From Turbulence to the Unification of Maxwell Field and Gravitational Field
(back to the roots)

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Abstract

The central focus of the theory lies on the solution of the since more than 165 years unsolved problem of turbulence. To achieve this aim the following interstations are reached successfully:

1. *definition of a pure continuum corresponding uniquely to a fluid,*
2. *stochastic turbulent particle transport by a fluctuating continuum,*
3. *context of deterministic turbulence and its stochastic counterpart in the sense of an ensemble theory.*

The result turns out to be geometrodynamics:

1. *a pure geometrodynamics of turbulence in a 1+3-dimensional Euclidian Space,*
2. *a pure geometrodynamics of deformation.*

Both geometrodynmics lead to

1. *evolution equations of General Relativity,*
2. *the quantitative unification of Maxwell Field and Gravitational Field,*
3. *the facilitation of quantizing gravitational fields,*
4. *considerations of general gravitational waves from a new perspective.*

The importance of the Einstein-Equations for microphysics is proved.

Contents

1. Preface

After his graduation in physics the author has got the impression that all the essential problems in physics had been solved. Accordingly William K. George (Professor of Turbulence Emeritus) states: "*I remember despairing as a student that it seemed that all the problems were solved, and there was really nothing fundamental left to do.*" [14] At an advanced age, after decades of research and development in industry, this impression has fundamentally changed. Last but not least the frequently occuring confrontations with problems of turbulence for example in connection with pollutant propagation in turbulently moved atmosphere or heat exchange processes in turbulently moved boundary layers[1] has shown, that much remains to be done. Furthermore the usual interpretation of quantum mechanics (the probability problem) and the question of its completeness got to thinking, too.

The variety of different "turbulence theories"[2] would never have had a chance, to find a useful application in Industry. As for turbulence not even a trace of a useful definition exists. Although treatises (for instance thesises) have been published by the most famous physicists of the 20th century like Heisenberg, it was nearly impossible for the industry concerning research and development to invest money and time. On the other hand hydrodynamics or rather turbulence didn't play a significant role studying theoretical physics.

The question, how an optimal theory of physics is characterized is qualitatively easy answered but difficult to realize in a practical and specific way: It has to give a 1 to 1 mapping of reality. Anyway there has to be a suitable definition of that part of reality, whose processes are to be analyzed. Sometimes the neccessary mathematics is not known. In this case one has to create formulations, which lead to calculable solutions. The identifying of their existence and definiteness may be sufficient for mathematics but for physics and especially industrial research they are not.

Theories in physics (especially in classical physics) are described by equations of motion with geometrodynamical connection (location, time, velocity and their space- and timelike partial derivations) and additional physical assumptions. These

[1] Reynold's law of similarity has proved to be entirely unsuitable in the case of turbulence.
[2] According to the opinion of the author: they all do not deserve this title.

assumptions include in the simplest case constants for instance the velocity of light in the Maxwell Equations of vacuum [3] , or pure physical terms like in the Navier-Stokes-Equations of hydrodynamics, which describe the accelleration field of the fluid continuum by state variables of thermodynamics. (At this an appropriate definition of the continuum is somewhat difficult.) These equations are in the laminar flow range well confirmed by experiments, the applications beyond these limiting cases remain at least dubious. The degree of hypothesizing in such theories is least, if the mathematical formulation can be reduced to geometrodynamics without assumptions and without loosing the **unique** reference to physical movements.

If it works calculating the velocity field of a heterogeneous mass as smeared mass distribution in a pure geometrodynamical way, the future smeared mass distribution can be determined exactly from the initially realized distribution.[4] This is the result of the well known continuity equation:

$$\frac{\partial}{\partial t}\boldsymbol{\rho} + \vec{\nabla} \cdot (\boldsymbol{\rho} \cdot \vec{\mathbf{v}}) = \mathbf{0}.$$

Calculations applying today's theory of hydrodynamics do not result in velocity fields of pure geometrodynamics. If the known hydrodynamic equations were correct [5], one could speak of a coupling of geometrodynamics and thermodynamics. Then there were an analogy to the theory of General Relativity, which means a coupling of space-time geometry with energy-momentum-density of matter. The development of suitable evolution equations in General Relativity is of great challenge for today's research.

Initially there was the intention to look for a mathematical formulation of deterministic transport processes of aerosols in turbulently moved fluids by a statistical transport theory within the meaning of an ensemble consideration. The developed transport equations would have been difficult to access experimentally, so that the second part of the research project arose, in which the relation of stochastic theory of natural causality with a suitable deterministic theory was established. Further theoretical researches of stochastic continuum fluctuations of 3-dimensional vector fields lead to pure geometrodynamics of turbulently fluctuating continua as well as fluctuating deformations. In doing so old line of thoughts of Einstein and Wheeler are not resumed. The resulting geometrodynamics is a consequent finding by avoiding

[3] these equations in this treatise are shown to be the consequence of geometrodynamics except the mentioned experimental velocity of light

[4] Generally, more cannot be achieved calculating a fluid. It requires defining a mathematical continuum corresponding to a discontinuously smeared matter

[5] Generally, they are not correct. Turbulence manifestations are the common case, laminar hydrodynamics is the limiting case.

hypotheses such that the unique relation of mathematical formulation and physical reference is not lost.
After having found these mathematical structures solutions for General Relativity fell into the laps:

1. *evolution equations of General Relativity*
2. *a unification of electromagnetic and gravitational fields*
3. *considerations of gravitational waves from a new perspective*
4. *the possibility of quantizing gravitational fields*[6]

Noting this required the fourth part of this treatise.

It is the opinion of the author, that the language of physics consists of two interacting parts-so to speak form and content-, whereby the form is linked to mathematical formulation and the content is linked to verbalized description.[7] In this way the present theory is developed. It is plausible that in the act of unifying certain fields of physics a joint mathematical formulation should take shape.

How mathematical formulation looks like after the great unification is not yet clear. The unification of Maxwell-and gravitational Field in this treatise is not yet the great unification, as the necessary linking to an elementary particle physics is missing. The author especially has in mind field theoretical explanations of electron, proton as well as neutron. Quantum mechanics, quantum field theory as well as the standard model of elementary particle physics of today turn up not to be complete theories. The claim of quantum field theory being a superior theory of physics is contradicted by the author in accordance with Penrose[34] as well as Einstein. Generally, classical physics is not the limiting case of quantum field theory ($\lim \hbar \to 0$) as maybe the limiting cases (of classical physics) Newton Mechanics and electrodynamics. In this treatise the electrodynamic field is shown to be a limiting case of formally classical physics but at the end of central importance. Einstein's General Relativity and the geometrodynamics of continuum physics of this treatise are no limiting cases of todays quantum field theory.

Overall, the author sees the way, which is persued, as a path to reality in analogy to the book "The Road to Reality“ of Penrose[34]. However Penrose doubts in the last chapter the correctness of the road outlined by himself (as it is nowadays generally tried in physics) not at least because the unification of Maxwell Field and Gravitational Field and quantization of the gravitational field failed. The author contends, that the prerequisit for understanding new developments is the understanding of their starting point, classical physics. Nowadays classical physics has serious shortcomings

[6] This option is not used in this treatise.
[7] the physicist is the moderator considering the interacting influences

on one side and on the other side contains methods to ensure a natural causality that is the simultaneous consideration of position and momentum, which is lost with the so called modern physics (quantum mechanics, quantum field theory, etc.). This treatise wants to cure the main deficits of classical physics via the way of solving the turbulence problem, the main problem of continuum physics, avoiding usual hypotheses. So the subtitle "back to the roots" has been chosen.

2. Introduction

> Feynman[13]: "*Nobody in physics has really been able to analyze it mathematically satisfactorily in spite of its importance to the sister sciences. It is the analysis of circulating or turbulent fluids.*"

The description of turbulent movements within the framework of continuum mechanics turned out to be difficult since more than 160 years. However, laminar fluid movements can be calculated by the known basic equations successfully confirmed in experiments: equation of continuity, Navier-Stokes-Equations and energy equation. The efforts, treating movements of turbulence in a similar way, must be considered as failures. There are substantial reasons for believing, that the above equations describing turbulent collective movements of **non-homogenously distributed molecular matter** are inadequate. This was the situation that inspired the idea, to explain the phenomenon of turbulence by stochastic methods. In that context, particularly approaches of Kolmogorov are to be mentioned, which lead to spectral energy distributions, assuming highly hypothetically, that turbulence is statistically isotropical and homogeneous. Between them there is a wide range of models with physically not well founded hypotheses. Overall, this leads to the statement of Feynman cited at the beginning, whereupon not much has changed since then.

This situation is characterized in recent treatises as for example by Trinh, Khanh Tuoc [24] in the following way:

" *the study of turbulence is immediately hampered by the surprising lack of a clear and concise definition of the physical process. Tsinober (2001) has published a long list of attempts at a definition by some of the most noted researchers in turbulence. The most common descriptions are vague: "a motion in which an irregular fluctuation (mixing, or eddying motion) is superimposed on the main stream" (Schlichting 1960), "a fluid motion of complex and irregular character" (Bayly, Orszag, Herbert, 1988)* *or negative as in the breakdown of laminar flow (Reynolds' experiment 1883). Some of the definitions are quite controversial like Saffman's (1981) "One of the best definition of turbulence is that it is a field of random chaotic vorticity" because the words random and chaotic would imply that a formal mathematical solution, which is necessarily deterministic, does not exist. Perhaps the most accurate definition can be attributed to Bradshaw (1971) "The only short but satisfactory answer to the question "what is turbulence" is that it is the general-solution of the Navier-Stokes equation". This definition cannot be argued with but it is singularly unhelpful since no general solution of the NS yet exists 160 years after they were formulated.*"

or by McDonough [30] " In particular, a turbulent flow can be expected to exhibit all of the following features:

1. *disorganized, chaotic, seemingly random behavior;*
2. *nonrepeatability (i.e., sensitivity to initial conditions);*
3. *extremely large range of length and time scales (but such that the smallest scales are still sufficiently large to satisfy the continuum hypothesis);*
4. *enhanced diffusion (mixing) and dissipation (both of which are mediated by viscosity at molecular scales);*
5. *three dimensionality, time dependence and rotationality (hence, potential flow cannot be turbulent because it is by definition irrotational);*
6. *intermittency in both space and time."*

Fluctuation elements of the presented theory always form a dense point set, i.e. a definition of a continuum of such fluctuation elements is important deducing equations of motion in form of partial differential equations. On the other hand a concept of a stochastic theory of a fluctuating continuum within the meaning of an ensemble theory is deduced. Fundametal principles of this treatise as well as in the whole classical physics are **locality, causality and deterministics**. In this treatise particular emphasis is placed on specially defined **natural causality**, which in contrary to **Newtonian causality** of point mechanics only knows finite velocities. Discussed stochastics arises from statistics with an in thought experiment supposed unlimited ensemble of locally equivalent deterministic processes.[1] Finally turbulence is derived as a deterministic process and traced back to geometrodynamics of turbulence. The fluidelement movements are described by interacting vortex- and vector-curvature

[1] If one wants, it can be seen as a many-world-theory of classical physics. However this is done creating beforehand unknown equations of the deterministic processes (in contrary to Everett's many-world-theory of quantum mechanics).

fields, whereas geometrodynamics of deformation $\vec{\boldsymbol{d}}$ is characterised by a fluctuating curvature-tensor-field of 2nd degree, alternatively decribed by interacting

$$\frac{\partial \vec{\boldsymbol{d}}}{\partial t}, \qquad rot(\vec{\boldsymbol{d}})$$

fields.

The interrelations of the deterministic and an associated stochastic theory enable a complete equation system of turbulently moved continua. The formulation of stochastically fluctuating processes of continua within the meaning of an ensemble theory is innovative for physics and mathematics. The known Navier-Stokes-Equations are not integrated in the complete equation system of turbulent moved fluids. The inclosed accelleration field $\frac{d\vec{\boldsymbol{v}}}{dt}$ of the associated momentum equation can not be described by thermodynamic variables and viscose frictions.

$$\frac{d\vec{\boldsymbol{v}}}{dt} \neq -\frac{1}{\rho}\vec{\boldsymbol{\nabla}}\boldsymbol{p} + \nu\Delta\vec{\boldsymbol{v}} + (\zeta + \frac{\nu}{3})\vec{\boldsymbol{\nabla}}\ (\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}})$$

The complete system of equations consists of 12 equations with 12 unknowns and contains only variables of motion in form of the vector fields: velocity, vortex, curvature and an accelleration field. So the developed theory of turbulence proves to be a geometrodynamics in a 3+1 dimensional Euclidian Space. Thermodynamics and matter distribution do not occur explicitly. This theory of variables of motion is principally exact. A smeared distribution of matter over Space-Time results by evalution of the calculated velocity vector field and the equation of continuity. Currently, in a parallel treatise, this system of equations will be solved numerically for different physical situations: pipes, concentric pipes and rectangular channels. The numerical solutions are two times continuously differentiable in space coordinates.[40]

Some potential solutions for General Relativity and Cosmology result unconstrainedly. They are preceded by definitions of dense fluid continua and distribution functions for an unlimited ensemble of locally equivalent continua.

Generally, the background theory describes two geometrodynamics of different structures, each faced by a stochastic theory of continuum fluctuations with **natural causality**. No additional physical model considerations enter into the theory.[2]
Overall, the following problems are treated or solved respectively:

- Transport Theory of turbulently moved Particles
  The stochastic transport equations considering consequently the continuum behaviour include coefficients, which stem from deterministic theory. The fluctuation variables are Euler Angles. These thoughts are important for example

[2] for example thermodynamics in the usual Navier-Stokes-Equations

considering pollutant dispersion in the lower atmosphere. The Theory offers several options of numerical evaluation depending on using the related partial differential equation, its presentation as integral equation or using particle trajectory-simulations in the related deterministic turbulence field (which can be calculated too).

- Continuum Theory of turbulent Fluctuations
  The physics of turbulence is described by certain geometrodynamics for related turbulent movements. The associated stochastic description represents a Markov-Process with **natural causality**. A Markov Process as the stochastic description of turbulence is generally denied, because the understanding of such a stochastic process is too restricted. Formulating a transition probability for space-time-points

  $$(\vec{x}, t) \longrightarrow (\vec{x}', t'),$$

  is not enough. It depends on the velocity in $(\vec{x}, t)$ reaching point $(\vec{x}', t')$. Arriving at the point $(\vec{x}', t')$ has to be the consequence of a velocity in $(\vec{x}, t)$, which will be regulated by means of a transition probability of velocities factored out in vortex and related curvature vectors. The existence of a velocity as causal factor for transport in space-time we call **natural causality**. Explicitly, this is forbidden in quantum mechanics. That is why there are principal differences between turbulence theory on one side and quantum mechanics and quantum field theory on the other side. The phase space appearing usually in Boltzmannlike formulations is replaced by an in every point $(\vec{x}, t)$ assumed velocity distribution of an ensemble of parallelly existing deterministic systems. Transition probabilities are explicitly calculated and are depending on the measurement accuracy of the measured motion quantities and degenerate exactly measured to δ-functions.

