1}{2}}\left(-i\hbar\partial_t - \hbar c(-\nabla^2)^{\frac{1}{2}}\right)F^{\mu\nu} = \\ N^{-\frac{1}{2}}(\hbar c)^2\left((1/c)^2\partial_t^2 - \nabla^2\right)F^{\mu\nu} = N^{-\frac{1}{2}}(\hbar c)^2(\partial_\lambda\partial^\lambda F^{\mu\nu}) = 0,\end{gathered} \tag{3d}$$

where the final equality of Eq. (3d) *follows from the* Eq. (3c) *lemma*. Having established that the Eq. (3b) $\Psi^{\mu\nu}$ *does indeed satisfy the* Eq. (2b) *Schrödinger equation, we can now legitimately select the particular value of the positive constant* $N^{-\frac{1}{2}}$ *in* Eq. (3b) *which normalizes* $\Psi^{\mu\nu}$,

$$\begin{gathered}1 = \sum_{\mu,\nu=0}^{3}\int|\Psi^{\mu\nu}(\mathbf{r}, t)|^2 d^3\mathbf{r} = \sum_{\mu,\nu=0}^{3}\int\left(\Psi^{\mu\nu}(\mathbf{r}, t)\right)^*\left(\Psi^{\mu\nu}(\mathbf{r}, t)\right)d^3\mathbf{r} = \\ N^{-1}\sum_{\mu,\nu=0}^{3}\int[(+i\hbar\partial_t F^{\mu\nu}) - (\hbar c(-\nabla^2)^{\frac{1}{2}}F^{\mu\nu})][(-i\hbar\partial_t F^{\mu\nu}) - (\hbar c(-\nabla^2)^{\frac{1}{2}}F^{\mu\nu})]d^3\mathbf{r} = \\ N^{-1}\sum_{\mu,\nu=0}^{3}\int\left[(\hbar\partial_t F^{\mu\nu})^2 + (\hbar c(-\nabla^2)^{\frac{1}{2}}F^{\mu\nu})^2\right]d^3\mathbf{r}.\end{gathered} \tag{3e}$$

Before evaluating N from the Eq. (3e) result, we establish the equality,

$$\sum_{\mu,\nu=0}^{3}\int(\hbar\partial_t F^{\mu\nu})^2 d^3\mathbf{r} = \sum_{\mu,\nu=0}^{3}\int(\hbar c(-\nabla^2)^{\frac{1}{2}}F^{\mu\nu})^2 d^3\mathbf{r}, \tag{3f}$$

from conservation of the well-known energy $\mathcal{E}$ of the source-free electromagnetic field $F^{\mu\nu}$, namely,

$$\mathcal{E} = (1/4)\sum_{\mu,\nu=0}^{3}\int(F^{\mu\nu})^2 d^3\mathbf{r} = (1/2)\int\left[|\mathbf{E}|^2 + |\mathbf{B}|^2\right]d^3\mathbf{r}. \tag{3g}$$

To establish that $\mathcal{E}$ is conserved, we insert the Source-Free Maxwell Law $\nabla\times\mathbf{B} = (1/c)(\partial_t\mathbf{E})$ and the Faraday Law $\nabla\times\mathbf{E} = -(1/c)(\partial_t\mathbf{B})$ *into the time derivative of the second* Eq. (3g) *expression for $\mathcal{E}$*,

$$\begin{aligned}\partial_t\mathcal{E} = (1/2)\textstyle\sum_{\mu,\nu=0}^{3}\int(\partial_t F^{\mu\nu})F^{\mu\nu}d^3\mathbf{r} = \int[(\partial_t\mathbf{E})\cdot\mathbf{E} + (\partial_t\mathbf{B})\cdot\mathbf{B}]\,d^3\mathbf{r} = \\ c\int[(\nabla\times\mathbf{B})\cdot\mathbf{E} - (\nabla\times\mathbf{E})\cdot\mathbf{B}]\,d^3\mathbf{r} = c\int[\mathbf{B}\cdot(\nabla\times\mathbf{E}) - (\nabla\times\mathbf{E})\cdot\mathbf{B}]\,d^3\mathbf{r} = 0,\end{aligned} \tag{3h}$$

where we in addition used $c\int[(\nabla\times\mathbf{B})\cdot\mathbf{E}\,]\,d^3\mathbf{r} = c\int[\mathbf{B}\cdot(\nabla\times\mathbf{E})]\,d^3\mathbf{r}$, which follows from integration by parts. Once more *taking the derivative with respect to t*, this time of $\hbar^2$ times the Eq. (3h) $\partial_t\mathcal{E}$, produces,

$$\hbar^2\partial_t^2\mathcal{E} = (1/2)\textstyle\sum_{\mu,\nu=0}^{3}\int(\hbar\partial_t F^{\mu\nu})^2d^3\mathbf{r} + (1/2)\sum_{\mu,\nu=0}^{3}\int(\hbar c)^2\big((1/c)^2\partial_t^2F^{\mu\nu}\big)F^{\mu\nu}d^3\mathbf{r} = 0. \tag{3i}$$

Eq. (3i), together with $0 = \partial_\lambda\partial^\lambda F^{\mu\nu} = (1/c)^2\partial_t^2F^{\mu\nu} - \nabla^2F^{\mu\nu}$, which is a consequence of Eq. (3c), yields,

$$\textstyle\sum_{\mu,\nu=0}^{3}\int(\hbar\partial_t F^{\mu\nu})^2d^3\mathbf{r} = \sum_{\mu,\nu=0}^{3}\int(\hbar c)^2\big(-\nabla^2F^{\mu\nu}\big)F^{\mu\nu}d^3\mathbf{r} = \sum_{\mu,\nu=0}^{3}\int(\hbar c(-\nabla^2)^{\frac{1}{2}}F^{\mu\nu})^2d^3\mathbf{r}, \tag{3j}$$

where the last equality *follows from the Hermitian nature of the free-photon Hamiltonian operator* $\widehat{H} = \hbar c(-\nabla^2)^{\frac{1}{2}}$. Eq. (3j) *establishes* Eq. (3f), *which together with* Eq. (3e) implies that,

$$N = 2\textstyle\sum_{\mu,\nu=0}^{3}\int(\hbar c(-\nabla^2)^{\frac{1}{2}}F^{\mu\nu})^2d^3\mathbf{r}. \tag{3k}$$

Insertion of the Eq. (3k) value of N into Eq. (3b) yields,

$$\Psi^{\mu\nu} = \big(-i\hbar\partial_t - \hbar c(-\nabla^2)^{\frac{1}{2}}\big)F^{\mu\nu}\big/\big(2\textstyle\sum_{\mu,\nu=0}^{3}\int\big(\hbar c(-\nabla^2)^{\frac{1}{2}}F^{\mu\nu}\big)^2d^3\mathbf{r}\big)^{\frac{1}{2}}, \tag{3l}$$

the free-photon complex-valued wave function *which corresponds to the source-free real-valued electromagnetic field $F^{\mu\nu}$*. It is readily seen that $\Psi^{\mu\nu}$ *is independent of both the scale of $F^{\mu\nu}$ and the value of $\hbar$.*

One might *speculate* that the gauge-invariant free-photon antisymmetric-tensor wave function $\Psi^{\mu\nu}$ could lead to the derivation *of a different class of Feynman rules for quantum electrodynamics which is gauge invariant at the fundamental propagator/vertex level*—the *existing* Feynman rules are gauge-invariant *only for sufficiently comprehensive sets of Feynman diagrams*.