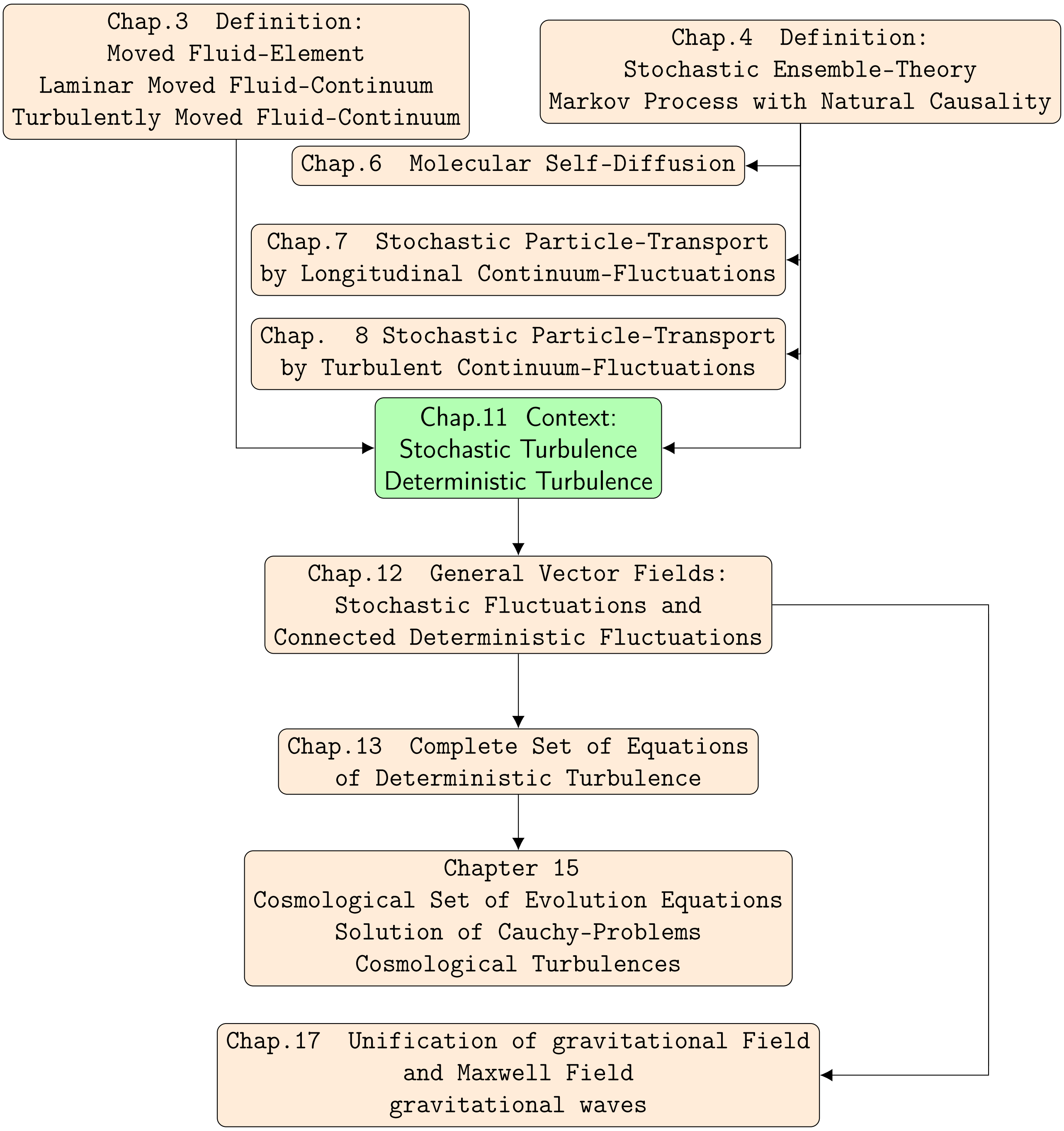
- Continuum Fluctuations of general 3-dimensional Vector Fields
  Their physical interpretation enables for example the calculation of
  - deformation fields (geometrodynamics of deformation)
  - turbulence fields (geometrodynamics of turbulence)
  - Maxwell Fields (they prove to be limiting cases of general deformation fields)

- Evolution Equations of General Relativity
  The complete equation system of turbulence proves to be an evolution equation system of General Relativity. That is why turbulence is described by pure geometrodynamics. The Cauchy-Problem of General Relativity is solved

by mapping the initial velocity vector field and its partial derivation with respect of time from space-time into a 3+1 dimensional Euclidian observer space (coordinate-space). From this all the necessary initial conditions for the system of 12 coupled equations of cosmological turbulence can be calculated. The determined accelleration field is the consequence of the Space-Time curvature field of General Relativity.

- <u>Fluctuations of the Riemannian hypersurface in Space-Time of General Relativity</u>
  Their formulation means the unification of General Relativity and Maxwell Field. This relation was not expected in the frame of classical physics. It results unconstrained by the mathematical structures of geometrodynamics of deformation. As the mapping is of key importance in mathematics so is the process in physics, in this case the deformation process of the Riemannian hypersurface of General Relativity. The solution succeeds not on founding the Riemannian geometry on an existing metric-tensor field but on a deformation vector field from which the metric tensor field can be constructed. So gravitational waves are seen from a new viewpoint. The Maxwell Fluctuations are quantitativly correlated with deformation fluctuations of Space (Riemannian hypersurface).

OVERVIEW: FLUCTUATIONS of the CONTINUUM



Part I.

Prerequisites

3. Definition of a moved fluid-continuum

3.1. Introduction

A proper definition of turbulence, which is based on a fluctuating, dense point set, does not exist. But this is neccessary establishing equations of movement in form of partial differential equations. The known Navier-Stokes equations are only providing sufficient solutions for laminar problems. Below a fluctuating fluid is defined, which is associated uniquely to a dense set of space points of the considered time. This definition is the prerequisite developing stochastic theories of turbulent transport of continously moved particles within the meaning of an ensemble theory, a deterministic theory and the connection of stochastic and a deterministic turbulence.

3.2. Definition of moved fluid-elements

At every time, space points ($\vec{\boldsymbol{x}}$) are assigned to fluid elements in a unique correspondence. As this applies to every space point ($\vec{\mathbf{x}}$) of the fluid field, the set of fluid elements is seen as a continuum. A Continuum of fluid element points (simply called fluid elements) is considered, where a fluid environment of non infinitesimal size is uniquely allocated to every fluid element point. Two infinitesimally neighboring fluid elements differ apart from their distance by their velocities and not quite identical material distributions of their neighborhoods. The neighborhoods of two nearby fluid elements overlap. A fluid element is shifted moving the material of its neighborhood. Though the material of such a fluid element may have changed marginally after an infinitesimal time interval t_ε, it can be identified principally by its prior material status. As every molecule possesses its own identity, there has to be at least an infinitesimally greater difference of material distribution to the neighborhoods of other fluid elements.
The neighborhoods exchange material with neighborhoods of adjacent fluid elements and vary their thermodynamic state (a local thermodynamic state does not necessarily exist). Their size is not infinitesimal, because a local thermodynamic state (if

physically existent) has to be detectable at least in thought experiment. The open neighborhoods have equally sized spherical shapes, generally. Near a solid border they are descibed by parts of spheres. Infinitesimally adjacent fluid elements possess overlapping neighborhoods. In an ε-surrounding they move in parallel. So one obtains a fluid, which is assumed to be a dense fluctuating point set, though there is no continuous matter distribution in Space-Time. That means it is possible to follow theoretically the history of every fluid element, though it has exchanged a lot of its initial material altering its local thermodynamic state.
Recapitulated:
Every space point ($\vec{\mathbf{x}}$) of the open point set of a considered fluid area is at every time in unique correspondence to a fluid element. The fluid is an abstract, dense set of fluctuating fluid elements, which do not generally correspond to material points.

3.3. Laminar moved fluids

A continuum of moved fluid elements is considered each uniquely assigned to a neighborhood and a velocity.

$$\vec{\mathbf{v}}_{t_\epsilon} = \frac{\vec{\mathbf{x}}_2 - \vec{\mathbf{x}}_1}{t_\epsilon} \tag{3.1}$$

The fluid elements move along sufficiently often continuously differentiable trajectories. The accuracies of the considered motion quantities are determined by t_ε-measurement processes t_ε characterising the accuracy. Deriving the transport equation of turbulent particle transport a limes consideration ($\lim \boldsymbol{t_\epsilon} \to 0$) is subjected. The whole of the velocities create a velocity vector field having $\boldsymbol{rot}(\vec{\mathbf{v}}) \neq \mathbf{0}$ generally.[1] Though $\boldsymbol{rot}(\vec{\mathbf{v}})$ has dimension $[\mathbf{1/sec}]$, it does not refer to a rotation of laminar flow. In an infinitesimally surrounding area of a space-time-point $(\vec{\mathbf{x}}_0, t_0)$ a fluid flow can be defined locally [2] by parallelly moved fluid elements. Considering without loss of generality a fluid movement of velocity $\vec{\mathbf{v}}(\vec{\mathbf{x}}_0) = (v_x, 0, 0)$ in a space point $\vec{\mathbf{x}}_0$ in cartesian coordinates, the velocity is described in an ϵ-neighborhood and parallel to the x-coordinate as follows:

$$\vec{\mathbf{v}}(\vec{\mathbf{x}}) = \begin{pmatrix} \mathbf{v}_x(\vec{\mathbf{x}}) \\ \mathbf{v}_y(\vec{\mathbf{x}}) \\ \mathbf{v}_z(\vec{\mathbf{x}}) \end{pmatrix} = \begin{pmatrix} \mathbf{v}_x(\vec{\mathbf{x}}_0) + & \frac{\partial \mathbf{v}_x}{\partial x}\Big|_{\vec{\mathbf{x}}_0} \cdot \Delta x + \frac{\partial \mathbf{v}_x}{\partial y}\Big|_{\vec{\mathbf{x}}_0} \cdot \Delta y + \frac{\partial \mathbf{v}_x}{\partial z}\Big|_{\vec{\mathbf{x}}_0} \cdot \Delta z + ... \\ & \frac{\partial \mathbf{v}_y}{\partial x}\Big|_{\vec{\mathbf{x}}_0} \cdot \Delta x + \frac{\partial \mathbf{v}_y}{\partial y}\Big|_{\vec{\mathbf{x}}_0} \cdot \Delta y + \frac{\partial \mathbf{v}_y}{\partial z}\Big|_{\vec{\mathbf{x}}_0} \cdot \Delta z + ... \\ & \frac{\partial \mathbf{v}_z}{\partial x}\Big|_{\vec{\mathbf{x}}_0} \cdot \Delta x + \frac{\partial \mathbf{v}_z}{\partial y}\Big|_{\vec{\mathbf{x}}_0} \cdot \Delta y + \frac{\partial \mathbf{v}_z}{\partial z}\Big|_{\vec{\mathbf{x}}_0} \cdot \Delta z + ... \end{pmatrix}$$

[1] in english literature $\boldsymbol{curl}(\vec{\mathbf{v}}) \neq \mathbf{0}$ is used but in turbulence the name $\boldsymbol{rot}$ is more adapted as will be seen

[2] except in stagnation points

The velocity components $\mathbf{v}_y(\vec{\mathbf{x}})$ and $\mathbf{v}_z(\vec{\mathbf{x}})$ **osculate** at the velocity $\vec{\mathbf{v}}(\vec{\mathbf{x}}_0) = (v_x, 0, 0)$ spatially approaching (constant time t_0),

$$\begin{array}{c} \mathbf{v}_y(x_0, y, z_0) \longrightarrow \mathbf{v}_y(x_0, y_0, z_0) = \mathbf{0} \\ \mathbf{v}_z(x_0, y_0, z) \longrightarrow \mathbf{v}_z(x_0, y_0, z_0) = \mathbf{0} \end{array}.$$

That means especially, that all the partial derivations by y- or z-coordinate of 1. order of $\mathbf{v}_y(\vec{\mathbf{x}})$ and $\mathbf{v}_z(\vec{\mathbf{x}})$ disappear in the point (x_0, y_0, z_0).

$$\begin{array}{c} \lim\limits_{z \to z_0} \frac{\Delta \mathbf{v}_y}{\Delta z}\Big|_{\vec{\mathbf{x}}_0} = \lim\limits_{y \to y_0} \frac{\Delta \mathbf{v}_z}{\Delta y}\Big|_{\vec{\mathbf{x}}_0} = \mathbf{0} \\ \vec{\mathbf{x}}_0 = (x_0, y_0, z_0) \end{array}. \tag{3.2}$$

Applying the differential quotients in the $\vec{\boldsymbol{\nabla}}\times$ -operator expresssed in cartesian coordinates gives for the fluid velocity

$$(\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{v}})_{|\vec{\mathbf{x}}_0} = \left(\begin{array}{c} 0 \\ \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \\ \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \end{array} \right)_{|\vec{\mathbf{x}}_0}, \vec{\mathbf{v}}(\vec{\mathbf{x}}_0) = (v_x, 0, 0) \tag{3.3}$$

The orthogonality of $\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{v}} \perp \vec{\mathbf{v}}$ is a fundamental quality [3][4] and a necessary condition for continuous fluid flow.

In this orthogonality velocity vector fields differ from deformation vector fields.

3.4. Turbulently moved fluids

Trying to identify the state of movement of a fluid element in turbulent fluids by a velocity $\vec{\mathbf{v}}_{t_\epsilon}$ it should be recognized, that the state of movement is not yet determined, as the path in every space point (except in turning points) is uniquely adapted by an infinitesimal circle segment. In the infinitesimal neighborhood of a path point the velocity is identified by an instantaneous axis of rotation $\vec{\boldsymbol{\omega}}_{t_\epsilon}$ and a radius vector $\vec{\mathbf{r}}_{t_\epsilon}$.[5]

$$\boxed{\vec{\mathbf{v}}_{t_\epsilon} = \vec{\boldsymbol{\omega}}_{t_\epsilon} \times \vec{\mathbf{r}}_{t_\epsilon}} \tag{3.4}$$

[3] this relation can not be found in literature.

[4] This is one reason why the known millenium prize question does not lead to a solution of the turbulence problem. However the validity problem of the Navier-Stokes-equations is more fatal.

[5] That is why turbulence can not be uniquely identified by experiments of local velocity statistics.

The considered vectorial motion quantities $\vec{\boldsymbol{\omega}}_{\boldsymbol{t_\epsilon}}$ and $\vec{\mathbf{r}}_{\boldsymbol{t_\epsilon}}$ are determined by t_ε-measurement processes, which are calculated later on by a limes process $\lim \boldsymbol{t_\epsilon} \to 0$. A fluid element originating from the point $\vec{\mathbf{x}}_0$ crossing $\vec{\mathbf{x}}_1$ after the time t_ε reaches $\vec{\mathbf{x}}_2$ after a further time t_ε.

$$\vec{\mathbf{x}}_0 \xrightarrow{\mathbf{t}_\epsilon} \vec{\mathbf{x}}_1 \xrightarrow{\mathbf{t}_\epsilon} \vec{\mathbf{x}}_2$$

By these 3 points a circle segment is uniquely drawn crossing point $\vec{\mathbf{x}}_1$ with radius vector $\vec{\mathbf{r}}_{t_\varepsilon}$ and speed of rotation $\vec{\boldsymbol{\omega}}_{t_\varepsilon}$. The local state of motion can not be described by velocity only, neither statistically nor deterministically. [6]
Thus the fluid element in the space-time-point $(\vec{\mathbf{x}}, t)$ is identified principally by the contents of the matter of its neighborhood and state of movement expressed by $\vec{\boldsymbol{\omega}}_{t_\epsilon}$ and $\vec{\mathbf{r}}_{t_\epsilon}$. In that way defined fluid elements move on sufficiently often continuously differentiable trajectories. They lead considering a continuum of fluctuating fluid elements to multiply continuously differentiable vector fields of motion. The continuum of moved fluid elements represent the turbulently collectiv movement of a discontinuously spaced Matter.
The field of turbulence is described by the two vector fields $\vec{\boldsymbol{\omega}}_{t_\epsilon}$ and $\vec{\mathbf{b}}_{t_\epsilon}$,

$$\vec{\mathbf{b}}_{t_\epsilon} = \vec{\mathbf{r}}_{t_\epsilon}/r_{t_\epsilon}^2 \qquad \text{-curvature vector field.} \tag{3.5}$$

In addition, the results show that

$$\vec{\boldsymbol{\omega}}_{t_\epsilon} = \frac{1}{2}\mathbf{rot}(\vec{\mathbf{v}}_{t_\epsilon}). \tag{3.6}$$

$\mathbf{rot}(\vec{\mathbf{v}})$ has the meaning of a local rotation in the frame of turbulence. An infinitesimal disturbance of stationary pipe flow leads to an change of the significance of $\mathbf{rot}(\vec{\mathbf{v}})$, where $\mathbf{rot}(\vec{\mathbf{v}})$ does not correspond to a rotation initially. Whether starting motions of turbulence are suppressed, depends on an existent viscosity. These decelerations are generally weak. The beginning of turbulent movements avoid Newtonian friction as well as pressure gradients by means of hereto orthogonal motions.

Vortex fields in turbulence (local rotation fields will be identified with vortex fields) and radius fields may have turning points $(\vec{\boldsymbol{x}}, t)$ along the paths of the fluid elements, which means $\vec{\boldsymbol{\omega}} = \mathbf{0}$ and $\vec{\boldsymbol{r}} = \infty$. [7] In this case the velocities are to be calculated by interpolation or extrapolation of the neighborhood, for example. In the theory a further method will be shown. The fluid elements are accompanied by a moving frame of $\vec{\boldsymbol{\omega}}$,$\vec{\boldsymbol{b}}$ and $\vec{\boldsymbol{v}}$ along their paths.
In the following it is outlined, how locally Lagrangian and Eulerian formulations of fluid dynamics are reassembled in the turbulence theory. So deterministic considerations are found via stochastic descriptions, which could be designated as Lagrangian. Nevertheless, Lagrangian paths are calculated only after the determin-

[6] This statement contadicts that of [44]
[7] The temporal and spatial neighborhood of a turning point does not have such singular properties.

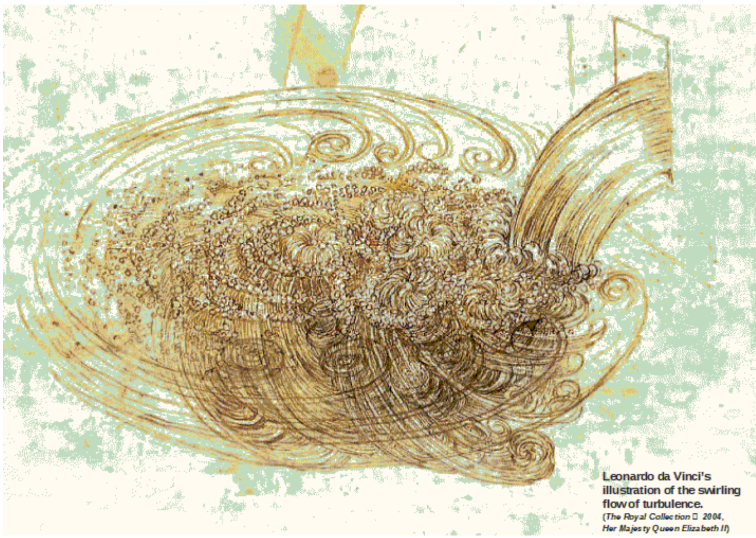


Figure 3.1.: Turbulences understood by Leonardo da Vinci

istic turbulence field is determined. These relations will become clear in later chapters.

4. Distribution functions

4.1. Introduction

Stochastically physical processes generally refer to random transports of physical quantities from $\vec{\boldsymbol{A}} = (\vec{\mathbf{x}}_1, t_1)$ to $\vec{\boldsymbol{B}} = (\vec{\mathbf{x}}_2, t_2)$, where a diffusion equation results at the end of all the discussions as can be seen in the well known treatise of Chandrasekhar [5]. This takes place in accordance with the Langevin equation, all known attempts characterizing Brownian motion and applying Fokker-Planck-equation, too. The diffusion equation is subjected to a Newtonian causality, that means the related propagation speed is unlimited. This is not the case in nonrelativistic physics beyond Newtonian mechanics, generally, as shown in the further course of this treatise.
In this context the Boltzmann Equation, which is only applicable for extensively diluted gases, constitutes a particularity. Despite surprising successes the importance of this equation is obviously not appropriately appreciated. In first approximation the Navier-Stokes equations are derived from this equation. A linear version can be classified as key-equation of nuclear reactor physics and is used for radioactive shielding problems in its stationary formulation.[1]These equations are based on a 6-dimensional phase-space with the apparent disadvantage, that using distribution functions $f(\vec{\mathbf{x}}, t, \vec{\boldsymbol{v}})$ a small but not infinitesimal phase space volume element $\triangle x \cdot \triangle y \cdot \triangle x \cdot \triangle v_x \cdot \triangle v_y \cdot \triangle v_z = \triangle \boldsymbol{V}$ is to be believed surrounding the phase space point $(\vec{\mathbf{x}}, \vec{\boldsymbol{v}})$. This situation is mathematically dissatisfying, as only a finite number of molecules can be existent inside this Volume, and executing $\lim \triangle \boldsymbol{V} \to 0$ there remain no molecules representing $f(\vec{\mathbf{x}}, t, \vec{\boldsymbol{v}})$. Despite this contrariness the Boltzmann Equation, in general or linear form, is successful considering the results.[2]
Such a situation exists in other fields of physics, too. For example, no mathematically satisfying definition of a continuum is existent justifying partial differential equations like the Navier-Stokes-equations. Nevertheless they have performed satisfactorily in the case of laminar fluid dynamics, but extending to the general case of turbulence the known equations of laminar fluid dynamics fail. Deficient mathematical justification is sometimes balanced by experiments, not always. In the special case of

[1]1968, associated with this the author has developed in his diploma thesis a numerical method (Monte-Carlo) for solving the linearely stationary Boltzmann Equation, without knowing this equation, simply by simulating the stochastic elementary processes.

[2]The Boltzmann Equation is the single equation describing the transition from a nonexisting local thermodynamic balance to a local thermodynamic balance.

kinetics, which is used for describing molecular self-diffusion, an ensemble theory is applied, which could be used for developing the Boltzmann Equation, too, avoiding the stated contrariness. On the other side the detailed mathematical formulations and their results do not alter by such a modified interpretation. But as phase space considerations are not possible for stochastically interpreted deterministic continuum fluctuations an equivalent treatment of ensemble theory will be used for all discussed problems.
The used distribution functions $f(\vec{\mathbf{x}}, t, \vec{\boldsymbol{v}})$ are not functions of the 6-dimensional phase space as usually applied in statiscal mechanics but regular functions of space time with a probability density distrubution of motion quantities in every point $(\vec{\mathbf{x}}, t)$ obtained by an unlimited ensemble of parallelly equivalent systems. In the case of the stationary linear radiation transport equation[3] very different elementary particles like neutrons, electrons, $\alpha - particles$, $\gamma - particles$ etc. are simultaneously calculated by this equation.

A suitable ensemble-consideration is helpful to avoid Newtonian causality (see section 4.5), and to get rid of the mathematical inconsistency of ΔV of limited size with limited number of included particles. So mathematically not justified applications of related partial differential equations are avoided. This interpretation not altering the mathematical formulations in connection with gas kinetics the turbulence is lead to new relations.

4.2. Ensemble consideration of molecular self-diffusion

The specifically used construction may appear somehow artificially, but it is supposed to illustrate the classification of the usual diffusion equation as an approximate equation of a primary, with natural causality endowed transport equation. The particle density distributions are gained in thought experiment by an unlimited number of ensemble systems, which exist simultaneously. Their functions are sufficiently often, continuously differentiable in space and time. Regarding the quantities of motion the continuity condition is sufficient. This situation may be generated as follows:

An ensemble of parallel, extensively diluted monomolecular systems is considered to be in local thermodynamic ballance. They are all seen as statistically equivalent. They generally differ locally in an $\boldsymbol{\epsilon}$- neighborhood. Permitting for some

[3] This is a slight modification of the linear Boltzmann Equation with a streaming function $\frac{1}{v} f(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}})$ instead of distribution function $f(\vec{\mathbf{x}}, t, \vec{\boldsymbol{v}}))$, and the different velocities before and after molecular collisions are considered in the differential cross sections of the collision integral only [43].

time equivalent molecules to enter in all systems by equally distributed sources such that the additional quantity of gas is insignificant in relation to the original quantity of the gas, the additional part of the gas will be in the same statistical balance in all the systems of the ensemble after a short time, though it has not reached a homogenous distribution. While the added part of gas consists in every single system of a limited number of molecules only, it is possible to formulate a sufficiently often continuously differentiable particle density distribution $f(\vec{\mathbf{x}}, \vec{\mathbf{v}}, t)$ for the added gas part, as the statistics of the velocities relates to the whole ensemble.

The expectation value of a suitable particle density may be constructed as follows. Around the point $(\vec{\mathbf{x}}, t)$ an equal-sized volume $\Delta\mathbf{V}_{(\vec{\mathbf{x}},t)}$ is chosen out of all representatives μ of the ensemble in which a subset $\Delta\mathbf{N}_\mu$ of molecules is located. μ identifies a single representative of the ensemble. μ passing all values from $\mathbf{1}$ to $\boldsymbol{\infty}$ an unlimited number of ensemble-representatives is taken into account. The expectation value of a particle density of this ensemble-consideration in point $(\vec{\mathbf{x}}, t)$ in the small volume $\Delta\mathbf{V}_{(\vec{\mathbf{x}},t)}$ of the neighborhood of $(\vec{\mathbf{x}}, t)$ results in

$$< \rho(\vec{\mathbf{x}}, t) > |_{\boldsymbol{\Delta V}} = \lim_{\boldsymbol{n \to \infty}} \frac{1}{n} \sum_{\mu=1}^{n} \frac{\Delta\mathbf{N}_\mu}{\Delta\mathbf{V}_{(\vec{\mathbf{x}},t)}}. \tag{4.1}$$

Contracting the volume $\Delta\mathbf{V}_{(\vec{\mathbf{x}},t)}$ to the point $(\vec{\mathbf{x}}, t)$ one has

$$< \rho(\vec{\mathbf{x}}, t) >= \lim_{\boldsymbol{\Delta V}_{(\vec{\mathbf{x}},t)} \to \mathbf{0}} < \rho(\vec{\mathbf{x}}, t) > |_{\boldsymbol{\Delta V}}. \tag{4.2}$$

This function of expectation values is sufficiently often continuously differentiable in its depending variables, especially in space and time. In accordance with the velocities, consisting of amount and direction of motion, their distribution density is separated in these quantities as follows [4]

$$\boxed{\begin{aligned} &< \rho(\vec{\mathbf{x}}, t) > = \int_0^\infty \int_{4\pi} f(\vec{\mathbf{x}}, t, \vec{\boldsymbol{v}}) d\vec{\boldsymbol{\Omega}} dv = \int_{4\pi} h(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}) d\vec{\boldsymbol{\Omega}} \\ &\quad f(\vec{\mathbf{x}}, t, \vec{\mathbf{v}}) = f(\vec{\mathbf{x}}, t, v\vec{\boldsymbol{\Omega}}) = h(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}) \mathbf{g}(v) \\ &\qquad\qquad \int_0^\infty \mathbf{g}(v) dv = \mathbf{1}, \qquad \bar{v} = \int_0^\infty \mathbf{g}(v) v dv \\ &\qquad\qquad \vec{\boldsymbol{v}} = \boldsymbol{v} \cdot \vec{\boldsymbol{\Omega}}. \end{aligned}} \tag{4.3}$$

So the necessary connection is given by

[4] A separation ansatz $f(\vec{\mathbf{x}}, t, v\vec{\boldsymbol{\Omega}}) = h(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}) g(\vec{\mathbf{x}}, t, v\vec{\boldsymbol{\Omega}})$ is generally to be chosen.

$$h(\vec{\mathbf{x}},t,\vec{\boldsymbol{\Omega}}) = \int_0^\infty f(\vec{\mathbf{x}},t,\boldsymbol{v}\vec{\boldsymbol{\Omega}})dv. \tag{4.4}$$

This enables the derivation of a transport equation of $h(\vec{\mathbf{x}},t,\vec{\boldsymbol{\Omega}})$ the amounts of the velocities occuring as constant coefficients only.

At this the distribution of the velocity amounts $\mathbf{g}(\mathbf{v})$ is separated from the direction distribution $h(\vec{\mathbf{x}},t,\vec{\boldsymbol{\Omega}})$. The diffusing particles possess a gaussian distribution and the equipartition law is applied. This is the prerequisite for deriving a suitable transport equation below and afterwards in less than first approximation a diffusion equation with constant diffusion coefficients.
This special distribution of the velocity amounts (equipartition law) corresponds to the assumed situation of molecular self-diffusion of chapter 6. The gained expectation value of the density does not exactly equal the value measured in an ensemble representative performed in a small volume. Thus one has

$$< \rho(\vec{\mathbf{x}},t) > \approx \rho(\vec{\mathbf{x}},t). \tag{4.5}$$

An exact measurement (this has nothing to do with a measurement of the macroscopic state quantity density) would result in

$$< \rho(\vec{\mathbf{x}},t) > \neq \rho(\vec{\mathbf{x}},t) = \begin{cases} \mathbf{1} & \text{one particle existent in point} \quad (\vec{\mathbf{x}},t) \\ \mathbf{0} & \text{else} \end{cases}. \tag{4.6}$$

4.3. Ensemble consideration of stochastic particle transport in a continuum of longitudinal fluctuations

Similarily, further considerations occur to section 4.2. Around the point $(\vec{\mathbf{x}},t)$ an equal-sized volume $\Delta\mathbf{V}_{(\vec{\mathbf{x}},t)}$ is chosen out of all representatives μ of the ensemble in which a subset $\Delta\mathbf{N}_\mu$ of particles [5] is located. μ identifies a single representative of the ensemble. The expectation value of a particle density results in

$$< \rho(\vec{\mathbf{x}},t) > = \lim_{\boldsymbol{\Delta V}_{(\vec{\mathbf{x}},t)}\to\mathbf{0}} \left(\lim_{\boldsymbol{n}\to\boldsymbol{\infty}} \frac{1}{n} \sum_{\mu=1}^{n} \frac{\Delta\mathbf{N}_\mu}{\Delta\mathbf{V}_{(\vec{\mathbf{x}},t)}} \right). \tag{4.7}$$

[5] the particles have to be of a size adapting the identical movement of present fluid element of the fluctuating continuum

This function of expectations is sufficiently often continuously differentiable in its variables of space and time. By the separation of the velocity in amount and direction the distribution density is described as follows

$$\boxed{\begin{aligned} < \rho(\vec{\mathbf{x}}, t) >= &\int_0^\infty \int_{4\pi} f(\vec{\mathbf{x}}, t, \vec{\boldsymbol{v}}) d\vec{\mathbf{\Omega}} dv = \int_{4\pi} \overline{f}(\vec{\mathbf{x}}, t, \vec{\mathbf{\Omega}}) d\vec{\mathbf{\Omega}} \\ f(\vec{\mathbf{x}}, t, \vec{\mathbf{v}}) =& G(\vec{\mathbf{x}}, t, v\vec{\mathbf{\Omega}}) \overline{f}(\vec{\mathbf{x}}, t, \vec{\mathbf{\Omega}}) \\ \int_0^\infty G(\vec{\mathbf{x}}, t, v\vec{\mathbf{\Omega}}) d\boldsymbol{v} =& \mathbf{1}, \qquad \overline{v}(\vec{\mathbf{x}}, t, \vec{\mathbf{\Omega}}) = \int_0^\infty G(\vec{\mathbf{x}}, t, v\vec{\mathbf{\Omega}}) v d\boldsymbol{v} \\ \vec{\boldsymbol{v}} =& \boldsymbol{v} \cdot \vec{\mathbf{\Omega}}. \end{aligned}} \tag{4.8}$$

So the necessary combination

$$\overline{f}(\vec{\mathbf{x}}, t, \vec{\mathbf{\Omega}}) = \int_0^\infty f(\vec{\mathbf{x}}, t, \boldsymbol{v}\vec{\mathbf{\Omega}}) dv. \tag{4.9}$$

is achieved deriving a transport equation for $\overline{f}(\vec{\mathbf{x}}, t, \vec{\mathbf{\Omega}})$. In this transport equation the velocities occur as coefficients of the averaged velocity amounts in dependence of space, time and direction.

The gained expectation value of the density fails to comply with the measured ensemble representative, that is

$$< \rho(\vec{\mathbf{x}}, t) > \neq \rho(\vec{\mathbf{x}}, t) \tag{4.10}$$

For a single ensemble representative the distribution function f degenerates to a delta-function

$$f \to \delta(\vec{\mathbf{v}}_{(\vec{\boldsymbol{x}}, t)}, \vec{\mathbf{v}}) \tag{4.11}$$

with

$$\begin{aligned} &\int_{\vec{\mathbf{v}}} \delta(\vec{\mathbf{v}}_{(\vec{\boldsymbol{x}}, t)}, \vec{\mathbf{v}}) d\vec{\mathbf{v}} = \mathbf{1} \\ &\int_{\vec{\mathbf{v}}} \delta(\vec{\mathbf{v}}_{(\vec{\boldsymbol{x}}, t)}, \vec{\mathbf{v}}) \vec{\mathbf{v}} d\vec{\mathbf{v}} = \vec{\mathbf{v}}_{(\vec{\boldsymbol{x}}, t)}. \end{aligned} \tag{4.12}$$

4.4. Ensemble consideration of stochastic particle transport in turbulently moved continua

The fluid fluctuations of different ensemble-representatives may arise at the same time by equivalent macroscopic-physical processes with different infinitesimal perturbations. That is why different fluent motions are created in $(\vec{\mathbf{x}}, t)$ in every of the

parallel-sytems. The simultaneous release of passive particels retracing uniquely the motions of fluid movements may have taken place by a distribution of similar point sources in all parallel systems. So an own particle distribution is developed in every individual system in space-time. The statistical recording running over the whole ensemble leads to continuously differentiable distribution functions of a limited number of particles [6] in a single system.
Further considerations follow analogously to section 4.2. Around the point $(\vec{\mathbf{x}}, t)$ an equal-sized volume $\Delta\mathbf{V}_{(\vec{\mathbf{x}},t)}$ is chosen out of all representatives μ of the ensemble in which a subset $\Delta\mathbf{N}_\mu$ of particles is located. μ identifies a single representative of the ensemble. The expectation value of a particle density results in

$$< \rho(\vec{\mathbf{x}}, t) >= \lim_{\boldsymbol{\Delta V}_{(\vec{\mathbf{x}},t)}\to\mathbf{0}} \left(\lim_{\boldsymbol{n\to\infty}} \frac{1}{n} \sum_{\mu=1}^{n} \frac{\Delta\mathbf{N}_\mu}{\Delta\mathbf{V}_{(\vec{\mathbf{x}},t)}} \right). \tag{4.13}$$

This function of expectation values arises out of a distribution function f of motion quantities

$$\begin{aligned} \vec{\omega} =& \vec{\omega}(\vec{\mathbf{x}}, t) \qquad \text{rotation speed} \\ \vec{\mathbf{r}} =& \vec{\mathbf{r}}(\vec{\mathbf{x}}, t) \qquad \text{radius vector} \\ \vec{\mathbf{v}} =& \vec{\omega}(\vec{\mathbf{x}}, t) \times \vec{\mathbf{r}}(\vec{\mathbf{x}}, t) \qquad \text{velocity vector} \end{aligned} \tag{4.14}$$

that means

$$f = f(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\omega}}, \vec{\mathbf{r}}). \tag{4.15}$$

A separation results in

$$\boxed{\begin{gathered} < \rho(\vec{\mathbf{x}}, t) >= \int_{\vec{\mathbf{r}}} \int_{\vec{\boldsymbol{\omega}}} f(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\omega}}, \vec{\mathbf{r}}) d\vec{\boldsymbol{\omega}} d\vec{\boldsymbol{r}} = \int_{2\pi} \int_{4\pi} \overline{f}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}) d\vec{\boldsymbol{\Omega}} d\vec{\boldsymbol{\Theta}} \\ f(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\omega}}, \vec{\mathbf{r}}) = G(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\omega}}, \vec{\mathbf{r}}) \overline{f}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}) \\ \int_0^\infty \int_0^\infty G(\vec{\mathbf{x}}, t, \boldsymbol{\omega}\vec{\boldsymbol{\Omega}}, r\vec{\boldsymbol{\Theta}}) d\omega dr = 1, \quad \int_0^\infty \int_0^\infty G(\vec{\mathbf{x}}, t, \boldsymbol{\omega}\vec{\boldsymbol{\Omega}}, r\vec{\boldsymbol{\Theta}}) \omega r d\omega dr = \overline{v}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}) \\ \int_0^\infty \int_0^\infty G(\vec{\mathbf{x}}, t, \boldsymbol{\omega}\vec{\boldsymbol{\Omega}}, r\vec{\boldsymbol{\Theta}}) r d\omega dr = \overline{r}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}), \quad \int_0^\infty \int_0^\infty G(\vec{\mathbf{x}}, t, \boldsymbol{\omega}\vec{\boldsymbol{\Omega}}, r\vec{\boldsymbol{\Theta}}) \omega d\omega dr = \overline{\omega}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}) \\ \overline{v}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}) = \overline{\omega}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}) \cdot \overline{r}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}) \\ \vec{\mathbf{v}}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}) = \vec{\omega}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}) \times \vec{\mathbf{r}}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}). \end{gathered}} \tag{4.16}$$

Such the necessary combination is given by

[6] the particles have to be of a size adapting the identical movement of present fluid element of the fluctuating continuum

$$\overline{f}(\vec{\mathbf{x}}, t, \vec{\mathbf{\Omega}}, \vec{\mathbf{\Theta}}) = \int_0^\infty \int_0^\infty f(\vec{\mathbf{x}}, t, \boldsymbol{\omega} \cdot \vec{\mathbf{\Omega}}, \mathbf{r} \cdot \vec{\mathbf{\Theta}}) d\omega dr. \tag{4.17}$$

This enables a transport equation of $\overline{f}(\vec{\mathbf{x}}, t, \vec{\mathbf{\Omega}}, \vec{\mathbf{\Theta}})$ with the averaged amounts of rotation velocities and radius-vectors as coefficients.

The resulting expectation value of the density does not equal the measured value of a single ensemble representative. That is

$$< \rho(\vec{\mathbf{x}}, t) > \neq \rho(\vec{\mathbf{x}}, t) \tag{4.18}$$

Limiting to one system of the ensemble the distribution function degenerates to a delta-function

$$f \longrightarrow \delta(\vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)}, \vec{\mathbf{r}}_{(\vec{\boldsymbol{x}},t)}; \vec{\boldsymbol{\omega}}, \vec{\mathbf{r}}) \tag{4.19}$$

with

$$\begin{aligned}
&\int_{\vec{\mathbf{r}}} \int_{\vec{\boldsymbol{\omega}}} \delta(\vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)}, \vec{\mathbf{r}}_{(\vec{\boldsymbol{x}},t)}; \vec{\boldsymbol{\omega}}, \vec{\mathbf{r}}) d\vec{\boldsymbol{\omega}} d\vec{\mathbf{r}} = \mathbf{1} \\
&\int_{\vec{\mathbf{r}}} \int_{\vec{\boldsymbol{\omega}}} \delta(\vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)}, \vec{\mathbf{r}}_{(\vec{\boldsymbol{x}},t)}; \vec{\boldsymbol{\omega}}, \vec{\mathbf{r}}) \vec{\boldsymbol{\omega}} d\vec{\boldsymbol{\omega}} d\vec{\mathbf{r}}' = \vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)} \\
&\int_{\vec{\mathbf{r}}} \int_{\vec{\boldsymbol{\omega}}} \delta(\vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)}, \vec{\mathbf{r}}_{(\vec{\boldsymbol{x}},t)}; \vec{\boldsymbol{\omega}}, \vec{\mathbf{r}}) \vec{\mathbf{r}} d\vec{\boldsymbol{\omega}} d\vec{\mathbf{r}}' = \vec{\mathbf{r}}_{(\vec{\boldsymbol{x}},t)}.
\end{aligned} \tag{4.20}$$

4.5. Definition of Markov Processes with natural causality

The probabilistic theory is related to random distributions of velocities $\vec{\boldsymbol{\pi}}$ moving from $(\vec{\mathbf{x}}, t)$ to $(\vec{\mathbf{x}} + \vec{\boldsymbol{\pi}} t_\epsilon, t + t_\epsilon)$. These velocity distributions may get together of vortex and curvature vector fields

$$\vec{\boldsymbol{\pi}} = \vec{\boldsymbol{\omega}} \times \frac{\vec{\boldsymbol{b}}}{b^2}.$$

The transport from $(\vec{\mathbf{x}} - t_\varepsilon \vec{\boldsymbol{\pi}}', \boldsymbol{t} - \boldsymbol{t_\epsilon})$ to $(\vec{\mathbf{x}}, t)$ is addionally controlled by transition probabilities

$$W_{\boldsymbol{t_\epsilon}} = W_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\pi}}, \vec{\boldsymbol{\pi}}'),$$

resulting in

$$\boxed{f_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, t, \vec{\pi}) = \int_{\vec{\pi}'} W_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\pi}}, \vec{\boldsymbol{\pi}}') f_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}} - t_\varepsilon \vec{\boldsymbol{\pi}}', t - \boldsymbol{t_\epsilon}, \vec{\pi}') d\vec{\boldsymbol{\pi}}'}.$$

Such a relation we call a Markov Process of natural causality. According to Sen [35] there is a so called Newtonian causality in nonrelativistic physics implying the possibility of unlimited velocities. However Newtonian causality is restricted to Newtonian mechanics and stochastic processes of physics ending with diffusion equations when applied practically. [7] This applies not for formulations of the general or linear Boltzmann Equation. In electrodynamics the velocity of light is the limiting velocity. In this treatise one essential statement is: classical physics is generally not Newtonian. Further on

1. is shown, that diffusion equations can only be approximations of an exact description. The diffusion equation is related to an unlimited propagation speed. The diffusion coefficient is correlated with the velocity of sound. Exact descriptions lead via Boltzmannlike formulations.
2. is shown, that the second Newtonian law applies to fluid dynamics in limiting cases only. In field theories as fluid dynamics not force- but accelleration fields are expressed. These are generally not free of rot (equivalently curl) in contrary to a Newtonian force field. That is why it is reasonable to distinguish conservative from non conservative accelleration fields. In classical physics one has normally non conservative fields.(Though for students a contrary impression may occur.)

The Newtonian causality proves to be a limiting case of non relativistic classical physics. Subsequently a causal Markov Process is continuously used or derived. Overarching master equations can not exist, physically. The transition probabilities $W_{\boldsymbol{t_\epsilon}}$ depend on a time quantity $\boldsymbol{t_\epsilon}$ related to continuum fluctuations of measurement accuracy according to vectorial motion quantities. For $\boldsymbol{t_\epsilon} \to \mathbf{0}$ (exact motion quantities) the transition probabillity $W_{\boldsymbol{t_\epsilon}}$ degenerates to a δ-function.

Simultaneous details of space and momentum are not possible in the context of quantum mechanics. The Schrödinger Equation for free particles

$$i\hbar \frac{\partial \psi(\vec{\mathbf{x}}, t)}{\partial t} = -\frac{\hbar^2}{2\mu} \vec{\nabla}^2 \psi(\vec{\mathbf{x}}, t) \tag{4.21}$$

can be transformed into a linear homogenuous integral eqution [16] [19]

[7] This statement applies to the Fokker-Planck and Langevin equation. See, for example, Chandrasekhar[5]

$$\psi(\vec{\mathbf{x}}, t) = i \int G(\vec{\mathbf{x}}, t; \vec{\mathbf{x}'}, t')\psi(\vec{\mathbf{x}'}, t')d\vec{\mathbf{x}'}. \tag{4.22}$$

The Green function

$$G(\vec{\mathbf{x}}, t; \vec{\mathbf{x}}', t') = \left\langle \vec{\mathbf{x}} \left| \boldsymbol{exp}\left(-\frac{\mathbf{i}}{\hbar}(t - t')\mathbf{H} \right) \right| \vec{\mathbf{x}}' \right\rangle \tag{4.23}$$

is called Feynman kernel, too.

In the case of the diffusion equation

$$\frac{\partial \rho(\vec{\mathbf{x}}, t)}{\partial t} = D\vec{\nabla}^2 \rho(\vec{\mathbf{x}}, t) \tag{4.24}$$

an equivalent integral equation the Green function understood as transition probabillity from $(\vec{\mathbf{x}}', t')$ to $(\vec{\mathbf{x}}, t)$ exists with

$$\rho(\vec{\mathbf{x}}, t) = \int_{V'} G(\vec{\mathbf{x}}, t; \vec{\mathbf{x}}', t')\rho(\vec{\mathbf{x}}', t')d\vec{\mathbf{x}}' \tag{4.25}$$

and the Green function

$$G(\vec{\mathbf{x}}, t; \vec{\mathbf{x}}', t') = \left(\frac{1}{4\pi D(t - t')} \right)^{\frac{3}{2}} \boldsymbol{e}^{-\frac{(\vec{\mathbf{x}} - \vec{\mathbf{x}}')^2}{4\pi D(t - t')}} \quad . \tag{4.26}$$

Equations based on a "heat-kernel"-structure are not exact in classical physics (as well as the Newtonian mechanics).

In quantum mechanics and quantum field theory natural causality is not possible because of the uncertainty principle. In Relativity there is the maximal possible velocity, the velocity of light. A geometrodynamic equation system of turbulence found further down does not contain such limiting velocities, explicitly. Velocity fields are calculated uniquely by an initial field giving to GR compatible results after mapping from Einstein Space into a suitable observer-space. Using other initial conditions higher velocities are possible.

Part II.

Stochastically continuous transport of passive scalar particles within the meaning of an ensemble-theory

5. Introduction

$$\boxed{\begin{array}{c} \dfrac{\partial \overline{f}}{\partial t} + \overline{\omega} \cdot \overline{r} \vec{\mathbf{\Omega}} \times \vec{\mathbf{\Theta}} \cdot \nabla \overline{f} = \dfrac{-1}{t_E} \sum_{l=0}^{+\infty} \sum_{m=-l}^{+l} \sum_{k=-\infty}^{+\infty} \dfrac{l(l+1)}{2} \overline{f}_{lmk}(\vec{\mathbf{x}}, t) P_{lm}(\vec{\mathbf{\Omega}}) H_k(\vec{\mathbf{\Theta}}) \\ \Updownarrow \\ \overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, \vec{\mathbf{\Omega}}, \vec{\mathbf{\Theta}}, t) = \displaystyle\int_{2\pi} \int_{4\pi} \widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \vec{\mathbf{\Omega}}, \vec{\mathbf{\Theta}}, \vec{\mathbf{\Omega}}', \vec{\mathbf{\Theta}}') \overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}} - t_\varepsilon \cdot \overline{v}' \vec{\mathbf{\Omega}}' \times \vec{\mathbf{\Theta}}', \vec{\mathbf{\Omega}}', \vec{\mathbf{\Theta}}'), t - t_{\boldsymbol{\epsilon}}) d\vec{\mathbf{\Omega}}' d\vec{\mathbf{\Theta}}' \end{array}}$$

The aim is the derivation of a stochastic transport equation of turbulent, passively moved scalar particles being based on particle balancing. Stochastics is always understood as the randomness of motions. From point $\vec{\mathbf{A}}$ originated a point $\vec{\mathbf{B}}$ is approached in consequence of a random velocity. The particle motions are totally adjusted to the fluid motions of the fluctuating continuum and reproduce single fluid motions in detail.[1] The used stochastics is based for one thing on an ensemble-consideration and on the other hand on a locally formulated motion process. So an equation is achieved owning local coefficients depending on space and time as well as the states of movement. The field of coefficients can be determined in principal in every desired level of detail by the deterministic turbulence theory of chapter 13. The transport equation is a partial differential equation shown to be equivalent to a derived integral equation. The respective stochastic process is immediately recognized as Markovian of natural causality. We call it causal Markov Process.

Chapter 6: Within the framework of kinetic theory a physical situation is selected handling the linear Boltzmann Equation. This equation is extensively studied in nuclear reactor physics called neutron Boltzmann Equation[43].[2] Using the above described ensemble consideration a statistical particle balance is formulated by local velocities and their unsteady changes by local cross sections. The resulting mathematical ties help the developments in further chapters as guideline and answer the question, which analogies exist between kinetic theory and turbulent stochastic continuum transport.

[1] The particles are assumed to have a suitable weight

[2] The insights of this theory had little impact on similar fields of physics.

Chapter 7: The motion of passive scalar particles by longitudinal continuum fluctuations is examined. In the centre of the consideration is the development of transition probability densities of velocities. They depend as well as the velocities and their particle density distributions on the accuracy of a measuring process indexed by t_ϵ. $\lim \boldsymbol{t_\epsilon} \rightarrow 0$ means exact measurements and the transition probabilities result into $\delta-$ functions. They have the property of test functions of the distribution theory with immediate physical meaning. Calculating them a transport equation in form of a partial differential equation as well as an equivalent integral equation is derived.

Chapter 8: In analogy to chapter 7 the motion by turbulent continuum fluctuations of passive particles is examined. The fluctuation directions are expressed by Eulerian angles and the distribution functions are developed by generalized spherical harmonics (we call them turbulence functions). A pair of equations is created consisting of a partial differential equation and an equivalent integral equation as in the cases of molecular self diffusion and the longitudinal (1+3)-dimensional continuum fluctuations. The three physical situations can be compared all the more as in the three cases the transition probabilities are explicitely formulated.

6. Brownian motion as molecular self-diffusion

$$\boxed{\begin{array}{c}\frac{\partial}{\partial t}h+\bar{v}\vec{\boldsymbol{\Omega}}\cdot\nabla h=\frac{1}{\tau}\sum_{l=1}^{+\infty}\boldsymbol{\gamma_l}\cdot\sum_{m=-l}^{+l}h_{lm}(\vec{\mathbf{x}},t)P^*_{lm}(\vec{\boldsymbol{\Omega}})\\ h_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}},\boldsymbol{v_{t_\epsilon}}\vec{\boldsymbol{\Omega}},t)=\int_{4\pi}\int_0^\infty W_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Omega}},v'\vec{\boldsymbol{\Omega}}')h_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}-v'\vec{\boldsymbol{\Omega}}'t_\varepsilon,v'\vec{\boldsymbol{\Omega}}',t-\boldsymbol{t_\epsilon})dv'd\vec{\boldsymbol{\Omega}}'\end{array}}$$

6.1. Introduction

Brownian motion is understood as a disordered thermic motion of molecules in gases or fluids creating a disordered motion of suspended, sufficiently small particles. This Brownian motion is all the more livelier the smaller the particle quantity is. With increasing particle sizes the detailed molecular influence on the particle movement disappears and having suitable sizes the particles reproduce the turbulent fluid motions. The phenomenon of small paricles was first examined by Einstein and Smoluchowski.

Subsequently, the case of very small particles that is the statistical development of the molecular distribution of a gas is evaluated. In the treatises of Einstein[12] and Smoluchowski[36] the considerations lead in each case to a diffusion equation, which contains two fundamental deficiencies, though beeing sufficient for the purpose at that time. The propagation speed concerning a diffusion equation is unlimited. Immediately after switching a point particle source on there is at least an infinitesimal influence in arbitrary distance. In close proximity to a point source the solution of a diffusion equation shows a $\sim \frac{1}{r}$-behaviour. But it should be $\sim \frac{1}{r^2}$.

6.2. Transport equation of molecular self-diffusion

Examining the molecular self-diffusion in a highly diluted gas in thermodynamic equilibrium the linear Boltzmann equation will be derived. It is a linear integro-differential equation statistically describing the transport of diffusing particles by cross sections of the interacting particles. The whole gas medium is regarded as devided into two parts, a main part and an additional very small part. The diffusing of the small part in the main part without changing the statistical properties of the main part is considered. (See section 4.2) Due to the low density of the diffusing molecules a relevant self interaction within the small part can be excluded. Regarding the spatiotemporal development the velocity distribution density $\mathbf{g}(\mathbf{v})$ is normalised to 1.

$$\int_0^\infty \mathbf{g}(\mathbf{v})d\boldsymbol{v} = \mathbf{1} \tag{6.1}$$

The diffusing part is depicted by

$$f(\vec{\mathbf{x}}, \vec{\mathbf{v}}, t) = f(\vec{\mathbf{x}}, t, v\vec{\mathbf{\Omega}}) = h(\vec{\mathbf{x}}, t, \vec{\mathbf{\Omega}})\mathbf{g}(\mathbf{v}) \tag{6.2}$$

I.e. the velocity distribution is independent of space-time $(\vec{\mathbf{x}}, t)$ and direction $(\vec{\mathbf{\Omega}})$ (equipartition theorem).

This allows to talk about an expectation value for every space-time point of the particle density

$$< \Phi(\vec{\mathbf{x}}, t) >= \int_0^\infty \int_{4\pi} f(\vec{\mathbf{x}}, t, \vec{\boldsymbol{v}})d\vec{\mathbf{\Omega}}dv = \int_{4\pi} h(\vec{\mathbf{x}}, t, \vec{\mathbf{\Omega}})d\vec{\mathbf{\Omega}}. \tag{6.3}$$

However the measured value of the density $\Phi(\vec{\mathbf{x}}, t)$ is only a good approximation of the expectation value

$$< \Phi(\vec{\mathbf{x}}, t) > \approx \Phi(\vec{\mathbf{x}}, t). \tag{6.4}$$

This ceases to apply for the particle transport by fluctuating continua.

The total derivative of the distribution function f in direction of the velocity $v\vec{\mathbf{\Omega}}$ results in

$$\frac{d}{dt}f(\vec{\mathbf{x}}, t, v\vec{\mathbf{\Omega}}) = \frac{\partial}{\partial t}f + v\vec{\mathbf{\Omega}}\nabla f. \tag{6.5}$$

The change of the particle density distribution for the velocity $\vec{\mathbf{v}} = v\vec{\mathbf{\Omega}}$ is balanced by collisions of molecules modifying the velocities with a certain probability expressed by differential cross sections. Defining $\frac{1}{v}f(\vec{\mathbf{x}}, \vec{\mathbf{v}}, t)$ as particle stream[1] the following

[1]Hereby no stream in the meaning of deterministic fluid dynamics is defined!

balance equation can be noted

$$\frac{1}{v}\frac{d}{dt}f(\vec{\mathbf{x}},t,v\vec{\mathbf{\Omega}}) = \frac{1}{v}\frac{\partial}{\partial t}f + \vec{\mathbf{\Omega}}\nabla f = I_+ - I_-. \tag{6.6}$$

I_+ corresponds to molecules coming from other directions $\vec{\Omega}'$.
I_- corresponds to molecules leaving direction $\vec{\Omega}$.

The particle distribution density varying in space this expression has to be different from zero. Otherwise the particle distribution density remains constant. So the assumed initial distribution is variously dispersed in space.

The momentum exchange is determined on one side by the cross sections of the impact partners and on the other side by the number of particles arriving at location $\vec{\mathbf{x}}$ and time t per unit area with the velocity $\vec{\mathbf{v}}'$ pivoting into the velocity $\vec{\mathbf{v}}$. This growth of the number of particles per time and unit-area is

$$I_+ = \rho \int_0^\infty \int_{4\pi} \sigma(\vec{\mathbf{v}}\cdot\vec{\mathbf{v}}')\cdot f(\vec{\mathbf{x}},t,v'\vec{\mathbf{\Omega}}')dv'd\vec{\mathbf{\Omega}}' \tag{6.7}$$

with

$$\begin{aligned}\rho &= \text{constant density of the main part of the gas}\\ \sigma(\vec{\mathbf{v}}\cdot\vec{\mathbf{v}}') &= \text{differential cross section, symmetrical in } \vec{\mathbf{v}} \text{ and } \vec{\mathbf{v}}'.\end{aligned}$$

The particles simultaneously changing their velocity $\vec{\mathbf{v}}$ into another $\vec{\mathbf{v}}'$ the appropriate decrease of particle number per time- and area-unit is expressed by

$$I_- = \rho \int_0^\infty \int_{4\pi} \sigma(\vec{\mathbf{v}}\cdot\vec{\mathbf{v}}')\cdot f(\vec{\mathbf{x}},t,v\vec{\mathbf{\Omega}})dv'd\vec{\mathbf{\Omega}}' = \Sigma(v)f(\vec{\mathbf{x}},t,v\vec{\mathbf{\Omega}}) \tag{6.8}$$

due to

$$\Sigma(v) = \rho \int_0^\infty \int_{4\pi} \sigma(\vec{\mathbf{v}}\cdot\vec{\mathbf{v}}')dv'd\vec{\mathbf{\Omega}}'' \quad [\boldsymbol{m^{-1}}]. \tag{6.9}$$

$\Sigma(v)$ represents the total macroscopic cross section and the transport equation results in the molecular self diffusion equation

$$\boxed{\frac{1}{v}\frac{\partial}{\partial t}f + \vec{\mathbf{\Omega}} \cdot \nabla f = \rho \int_0^\infty \int_{4\pi} \sigma(\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}') \cdot f(\vec{\mathbf{x}}, t, v'\vec{\mathbf{\Omega}}')dv'd\vec{\mathbf{\Omega}}' - \Sigma(v) f}. \tag{6.10}$$

Further on, considering the molecules being in statistical balance throughout the whole gas, i.e. the same Boltzmann Distribution $\mathbf{g}(\mathbf{v})$ is existing everywhere, an integration of $\int_0^\infty (6.10) dv$ results in a manageable equation as follows.

Defining

$$\overline{\Sigma} = \int_0^\infty \Sigma(v) g(v) dv \tag{6.11}$$

$$\overline{\sigma}(\vec{\mathbf{\Omega}} \cdot \vec{\mathbf{\Omega}}') = \int_0^\infty \int_0^\infty \sigma(\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}') g(v') dv dv' \tag{6.12}$$

$$\overline{v} = \int_0^\infty v g(v) dv \tag{6.13}$$

gives[2]

$$\frac{\partial}{\partial t}h + \overline{v}\vec{\mathbf{\Omega}} \cdot \nabla h = \overline{v}\rho \int_{4\pi} \overline{\sigma}(\vec{\mathbf{\Omega}} \cdot \vec{\mathbf{\Omega}}') \cdot h(\vec{\mathbf{x}}, t, \vec{\mathbf{\Omega}}')d\vec{\mathbf{\Omega}}' - \overline{v}\overline{\Sigma} \cdot h(\vec{\mathbf{x}}, t, \vec{\mathbf{\Omega}}). \tag{6.14}$$

Developing by spherical harmonics (see appendix 9.2) yield in

$$\overline{\sigma}(\vec{\mathbf{\Omega}} \cdot \vec{\mathbf{\Omega}}') = \sum_{l=0}^{+\infty} \sigma_l P_l(cos(\alpha)) = \sum_{l=0}^{+\infty} \sigma_l \sum_{m=-l}^{m=+l} P_{lm}(\vec{\mathbf{\Omega}}) P^*_{lm}(\vec{\mathbf{\Omega}}') \tag{6.15}$$

and

$$\begin{aligned} h(\vec{\mathbf{x}}, t, \vec{\mathbf{\Omega}}) &= \sum_{l=0}^{+\infty} \sum_{m=-l}^{m=+l} h_{lm}(\mathbf{x}, t) P_{lm}(\vec{\mathbf{\Omega}}) \\ &= \sum_{l=0}^{+\infty} \sum_{m=-l}^{m=+l} h_{lm}(\mathbf{x}, t) P^*_{lm}(\vec{\mathbf{\Omega}}). \end{aligned} \tag{6.16}$$

[2]Such an equation corresponds in nuclear reactor physics to the one group neutron-transport-equation regardless of absorbtions-and fission effects.

These developments inserted into (6.14) and executing the respective integrations lead to

$$\begin{aligned}\frac{\partial}{\partial t}h + \overline{v}\vec{\mathbf{\Omega}}\cdot\nabla h =& \rho\overline{v}\sum_{l=1}^{+\infty}\sigma_l\frac{4\pi}{2l+1}\sum_{m=-l}^{m=+l}h_{lm}P^*_{lm}(\vec{\mathbf{\Omega}}) - \overline{v}\overline{\Sigma}\cdot\sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}h_{lm}(\vec{\mathbf{x}},t)P^*_{lm}(\vec{\mathbf{\Omega}})\\ =& \sum_{l=1}^{+\infty}\overline{v}\{\rho\sigma_l\frac{4\pi}{2l+1} - \overline{\Sigma}\}\cdot\sum_{m=-l}^{+l}h_{lm}(\vec{\mathbf{x}},t)P^*_{lm}(\vec{\mathbf{\Omega}})\end{aligned} \tag{6.17}$$

It holds

$$\overline{\Sigma} = 4\pi\rho\sigma_0 = \rho\int_{4\pi}\overline{\sigma}(\vec{\mathbf{\Omega}}\cdot\vec{\mathbf{\Omega}}')d\vec{\mathbf{\Omega}}' \quad [m^{-1}] \tag{6.18}$$

and defining

$$\begin{aligned}\boldsymbol{\tau^{-1}} &= \overline{v}\overline{\Sigma} \quad [\boldsymbol{sec^{-1}}]\\ \boldsymbol{\gamma_l} &= \left(\frac{\sigma_l}{\sigma_0}\frac{1}{2l+1} - 1\right) < 0 \quad \text{für} \quad l\geq 1 \quad [/]\end{aligned} \tag{6.19}$$

$$\Longrightarrow \boldsymbol{\gamma_0 = 0} \tag{6.20}$$

one gets

$$\boxed{\frac{\partial}{\partial t}h + \overline{v}\vec{\mathbf{\Omega}}\cdot\nabla h = \frac{1}{\tau}\sum_{l=1}^{+\infty}\boldsymbol{\gamma_l}\cdot\sum_{m=-l}^{+l}h_{lm}(\vec{\mathbf{x}},t)P^*_{lm}(\vec{\mathbf{\Omega}})}. \tag{6.21}$$

As $\boldsymbol{\gamma_l} \leq 0$, the development components of order l are the more rapidly decaying the l becoming greater. The total derivation in the direction of the velocity $\overline{v}\vec{\mathbf{\Omega}}$ leads to

$$\frac{d}{dt}h(\vec{\mathbf{x}},t,\vec{\mathbf{\Omega}}) = \sum_{l=0}^{+\infty}\sum_{m=-l}^{m=+l}\frac{d}{dt}h_{lm}(\mathbf{x},t)P_{lm}(\vec{\mathbf{\Omega}}) = \frac{1}{\tau}\sum_{l=1}^{+\infty}\boldsymbol{\gamma_l}\cdot\sum_{m=-l}^{+l}h_{lm}(\vec{\mathbf{x}},t)P_{lm}(\vec{\mathbf{\Omega}}). \tag{6.22}$$

That is why the time behaviour of the single development components result in

$$\frac{d}{dt}h_{lm}(t) = \frac{\gamma_l}{\tau}h_{lm} \tag{6.23}$$

and

$$h_{lm}(t) \sim \mathbf{exp}(\frac{\gamma_l}{\tau} \cdot t). \tag{6.24}$$

So approximations of first order turn out to approach exact solutions, asymptotically.[3]

6.3. Brownian motion as Markov Process with natural causality

Defining the transition probability density of directions

$$\boxed{\overline{W}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{\Omega}} \cdot \vec{\mathbf{\Omega}}') = \sum_{l=0}^{+\infty} \frac{2l+1}{4\pi} e^{+\Upsilon_l \cdot \frac{t_\epsilon}{\tau}} \cdot \sum_{m=-l}^{+l} P^*_{lm}(\vec{\mathbf{\Omega}}) P_{lm}(\vec{\mathbf{\Omega}}')} \tag{6.25}$$

and determining the following relationships

$$\epsilon = \frac{t_\epsilon}{\tau}, \qquad \frac{1}{\tau} = \overline{v} \cdot \overline{\Sigma} = \overline{v} \cdot 4\pi\rho\sigma_0 = const \tag{6.26}$$

$$\overline{\sigma}(\vec{\mathbf{\Omega}} \cdot \vec{\mathbf{\Omega}}') = \sum_{l=0}^{+\infty} \sigma_l P_l(cos(\alpha)) = \sum_{l=0}^{+\infty} \sigma_l \sum_{m=-l}^{m=+l} P_{lm}(\vec{\mathbf{\Omega}}) P^*_{lm}(\vec{\mathbf{\Omega}}') \tag{6.27}$$

$$\Upsilon_l = \left(\frac{\sigma_l}{\sigma_0} \cdot \frac{1}{2l+1} - 1\right) \tag{6.28}$$

an integral equation of self-diffusion results in dependence of directions of motions, cross sections and locally averaged absolute values of velocities as coefficients,

$$\boxed{h_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, t, \vec{\mathbf{\Omega}}) = \int_{4\pi} \overline{W}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{\Omega}} \cdot \vec{\mathbf{\Omega}}') \cdot h_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}} - \overline{v}\vec{\mathbf{\Omega}}' \boldsymbol{t_\epsilon}, t - \boldsymbol{t_\epsilon}, \vec{\mathbf{\Omega}}') d\vec{\mathbf{\Omega}}'} \tag{6.29}$$

from which equation

$$\frac{\partial}{\partial t} h + \overline{v}\vec{\mathbf{\Omega}} \cdot \nabla h = \frac{1}{\tau} \sum_{l=1}^{+\infty} \Upsilon_l \cdot \sum_{m=-l}^{+l} h_{lm}(\vec{\mathbf{x}}, t) P^*_{lm}(\vec{\mathbf{\Omega}}) \tag{6.21}$$

[3] This is correct in the case of lacking absorbtion processes. We are only regarding scattering.

may be reconstructed.

<u>Proof:</u>

$h_{\boldsymbol{t_\epsilon}}$ developed around $\vec{\mathbf{x}}$ and t until first order one gets

$$h_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}} - \overline{v}'_{t_\epsilon}\vec{\boldsymbol{\Omega}}' \cdot t_\varepsilon, \vec{\boldsymbol{\Omega}}', t - t_{\boldsymbol{\epsilon}}) = h_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, \vec{\boldsymbol{\Omega}}', t) - \tau \cdot \epsilon \cdot [\frac{\partial h'_{\boldsymbol{t_\epsilon}}}{\partial t} + \vec{\boldsymbol{v}}'_{t_\epsilon} \cdot \vec{\boldsymbol{\nabla}} h'_{\boldsymbol{t_\epsilon}} + O(\epsilon^2)] \quad (6.30)$$

with $t_\epsilon = \tau \cdot \epsilon$. Inserted into (6.29) this leads to

$$h_{\boldsymbol{t_\epsilon}} = \int_{4\pi} \overline{W}_{\boldsymbol{t_\epsilon}} h'_{\boldsymbol{t_\epsilon}} d\vec{\boldsymbol{\Omega}}' - \int\limits_{4\pi} \overline{W}_{\boldsymbol{t_\epsilon}} \cdot \tau \cdot \epsilon \cdot [\frac{\partial h'_{\boldsymbol{t_\epsilon}}}{\partial t} + \vec{\overline{\boldsymbol{v}}}'_{t_\epsilon} \cdot \vec{\boldsymbol{\nabla}} h'_{\boldsymbol{t_\epsilon}} + O(\epsilon^2)] d\vec{\boldsymbol{\Omega}}' \quad (6.31)$$

and simple conversions give

$$\frac{\int_{4\pi} \overline{W}_{\boldsymbol{t_\epsilon}} h'_{\boldsymbol{t_\epsilon}} d\vec{\boldsymbol{\Omega}}' - h_{\boldsymbol{t_\epsilon}}}{\epsilon} = \int\limits_{4\pi} \overline{W}_{\boldsymbol{t_\epsilon}} \cdot \tau [\cdot \frac{\partial h'_{\boldsymbol{t_\epsilon}}}{\partial t} + \vec{\overline{\boldsymbol{v}}}'_{t_\epsilon} \cdot \vec{\boldsymbol{\nabla}} h'_{\boldsymbol{t_\epsilon}} + O(\epsilon^2)] d\vec{\boldsymbol{\Omega}}'. \quad (6.32)$$

Executing the limiting process $t_\epsilon \to 0$ the transition probability $\overline{W}_{\boldsymbol{t_\epsilon}}$ results in a δ-function and the particle density distribution $h_{\boldsymbol{t_\epsilon}}$ achieves the limiting function h.

$$\begin{aligned}
&\lim_{\boldsymbol{t_\epsilon} \to 0} \overline{W}_{\boldsymbol{t_\epsilon}} = \delta(\vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Omega}}') \quad \text{:delta-Function} \\
&\lim_{\boldsymbol{t_\epsilon} \to 0} h_{\boldsymbol{t_\epsilon}} = h \\
&\lim_{\boldsymbol{t_\epsilon} \to 0} \vec{\overline{\boldsymbol{v}}}_{t_\epsilon} = \vec{\overline{\boldsymbol{v}}} = \overline{\boldsymbol{v}} \cdot \vec{\boldsymbol{\Omega}} \\
&\lim_{\boldsymbol{t_\epsilon} \to 0} \vec{\boldsymbol{\Omega}}_{t_\epsilon} = \vec{\boldsymbol{\Omega}}
\end{aligned} \quad (6.33)$$

Executing the limiting process $\boldsymbol{t_\epsilon} \to \mathbf{0}$ on equation (6.32) the $\boldsymbol{t_\epsilon}$-indexing disappears in accordance with the distribution functions.

Before the limes process is carried out the following integrations lead to

$$\begin{aligned}
&\int_{4\pi} \overline{W}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{\Omega}} \cdot \vec{\mathbf{\Omega}}')h_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, t, \vec{\mathbf{\Omega}}')d\vec{\mathbf{\Omega}}' = \\
&\int_{4\pi} \Big[\sum_{l=0}^{+\infty} \frac{2l+1}{4\pi} e^{+\Upsilon_l \cdot \frac{t_\epsilon}{\tau}} \cdot \sum_{m=-l}^{+l} P^*_{lm}(\vec{\mathbf{\Omega}}) P_{lm}(\vec{\mathbf{\Omega}}')\Big] h_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, t, \vec{\mathbf{\Omega}}')d\vec{\mathbf{\Omega}}' \\
&= \int_{4\pi} \Big[\sum_{l=0}^{+\infty} \frac{2l+1}{4\pi} e^{+\Upsilon_l \cdot \frac{t_\epsilon}{\tau}} \cdot \sum_{m=-l}^{+l} P^*_{lm}(\vec{\mathbf{\Omega}}) P_{lm}(\vec{\mathbf{\Omega}}')\Big] \sum_{l=0}^{+\infty} \sum_{m=-l}^{m=+l} h_{\boldsymbol{t_\epsilon} lm}(\mathbf{x}, t) P^*_{lm}(\vec{\mathbf{\Omega}}')d\vec{\mathbf{\Omega}}' \\
&= \sum_{l=0}^{+\infty} e^{+\Upsilon_l \cdot \frac{t_\epsilon}{\tau}} \sum_{m=-l}^{m=+l} h_{\boldsymbol{t_\epsilon} lm}(\mathbf{x}, t) P^*_{lm}(\vec{\mathbf{\Omega}}).
\end{aligned} \tag{6.34}$$

Thus one gets

$$\begin{aligned}
&\frac{\int_{4\pi} \overline{W}_{\boldsymbol{t_\epsilon}} h'_{\boldsymbol{t_\epsilon}} d\vec{\mathbf{\Omega}}' - h_{\boldsymbol{t_\epsilon}}}{\epsilon \cdot \tau} \\
&= \frac{\int_{4\pi} \overline{W}_{\boldsymbol{t_\epsilon}} h'_{\boldsymbol{t_\epsilon}} d\vec{\mathbf{\Omega}}' - \sum_{l=0}^{+\infty} \sum_{m=-l}^{m=+l} h_{\boldsymbol{t_\epsilon} lm}(\mathbf{x}, t) P^*_{lm}(\vec{\mathbf{\Omega}})}{\epsilon \cdot \tau} \\
&= \frac{\sum_{l=0}^{+\infty} (e^{+\Upsilon_l \cdot \frac{t_\epsilon}{\tau}} - 1) \sum_{m=-l}^{m=+l} h_{\boldsymbol{t_\epsilon} lm}(\mathbf{x}, t) P^*_{lm}(\vec{\mathbf{\Omega}})}{\epsilon \cdot \tau}.
\end{aligned} \tag{6.35}$$

Setting

$$\Upsilon_l = \lim_{\boldsymbol{t_\epsilon} \to 0} \frac{e^{+\Upsilon_l \cdot \frac{t_\epsilon}{\tau}} - 1}{\epsilon}, \qquad t_\epsilon = \epsilon \cdot \tau \tag{6.36}$$

creates

$$\lim_{\boldsymbol{\epsilon} \to 0} \frac{\int_{4\pi} \overline{W}_{\boldsymbol{t_\epsilon}} h'_{\boldsymbol{t_\epsilon}} d\vec{\mathbf{\Omega}}' - h_{\boldsymbol{t_\epsilon}}}{\epsilon \cdot \tau} = \frac{1}{\tau} \sum_{l=1}^{+\infty} \Upsilon_l \cdot \sum_{m=-l}^{+l} h_{lm}(\vec{\mathbf{x}}, t) P^*_{lm}(\vec{\mathbf{\Omega}}) \tag{6.37}$$

and

$$\lim_{\boldsymbol{\epsilon} \to 0} \frac{\int_{4\pi} \overline{W}_{\boldsymbol{t_\epsilon}} h'_{\boldsymbol{t_\epsilon}} d\vec{\mathbf{\Omega}}' - h_{\boldsymbol{t_\epsilon}}}{\epsilon \cdot \tau} = \frac{\partial h}{\partial t} + \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{\nabla}} h, \tag{6.38}$$

which results in

$$\boxed{\frac{\partial h}{\partial t} + \overline{v}\vec{\boldsymbol{\Omega}} \cdot \vec{\boldsymbol{\nabla}} h = \frac{1}{\tau} \sum_{l=1}^{+\infty} \Upsilon_l \cdot \sum_{m=-l}^{+l} h_{lm}(\vec{\mathbf{x}}, t) P^*_{lm}(\vec{\boldsymbol{\Omega}})}. \tag{6.39}$$

q.e.d.

Extending the transition probability density $\overline{W}$ by the velocity distribution $g(v')$

$$W_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Omega}}, v'\vec{\boldsymbol{\Omega}}') = g(v')\overline{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Omega}} \cdot \vec{\boldsymbol{\Omega}}'), \tag{6.40}$$

one gets the Brownian molecular motion under the terms of the described model in the most general form.

$$\boxed{f_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, \boldsymbol{v_{t_\epsilon}}\vec{\boldsymbol{\Omega}}, t) = \int_{4\pi} \int_0^\infty W_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Omega}}, v'\vec{\boldsymbol{\Omega}}') f_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}} - v'\vec{\boldsymbol{\Omega}}' t_\varepsilon, v'\vec{\boldsymbol{\Omega}}', t - t_{\boldsymbol{\epsilon}}) dv' d\vec{\boldsymbol{\Omega}}'} \tag{6.41}$$

The transition probabilities are not symmetric in contrary to the differential cross section!

6.4. Approximation formula

An approximation formula of 1. order of the equation

$$\frac{\partial}{\partial t} h + \overline{v}\vec{\boldsymbol{\Omega}} \cdot \nabla h = \frac{1}{\tau} \sum_{l=1}^{\infty} \boldsymbol{\gamma_l} \cdot \sum_{m=-l}^{+l} h_{1m}(\vec{\mathbf{x}}, t) P_{1m}(\vec{\boldsymbol{\Omega}}) \tag{6.42}$$

is beeing looked for. The approach accounts for the methods of the transport theory of nuclear reactor physics [43]. In cartesian coordinates this leads to

$$\frac{\partial}{\partial t} h + \overline{v} \cdot \left(\boldsymbol{\Omega}_x \frac{\partial}{\partial x} h + \boldsymbol{\Omega}_y \frac{\partial}{\partial y} h + \boldsymbol{\Omega}_z \frac{\partial}{\partial z} h \right) = \frac{1}{\tau} \sum_{l=1}^{\infty} \boldsymbol{\gamma_l} \cdot \sum_{m=-l}^{+l} h_{1m}(\vec{\mathbf{x}}, t) P_{1m}(\vec{\boldsymbol{\Omega}}). \tag{6.43}$$

$$\begin{aligned}
\mathbf{\Omega}_x &= sin(\vartheta)cos(\varphi) \\
\mathbf{\Omega_y} &= sin(\vartheta)sin(\varphi) \\
\mathbf{\Omega_z} &= cos(\vartheta)
\end{aligned} \tag{6.44}$$

The spherical harmonics of 0th and 1st Order are

$$\begin{aligned}
&P_{00} = 1 \qquad P_{1-1} = 2^{-\frac{1}{2}} e^{-i\varphi} sin\vartheta \qquad P_{10} = cos\vartheta \qquad P_{11} = -2^{-\frac{1}{2}} e^{i\varphi} sin\vartheta \\
&P^*_{00} = 1 \qquad P^*_{1-1} = 2^{-\frac{1}{2}} e^{+i\varphi} sin\vartheta \qquad P^*_{10} = cos\vartheta \qquad P^*_{11} = -2^{-\frac{1}{2}} e^{-i\varphi} sin\vartheta.
\end{aligned} \tag{6.45}$$

In cartesian coordinates until 1st order this leads to

$$\frac{\partial}{\partial t} h + \overline{v} \cdot \left(\mathbf{\Omega}_x \frac{\partial}{\partial x} h + \mathbf{\Omega}_y \frac{\partial}{\partial y} h + \mathbf{\Omega}_z \frac{\partial}{\partial z} h \right) = \frac{1}{\tau} \boldsymbol{\gamma_1} \cdot \sum_{m=-1}^{+1} h_{1m}(\vec{\mathbf{x}}, t) P_{1m}(\vec{\mathbf{\Omega}}) \tag{6.46}$$

The direction vectors in cartesian coordinates expressed by spherical harmonics are written

$$\begin{aligned}
\mathbf{\Omega}_x &= 2^{-\frac{1}{2}} [P_{1-1} - P_{11}] \\
\mathbf{\Omega_y} &= -i 2^{-\frac{1}{2}} [P_{1-1} + P_{11}] \\
\mathbf{\Omega_z} &= P_{10}.
\end{aligned} \tag{6.47}$$

The transport equation in 1st approximation is reduced to

$$\begin{aligned}
&\frac{\partial}{\partial t}(h_{00}P_{00} + h_{1-1}P_{1-1} + h_{10}P_{10} + h_{11}P_{11}) \\
&+ \overline{v} \Big[\cdot 2^{-\frac{1}{2}} [P_{1-1} - P_{11}] \frac{\partial}{\partial x}(h_{00}P_{00} + h_{1-1}P_{1-1} + h_{10}P_{10} + h_{11}P_{11}) \\
&- i 2^{-\frac{1}{2}} [P_{1-1} + P_{11}] \frac{\partial}{\partial y}(h_{00}P_{00} + h_{1-1}P_{1-1} + h_{10}P_{10} + h_{11}P_{11}) \\
&+ P_{10} \frac{\partial}{\partial z}(h_{00}P_{00} + h_{1-1}P_{1-1} + h_{10}P_{10} + h_{11}P_{11}) \Big] \\
&= \frac{1}{\tau} \boldsymbol{\gamma_1} \cdot (h_{1-10}P_{1-1} + h_{10}P_{10} + h_{11}P_{11}).
\end{aligned} \tag{6.48}$$

After integrating $\int(6.48 P^*_{lm}(\vec{\mathbf{\Omega}}) d\vec{\mathbf{\Omega}}$ for $l = 0, 1$ the evolution equation set until 1st order

$$\frac{\partial h_{00}}{\partial t}+\frac{\overline{v}}{3}\left[2^{-\frac{1}{2}}\left(-\frac{\partial h_{11}}{\partial x}+\frac{\partial h_{1-1}}{\partial x}\right)-i2^{-\frac{1}{2}}\left(\frac{\partial h_{11}}{\partial y}+\frac{\partial h_{1-1}}{\partial y}\right)+\frac{\partial h_{10}}{\partial z}\right]=0 \tag{6.49}$$

$$\frac{\partial h_{10}}{\partial t}+\overline{v}\frac{\partial h_{00}}{\partial z}-\frac{\Upsilon_1}{\tau}h_{10}=0 \tag{6.50}$$

$$\frac{\partial h_{1-1}}{\partial t}+\overline{v}2^{-\frac{1}{2}}\left(\frac{\partial h_{00}}{\partial x}+i\frac{\partial h_{00}}{\partial y}\right)-\frac{\Upsilon_1}{\tau}h_{1-1}=0 \tag{6.51}$$

$$\frac{\partial h_{11}}{\partial t}+\overline{v}2^{-\frac{1}{2}}\left(-\frac{\partial h_{00}}{\partial x}+i\frac{\partial h_{00}}{\partial y}\right)-\frac{\Upsilon_1}{\tau}h_{11}=0 \tag{6.52}$$

is approached.

Now we define a vector field $\vec{\boldsymbol{J}}$.

$$\begin{aligned}
J_x =& \frac{4\pi}{3}2^{-\frac{1}{2}}h(_{1-1}-h_{11})\\
J_y =& -i\frac{4\pi}{3}2^{-\frac{1}{2}}(h_{1-1}+h_{11})\\
J_z =& \frac{4\pi}{3}h_{10}\\
\boldsymbol{\Phi} =& 4\pi h_{00}
\end{aligned} \tag{6.53}$$

Insertion (6.53) into (6.49) gives

$$\frac{\partial\boldsymbol{\Phi}}{\partial t}+\overline{v}\cdot\vec{\nabla}\cdot\vec{\boldsymbol{J}}=0. \tag{6.54}$$

$\boldsymbol{\Phi}$ is the particle density of an in a thought experiment assumed small part of the molecular set. Inserting (6.53) into (6.50) until (6.52) leads to

$$\vec{\boldsymbol{J}}=\frac{\tau}{\Upsilon_1}\left[\frac{\overline{v}}{3}\vec{\nabla}\Phi+\frac{\partial\vec{\boldsymbol{J}}}{\partial t}\right] \tag{6.55}$$

with $\Upsilon_1=\left(\frac{1}{3}\frac{\sigma_1}{\sigma_0}-1\right)=-\eta$

So a telegrapher's equation arises[4]

$$\frac{\tau}{\eta}\frac{\partial^2\Phi}{\partial t^2}+\frac{\partial\Phi}{\partial t}=\frac{\tau}{\eta}\overline{v}\vec{\nabla}\cdot\frac{\overline{v}}{3}\vec{\nabla}\Phi. \tag{6.56}$$

[4]To derive telegrapher's equation relativistic considerations are not necessary as is stated in [10]. The propagation speed is closely connected with the speed of sound.

As η, τ and $\overline{v}$ represent constants, the telegrapher's equation is written

$$\boxed{\begin{array}{l} \frac{\tau}{\eta}\frac{\partial^2\Phi}{\partial t^2}+\frac{\partial\Phi}{\partial t} = \boldsymbol{D}\Delta\boldsymbol{\Phi} \\ \textit{with} \qquad \boldsymbol{D} = \frac{\tau}{\eta}\frac{\overline{v}^2}{3} \end{array}} \tag{6.57}$$

D=diffusion coefficient, η dimensionless, $\tau = (\overline{v}\cdot\overline{\Sigma})^{-1}$= mean free collision time
$\overline{v}$=mean amount of velocity
Compared to the 1st derivation the term with temporal derivation of 2nd order can normally be neglected.

The dependence of the diffusion coefficient from macroscopic state variables of an ideal gas may happen as follows:

The equation of state of the ideal gas becomes

$$\boldsymbol{p} = \boldsymbol{\rho R T}. \tag{6.58}$$

The mean quadratic velocity of a Maxwellian velocity distribution of particles with mass m is [9]

$$\overline{\boldsymbol{v^2}} = \frac{3kT}{m} \tag{6.59}$$

$\Longrightarrow$

$$\overline{\boldsymbol{v}} = \sqrt{\frac{8}{\pi}\frac{kT}{m}} = \sqrt{\frac{8}{\pi}\frac{p}{\rho}} \tag{6.60}$$

m means the mass of a molecule.
k is the Boltzmann constant.

To get a comparison with the speed of sound at a Gaussian velocity distribution

$$\boldsymbol{c} = \sqrt{\left.\frac{\partial\boldsymbol{p}}{\partial\boldsymbol{\rho}}\right|_{\boldsymbol{T}}} \tag{6.61}$$

one obtains

$$\overline{\boldsymbol{v}} = \sqrt{\frac{8}{\pi}}\boldsymbol{c} \tag{6.62}$$

and

$$\boxed{\boldsymbol{D} = \frac{1}{\eta\overline{\Sigma}}\frac{8}{3\pi}\boldsymbol{c^2}} \;. \tag{6.63}$$

The propagation speed for Brownian molecular motion is $\overline{\boldsymbol{v}}$. This in particular becomes apparent by equation (6.29). In connection with the diffusion approximation an unlimited propagation speed is assigned. This leads to solutions approaching asymptotically to those of the linear Boltzmann Equation. In close proximity to point sources (less than 3 average free lengths afar[5]) one obtains the following characteristics of the exact and the diffusional solution.

$$\Phi \sim \frac{1}{r^2} \quad \text{solution of the transport equation in the proximity of a point source} \tag{6.64}$$

Anticipating this result from an exact theory appears directly plausible.

$$\Phi \sim \frac{1}{r} \quad \text{solution of the diffusion approximation in the proximity of a point source} \tag{6.65}$$

Avoiding such deficiencies it is neccessary to take a stochastic velocity distribution into account as root of the diffusion process. Analyzing turbulent particle transport this does not satisfy.

6.5. Appendix: equations for the spherical hamonics components

The general equations arise out of

$$\int (6.43) P^*_{lm}(\vec{\boldsymbol{\Omega}}) d\vec{\boldsymbol{\Omega}} \tag{6.66}$$

$\Longrightarrow$

[5]An experience of the neutron transport theory [43]

$$
\begin{aligned}
\frac{\partial h_{lm}(\vec{\mathbf{x}},t)}{\partial t} = -\overline{v}\Bigg[&\frac{\sqrt{(l+2+m)(l+1+m)}}{2l+3}\left(-\frac{1}{2}\frac{\partial h_{l+1,m+1}}{\partial x}-\frac{i}{2}\frac{\partial h_{l+1,m+1}}{\partial y}\right)\\
&+\frac{\sqrt{(l+1-m)(l+2-m)}}{2l+3}\left(\frac{1}{2}\frac{\partial h_{l+1,m-1}}{\partial x}-\frac{i}{2}\frac{\partial h_{l+1,m-1}}{\partial y}\right)\\
&+\frac{\sqrt{(l-1-m)(l-m)}}{2l-1}\left(\frac{1}{2}\frac{\partial h_{l-1,m+1}}{\partial x}+\frac{i}{2}\frac{\partial h_{l-1,m+1}}{\partial y}\right)\\
&+\frac{\sqrt{(l+m)(l+m-1)}}{2l-1}\left(-\frac{1}{2}\frac{\partial h_{l-1,m-1}}{\partial x}+\frac{i}{2}\frac{\partial h_{l-1,m-1}}{\partial y}\right)\\
&+\frac{\sqrt{(l+1+m)(l-m+1)}}{2l+3}\frac{\partial h_{l+1,m}}{\partial z}+\frac{\sqrt{(l+m)(l-m)}}{2l-1}\frac{\partial h_{l-1,m}}{\partial z}\Bigg]-\frac{\Upsilon_l}{\tau}h_{lm}
\end{aligned}
\tag{6.67}
$$

7. Stochastic transport by longitudinal fluctuations of a continuum

$$\boxed{\begin{gathered}\overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}},\vec{\boldsymbol{\Omega}},t)=\int_{4\pi}\widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}},t,\vec{\boldsymbol{\Omega}},\vec{\boldsymbol{\Omega}}')\overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}-t_\varepsilon\cdot\overline{v}'\vec{\boldsymbol{\Omega}}',\vec{\boldsymbol{\Omega}}',t-t_{\boldsymbol{\epsilon}})d\vec{\boldsymbol{\Omega}}'\\ \Updownarrow\\ \frac{\partial\overline{f}}{\partial t}+\overline{v}\vec{\boldsymbol{\Omega}}\cdot\vec{\boldsymbol{\nabla}}\overline{f}=-\frac{1}{t_E}\sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}\frac{l(l+1)}{2}\overline{f}_{l,m}(\vec{\mathbf{x}},t)P_{l,m}(\vec{\boldsymbol{\Omega}})\end{gathered}}$$

7.1. Introduction

The motion of passive particles by longitudinal continuum fluctuations is examined. The particles are moved in this field without interaction.[1] In accordance with section 4.3 they perform detailed motions of single fluid elements of fluid continua. The considered velocities of the particles are determined by measure processes. The particles coming from point x_1 and moving further for a time t_ε are detected in x_2. So the velocity $\vec{\mathbf{v}}_{t_\epsilon}$ may be assigned to

$$\vec{\mathbf{v}}_{t_\epsilon}=\frac{\vec{\mathbf{x}}_2-\vec{\mathbf{x}}_1}{t_\epsilon}=v_{t_\epsilon}\vec{\boldsymbol{\Omega}}_{t_\epsilon}. \tag{7.1}$$

This corresponds to $\boldsymbol{(\vec{\mathbf{x}}_1,t)\longrightarrow(\vec{\mathbf{x}}_2,t+t_\varepsilon)=(\vec{\mathbf{x}}_1+\vec{\mathbf{v}}_{t_\epsilon}\cdot t_\varepsilon,t+t_\varepsilon)}$.

According to an ensemble consideration (see chapter 4) for every point $(\vec{\boldsymbol{x}},t)$ a continuously differentiable particle density distribution of velocities $\vec{\mathbf{v}}_{t_\epsilon}$ is assigned in accordance with

$$f_{\boldsymbol{t_\epsilon}}=f_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}},\vec{\mathbf{v}}_{t_\epsilon},t). \tag{7.2}$$

[1]Such conditions generally lead to linear equations.

The functions indexed with $\boldsymbol{t_\epsilon}$ or $\boldsymbol{\epsilon}$ enclose motion quantities $\vec{\mathbf{v}}_{t_\epsilon}$ or their motion directions $\vec{\boldsymbol{\Omega}}_{\boldsymbol{t_\epsilon}}$ as variables subordinated to an understanding of measurement accuracy. The indexing of motion quantities with $\boldsymbol{t_\epsilon}$ or $\boldsymbol{\epsilon}$ may be dropped if their functions are indexed. Executing a limiting process, for instance

$$\lim_{\boldsymbol{t_\epsilon}\to 0} f_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, \vec{\mathbf{v}}, t) = f(\vec{\mathbf{x}}, \vec{\mathbf{v}}, t) \tag{7.3}$$

f and $\vec{\mathbf{v}}$ are literally understood as results of exact measurement processes. [2] Integrating the particle density distribution over the velocity one obtains an expectation value of a particle density not generally coinciding with the actually measured value ρ.

$$< \boldsymbol{\rho_{t_\epsilon}(\vec{\mathbf{x}}, t)} >= \int_{4\pi}\int_0^\infty f_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, v\vec{\boldsymbol{\Omega}}, t)dvd\vec{\boldsymbol{\Omega}} \neq \boldsymbol{\rho_{t_\epsilon}}(\vec{\mathbf{x}}, t) \tag{7.4}$$

This is contradicting the molecular self-diffusion beeing an inherent stochastic process.
It results into a rigorously derived partial differential equation calculating particle density distributions in dependence on space-time and motion directions. The initially unlimited number of unknown coefficients is reduced to one, a local time-scaling. The initially abstractly formulated transition probabilities obtain their precise functional dependencies alternativly generating an integral equation. For numerical solutions there are always suitable Monte-Carlo methods possible.

Equations of 1st approximation substantially differ from usual diffusion equations.

7.2. Transport by Markov Processes with natural causality

The probability particles at location $\vec{x}$ and time t changing their velocity from $\vec{\boldsymbol{v}}'_{t_\epsilon} = v'\vec{\boldsymbol{\Omega}}'$ to $\vec{\boldsymbol{v}}_{t_\epsilon} = v\vec{\boldsymbol{\Omega}}$ is given by the transition probability

$$W_{\boldsymbol{t_\epsilon}} = W_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, t, v\vec{\boldsymbol{\Omega}}, v'\vec{\boldsymbol{\Omega}}') \tag{7.5}$$

with

$$\int_{4\pi}\int_0^\infty W_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, t, v\vec{\boldsymbol{\Omega}}, v'\vec{\boldsymbol{\Omega}}')dv'd\vec{\boldsymbol{\Omega}}' = 1. \tag{7.6}$$

[2] With the indexing $\boldsymbol{t_\epsilon}$ an assigned measurement process is always understood according to accuracy.

So the following Markov Process is defined by

$$\boxed{f_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, v\vec{\mathbf{\Omega}}, t) = \int_{4\pi}\int_0^\infty W_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, t, v\vec{\mathbf{\Omega}}, v'\vec{\mathbf{\Omega}}') f_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}} - t_\varepsilon \cdot v'\vec{\mathbf{\Omega}}', v'\vec{\mathbf{\Omega}}', t - \boldsymbol{t_\epsilon}) dv' d\vec{\mathbf{\Omega}}'} \tag{7.7}$$

$$\Delta\vec{\mathbf{x}} = \vec{\mathbf{x}} - \vec{\mathbf{x_1}} = v'_{t_\epsilon} \vec{\mathbf{\Omega}}'_{t_\epsilon} \cdot \boldsymbol{t_\epsilon} \tag{7.8}$$

For the transition probability $W_{\boldsymbol{t_\epsilon}}$ merely steadiness is reqired regarding all variables. The sequence of the velocities $\vec{\boldsymbol{v}}'_{t_\epsilon}, \vec{\boldsymbol{v}}_{t_\epsilon}$ means a motion

$$(\vec{\boldsymbol{x}} - \vec{\boldsymbol{v}}'_{t_\epsilon} \cdot \boldsymbol{t_\epsilon}, t - t_\epsilon, \vec{\boldsymbol{v}}'_{t_\epsilon}) \longrightarrow (\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{v}}_{t_\epsilon}). \tag{7.9}$$

At the process $t_\varepsilon \to 0$ the transition probabilities W_{t_ϵ} prove to be physical realisations of test functions of distribution theory.
The passive particles have to reproduce the motions of the fluctuation field, exactly. For the particle density distribution $f_{t_\varepsilon}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{v}})$ a separation approach is formulated without restriction of generality:

$$\begin{aligned} f_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}} - v\vec{\mathbf{\Omega}} \cdot t_\varepsilon, v\vec{\mathbf{\Omega}}, t) =& G_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}} - v\vec{\mathbf{\Omega}} \cdot t_\varepsilon, v\vec{\mathbf{\Omega}}, t)\overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}} - \overline{v}\vec{\mathbf{\Omega}} \cdot t_\varepsilon, \vec{\mathbf{\Omega}}, t) \\ &\int_0^\infty G_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, v\vec{\mathbf{\Omega}}, t) dv = \mathbf{1} \end{aligned} \tag{7.10}$$

$\Longrightarrow$

$$\overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}} - \overline{v}\vec{\mathbf{\Omega}} \cdot t_\varepsilon, \vec{\mathbf{\Omega}}, t) = \int_0^\infty f_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}} - v\vec{\mathbf{\Omega}} \cdot t_\varepsilon, v\vec{\mathbf{\Omega}}, t) dv \tag{7.11}$$

This results in

$$\overline{\mathbf{v}} = \overline{\mathbf{v}}(\vec{\mathbf{x}}, t, \vec{\mathbf{\Omega}}) = \int_0^\infty G_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, v\vec{\mathbf{\Omega}}, t) \cdot \mathbf{v} dv. \tag{7.12}$$

I.e. $\overline{\mathbf{v}}$ is dependent on $(\vec{\mathbf{x}}, t, \vec{\mathbf{\Omega}})$.

A transition probability only in dependence on the directions and space-time $\overline{W}_{\boldsymbol{t_\epsilon}}$ is obtained by integration of $W_{\boldsymbol{t_\epsilon}}$ over the velocity amounts v'_{t_ϵ} and v_{t_ϵ}

$$\boxed{\overline{W}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}},\vec{\boldsymbol{\Omega}},\vec{\boldsymbol{\Omega}}',t) = \int\limits_0^\infty\int\limits_0^\infty W_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}},t,v\vec{\boldsymbol{\Omega}},v'\vec{\boldsymbol{\Omega}}')G_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}-v'\vec{\boldsymbol{\Omega}}'\cdot t_\varepsilon,v'\vec{\boldsymbol{\Omega}}',t-t_{\boldsymbol{\epsilon}})dv'dv}. \tag{7.13}$$

Now an integration of $\int_0^\infty(7.7)dv$ is leading to

$$\overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}},\vec{\boldsymbol{\Omega}},t) = \int\limits_0^\infty\int\limits_0^\infty\int\limits_{4\pi} W_{\boldsymbol{t_\epsilon}}G_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}-v'\vec{\boldsymbol{\Omega}}'\cdot t_\varepsilon,v'\vec{\boldsymbol{\Omega}}',t-t_{\boldsymbol{\epsilon}})\overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}-\overline{v}'\vec{\boldsymbol{\Omega}}'\cdot t_\varepsilon,\vec{\boldsymbol{\Omega}}',t-t_{\boldsymbol{\epsilon}})dv'dvd\vec{\boldsymbol{\Omega}}' \tag{7.14}$$

respectively

$$\boxed{\overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}},\vec{\boldsymbol{\Omega}},t) = \int\limits_{4\pi}\overline{W}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}},t,\vec{\boldsymbol{\Omega}},\vec{\boldsymbol{\Omega}}')\overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}-\overline{v}'\vec{\boldsymbol{\Omega}}'\cdot t_\varepsilon,\vec{\boldsymbol{\Omega}}',t-t_{\boldsymbol{\epsilon}})d\vec{\boldsymbol{\Omega}}'}. \tag{7.15}$$

$\overline{f}_{\boldsymbol{t_\epsilon}}$ in the integrand is developed about $\vec{\mathbf{x}}$ and t until 1st order and one obtains

$$\begin{aligned}\overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}-\overline{v}'\vec{\boldsymbol{\Omega}}'\cdot t_\varepsilon,\vec{\boldsymbol{\Omega}}',t-t_{\boldsymbol{\epsilon}}) &= \overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}},\vec{\boldsymbol{\Omega}}',t)-\tau_E\cdot\epsilon\cdot[\frac{\partial\overline{f}'_{\boldsymbol{t_\epsilon}}}{\partial t}+\overline{v}'_{\boldsymbol{t_\epsilon}}\vec{\boldsymbol{\Omega}}'_{\boldsymbol{t_\epsilon}}\cdot\vec{\boldsymbol{\nabla}}\overline{f}'_{\boldsymbol{t_\epsilon}}+O(\epsilon^2)]\\ \overline{f}'_{\boldsymbol{t_\epsilon}} &= \overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}},\vec{\boldsymbol{\Omega}}',t)\end{aligned} \tag{7.16}$$

with $t_\epsilon = \tau_E\cdot\epsilon$ and $\tau_E = const.$ Inserted in (7.15) this leads to

$$\overline{f}_{\boldsymbol{t_\epsilon}} = \int_{4\pi}\overline{W}_{\boldsymbol{t_\epsilon}}\overline{f}'_{\boldsymbol{t_\epsilon}}d\vec{\boldsymbol{\Omega}}' - \int\limits_{4\pi}\overline{W}_{\boldsymbol{t_\epsilon}}\cdot\tau_E\cdot\epsilon\cdot[\frac{\partial\overline{f}'_{\boldsymbol{t_\epsilon}}}{\partial t}+\vec{\overline{\boldsymbol{v}}}'_{\boldsymbol{t_\epsilon}}\cdot\vec{\boldsymbol{\nabla}}\overline{f}'_{\boldsymbol{t_\epsilon}}+O(\epsilon^2)]d\vec{\boldsymbol{\Omega}}'. \tag{7.17}$$

and simple conversions give

$$\frac{\int_{4\pi} \overline{W}_{\boldsymbol{t_\epsilon}} \overline{f}'_{\boldsymbol{t_\epsilon}} d\vec{\boldsymbol{\Omega}}' - \overline{f}_{\boldsymbol{t_\epsilon}}}{\epsilon} = \int_{4\pi} \overline{W}_{\boldsymbol{t_\epsilon}} \cdot \tau_E [\cdot \frac{\partial \overline{f}'_{\boldsymbol{t_\epsilon}}}{\partial t} + \vec{\overline{\boldsymbol{v}}}'_{t_\epsilon} \cdot \vec{\boldsymbol{\nabla}} \overline{f}'_{\boldsymbol{t_\epsilon}} + O(\epsilon)] d\vec{\boldsymbol{\Omega}}'. \tag{7.18}$$

The process $t_\epsilon \to 0$ applied to the transition probability $\overline{W}_{\boldsymbol{t_\epsilon}}$ ceates a δ-function and the particle density distribution $\overline{f}_{\boldsymbol{t_\epsilon}}$ results in $\overline{f}$.

$$\begin{aligned}
&\lim_{\boldsymbol{t_\epsilon} \to 0} \overline{W}_{\boldsymbol{t_\epsilon}} = \delta(\vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Omega}}') \quad \text{:delta-Function} \\
&\lim_{\boldsymbol{t_\epsilon} \to 0} \overline{f}_{\boldsymbol{t_\epsilon}} = \overline{f} \\
&\lim_{\boldsymbol{t_\epsilon} \to 0} \vec{\overline{\boldsymbol{v}}}_{t_\epsilon} = \vec{\overline{\boldsymbol{v}}} = \overline{\boldsymbol{v}} \cdot \vec{\boldsymbol{\Omega}} \\
&\lim_{\boldsymbol{t_\epsilon} \to 0} \vec{\boldsymbol{\Omega}}_{t_\epsilon} = \vec{\boldsymbol{\Omega}}
\end{aligned} \tag{7.19}$$

These relations applied to equation (7.18) give

$$\boxed{\lim_{\boldsymbol{\epsilon} \to 0} \frac{\int_{4\pi} \overline{W}_{\boldsymbol{t_\epsilon}} \overline{f}'_{\boldsymbol{t_\epsilon}} d\vec{\boldsymbol{\Omega}}' - \overline{f}_{\boldsymbol{t_\epsilon}}}{\epsilon \cdot \boldsymbol{\tau_E}} = \frac{\partial \overline{f}}{\partial t} + \vec{\overline{\boldsymbol{v}}} \cdot \vec{\boldsymbol{\nabla}} \overline{f}} \tag{7.20}$$

and

$$\lim_{\boldsymbol{\epsilon} \to 0} \frac{\int_{4\pi} \overline{W}_{\boldsymbol{t_\epsilon}} \overline{f}'_{\boldsymbol{t_\epsilon}} d\vec{\boldsymbol{\Omega}}' - \overline{f}_{\boldsymbol{t_\epsilon}}}{\epsilon \cdot \boldsymbol{\tau_E}} \tag{7.21}$$

subsequently called exchange-term.

7.3. Calculation of the exchange-term

The dependencies of the transition probability $\overline{W}_{\boldsymbol{t_\epsilon}}$ on the initially uncorrelated movement directions $\vec{\boldsymbol{\Omega}}$ and $\vec{\boldsymbol{\Omega}}'$ may be expressed by the scalar product of the movement directions $\vec{\boldsymbol{\Omega}} \cdot \vec{\boldsymbol{\Omega}}'$ and the simultaneous interchange of the constant time scaling $\boldsymbol{\tau_E}$ by a time scaling depending on location, time and direction $\boldsymbol{t_E}(\vec{\mathbf{x}}, \vec{\boldsymbol{\Omega}}, t)$, i.e.

$$\begin{aligned}
\lim_{\boldsymbol{\epsilon} \to 0} \frac{\int_{4\pi} \overline{W}_{\boldsymbol{t_\epsilon}} \overline{f}'_{\boldsymbol{t_\epsilon}} d\vec{\boldsymbol{\Omega}}' - \overline{f}_{\boldsymbol{t_\epsilon}}}{\epsilon \cdot \boldsymbol{\tau_E}} &= \lim_{\boldsymbol{\epsilon} \to 0} \frac{\int_{4\pi} \widetilde{W}_{\boldsymbol{t_\epsilon}} (\vec{\boldsymbol{\Omega}} \cdot \vec{\boldsymbol{\Omega}}') \overline{f}'_{\boldsymbol{t_\epsilon}} d\vec{\boldsymbol{\Omega}}' - \overline{f}_{\boldsymbol{t_\epsilon}}}{\epsilon \cdot \boldsymbol{t_E}(\vec{\mathbf{x}}, \vec{\boldsymbol{\Omega}}, t)} \\
\overline{W}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Omega}}') &\longrightarrow \widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Omega}} \cdot \vec{\boldsymbol{\Omega}}') \\
\boldsymbol{\tau_E} &\longrightarrow \boldsymbol{t_E}(\vec{\mathbf{x}}, \vec{\boldsymbol{\Omega}}, t).
\end{aligned} \tag{7.22}$$

The direction distribution of the particles is developed by complex spherical harmonics P_{lm}, the transition probability by legendre polynomials P_l.

$$\overline{f}(\vec{\mathbf{x}},\vec{\mathbf{\Omega}},t)=\sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}\overline{f}_{lm}(\vec{\mathbf{x}},t)P_{lm}(\vec{\mathbf{\Omega}})=\sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}\overline{f}_{lm}(\vec{\mathbf{x}},t)P^*_{lm}(\vec{\mathbf{\Omega}}) \tag{7.23}$$

$$\widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{\Omega}}'\cdot\vec{\mathbf{\Omega}})=\sum_{l=0}^{+\infty}\widetilde{W}_{\boldsymbol{t_\epsilon}l}P_l(cos(\alpha))=\sum_{l=0}^{+\infty}\sum_{m=-l}^{m=+l}\widetilde{W}_{\boldsymbol{t_\epsilon}l}P_{lm}(\vec{\mathbf{\Omega}}')P^*_{lm}(\vec{\mathbf{\Omega}}) \tag{7.24}$$

The spherical harmonics P_{lm} are

$$P_{lm}(\vec{\mathbf{\Omega}})=e^{im\varphi}\frac{(-sin(\vartheta))^m}{l!2^l}\cdot\left(\frac{(l-m)!}{(l+m)!}\right)^{\frac{1}{2}}\frac{d^{l+m}(cos^2\vartheta-1)^l}{(dcos\vartheta)^{l+m}} \tag{7.25}$$

The normalisation holds:

$$\int_{4\pi}P_{lm}P^*_{lm}d\vec{\mathbf{\Omega}}=\begin{cases}\frac{4\pi}{2l+1} & \text{l = l' and m=m'}\\ 0 & \text{else}\end{cases} \tag{7.26}$$

There is the relation between spherical harmonics P_{lm} and Legendre polynomials P_l:

$$P_l(\vec{\mathbf{\Omega}}'\cdot\vec{\mathbf{\Omega}})=P_l(cos(\alpha))=\sum_{m=-l}^{m=+l}P_{lm}(\vec{\mathbf{\Omega}}')P^*_{lm}(\vec{\mathbf{\Omega}}) \tag{7.27}$$

Thus one has

$$\begin{aligned}\int_{4\pi}\widetilde{W}_{\boldsymbol{t_\epsilon}}\overline{f}'_{\boldsymbol{\epsilon}}d\vec{\mathbf{\Omega}}'&=\int_{4\pi}\sum_{l=0}^{+\infty}\sum_{m=-l}^{m=+l}\widetilde{W}_{\boldsymbol{t_\epsilon}l}P_{lm}(\vec{\mathbf{\Omega}}')P^*_{lm}(\vec{\mathbf{\Omega}})\cdot\sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}\overline{f}_{\boldsymbol{t_\epsilon}lm}(\vec{\mathbf{x}},t)P^*_{lm}(\vec{\mathbf{\Omega}}')d\vec{\mathbf{\Omega}}'\\&=\sum_{l=0}^{+\infty}\widetilde{W}_{\boldsymbol{t_\epsilon}l}\frac{4\pi}{2l+1}\sum_{m=-l}^{+l}P^*_{lm}(\vec{\mathbf{\Omega}})\overline{f}_{\boldsymbol{t_\epsilon}lm}(\vec{\mathbf{x}},t).\end{aligned} \tag{7.28}$$

The left side of equation (7.20) results in

$$\begin{aligned}\lim_{\epsilon\to 0}\frac{\int_{4\pi}\widetilde{W}_{\boldsymbol{t_\epsilon}}\overline{f}'_{\boldsymbol{t_\epsilon}}d\vec{\boldsymbol{\Omega}}'-\overline{f}_{\boldsymbol{t_\epsilon}}}{t_E\cdot\epsilon} &= \lim_{\epsilon\to 0}\sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}\frac{(\widetilde{W}_{\boldsymbol{t_\epsilon}l}\frac{4\pi}{2l+1}-1)}{t_E\cdot\epsilon}\overline{f}_{\boldsymbol{t_\epsilon}lm}(\vec{\mathbf{x}},t)P^*_{lm}(\vec{\boldsymbol{\Omega}})\\ &= \frac{1}{t_E}\sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}\Upsilon_l\overline{f}_{\boldsymbol{t_\epsilon}lm}(\vec{\mathbf{x}},t)P_{lm}(\vec{\boldsymbol{\Omega}})\end{aligned} \tag{7.29}$$

with

$$\Upsilon_l = \lim_{\epsilon\to 0}\frac{(\widetilde{W}_{\boldsymbol{t_\epsilon}l}\frac{4\pi}{2l+1}-1)}{\boldsymbol{\epsilon}} \tag{7.30}$$

as exchange coefficient.

Now equation (7.20) yields

$$\boxed{\frac{1}{t_E}\sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}\Upsilon_l\overline{f}_{lm}(\vec{\mathbf{x}},t)P_{lm}(\vec{\boldsymbol{\Omega}}) = \frac{\partial\overline{f}}{\partial t}+\overline{v}\vec{\boldsymbol{\Omega}}\cdot\vec{\boldsymbol{\nabla}}\overline{f}}. \tag{7.31}$$

7.4. Calculation of the exchange-coefficients Υ_l

The transition probability is outlined by Legendre-polynomials respectively spherical harmonics:

$$\begin{aligned}\widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Omega}}\cdot\vec{\boldsymbol{\Omega}}') = \sum_{l=0}^{+\infty}\widetilde{W}_{\boldsymbol{t_\epsilon}l}P_l(cos(\vartheta)) &= \sum_{l=0}^{+\infty}\widetilde{W}_{\boldsymbol{t_\epsilon}l}\sum_{m=-l}^{m=+l}P_{lm}(\vec{\boldsymbol{\Omega}})P^*_{lm}(\vec{\boldsymbol{\Omega}}')\\ cos(\vartheta) = \vec{\boldsymbol{\Omega}}\cdot\vec{\boldsymbol{\Omega}}' &= \mu.\end{aligned} \tag{7.32}$$

On the other hand is

$$\lim_{\boldsymbol{t_\epsilon}\to 0} \widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Omega}}\cdot\vec{\boldsymbol{\Omega}}') = \delta(\vec{\boldsymbol{\Omega}}\cdot\vec{\boldsymbol{\Omega}}')$$
$$\delta(\vec{\boldsymbol{\Omega}}\cdot\vec{\boldsymbol{\Omega}}') = \sum_{l=0}^{+\infty}\frac{2l+1}{4\pi}\sum_{m=-l}^{m=+l}P_{lm}(\vec{\boldsymbol{\Omega}})P^*_{lm}(\vec{\boldsymbol{\Omega}}') = \sum_{l=0}^{+\infty}\frac{2l+1}{4\pi}\boldsymbol{P_l} \qquad \text{see(9.20)}. \tag{7.33}$$

$\widetilde{W}_{\boldsymbol{t_\epsilon}}(\mu) \geqslant 0$ is only in the range $\mu \in [1-\varepsilon, 1]$ essentially different from 0. So the Legendre polynomials are approximated by

$$\begin{aligned}
&P_l(\mu) = 1 - \frac{dP_l}{d\mu}|_1 \cdot \varepsilon + O(\varepsilon^2) \qquad \varepsilon = 1-\mu\\
&\frac{dP_l}{d\mu}|_1 = \frac{l(l+1)}{2} \quad \text{ see (9.1) } \boldsymbol{P_0} = 1, \boldsymbol{P_1} = \mu\\
&\Longrightarrow\\
&P_l(\mu) = P_0 - (P_0 - P_1)\frac{l(l+1)}{2} + O(\varepsilon^2).
\end{aligned} \tag{7.34}$$

Using

$$\int_{-1}^{+1} P_l P_{l'} d\mu = \delta_{ll'}\frac{2}{2l+1} \tag{7.35}$$

follows

$$\int_{-1}^{+1} \widetilde{W}_{\boldsymbol{t_\epsilon}} P_l d\mu = 2\widetilde{W}_{\boldsymbol{t_\epsilon}0} - l(l+1)\widetilde{W}_{\boldsymbol{t_\epsilon}0} + \frac{l(l+1)}{3}\widetilde{W}_{\boldsymbol{t_\epsilon}1} = \frac{2}{2l+1}\widetilde{W}_{\boldsymbol{t_\epsilon}l}. \tag{7.36}$$

Furthermore is

$$\begin{aligned}
&\int_{4\pi} \widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Omega}}\cdot\vec{\boldsymbol{\Omega}}')d\vec{\boldsymbol{\Omega}}' = \int_{4\pi}\widetilde{W}_{\boldsymbol{t_\epsilon}0}d\vec{\boldsymbol{\Omega}}' = \boldsymbol{4\pi}\widetilde{W}_{\boldsymbol{t_\epsilon}0} = \boldsymbol{1}\\
&\qquad\Longrightarrow \qquad \widetilde{W}_{\boldsymbol{t_\epsilon}0} = \frac{1}{4\pi},
\end{aligned} \tag{7.37}$$

as $\widetilde{W}_{\boldsymbol{t_\epsilon}}$ for $\boldsymbol{t_\epsilon} \to \boldsymbol{0}$ degenerates to a δ-function. That is why the $\widetilde{W}_{\boldsymbol{t_\epsilon}l}$ are expressed by $\widetilde{W}_{\boldsymbol{t_\epsilon}1}$ and the determination of $\widetilde{W}_{\boldsymbol{t_\epsilon}1}$ remains to be calculated. We set

$$\lim_{\boldsymbol{\epsilon}\to 0}\frac{(\widetilde{W}_{\boldsymbol{t_\epsilon}1}\frac{4\pi}{3}-1)}{\boldsymbol{\epsilon}}=\boldsymbol{\zeta}. \tag{7.38}$$

Multiplying equation (7.36) with 2π leads to

$$\frac{4\pi}{2l+1}\widetilde{W}_{\boldsymbol{t_\epsilon}l}=4\pi\widetilde{W}_{\boldsymbol{t_\epsilon}0}-(4\pi)\frac{l(l+1)}{2}\widetilde{W}_{\boldsymbol{t_\epsilon}0}+\frac{4\pi}{3}\frac{l(l+1)}{2}\widetilde{W}_{\boldsymbol{t_\epsilon}1}. \tag{7.39}$$

I.e.

$$\begin{aligned}\frac{4\pi}{2l+1}\widetilde{W}_{\boldsymbol{t_\epsilon}l}-\mathbf{1}=\frac{l(l+1)}{2}(\frac{4\pi}{3}\widetilde{W}_{\boldsymbol{t_\epsilon}1}-1)=-\frac{l(l+1)}{2}\boldsymbol{\zeta}+O(\epsilon^2)=\boldsymbol{\Upsilon_l}+O(\epsilon^2)\\ \boldsymbol{\Upsilon_l}=-\frac{l(l+1)}{2}\boldsymbol{\zeta}\end{aligned} \tag{7.40}$$

$\Longrightarrow$

$$\boxed{\frac{\partial\overline{f}}{\partial t}+\overline{v}\vec{\boldsymbol{\Omega}}\cdot\vec{\boldsymbol{\nabla}}\overline{f}=-\frac{1}{t_E}\sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}\frac{l(l+1)}{2}\overline{f}_{l,m}(\vec{\mathbf{x}},t)P_{l,m}(\vec{\boldsymbol{\Omega}})} \tag{7.41}$$

This equation only contains the unknown coefficients t_E and $\overline{v}$ principally depending upon the space-time-point $(\vec{\mathbf{x}},t)$ and the fluctuation direction $\vec{\boldsymbol{\Omega}}$

$$\begin{aligned}t_E=&t_E(\vec{\mathbf{x}},t,\vec{\boldsymbol{\Omega}})\\ \overline{v}=&\overline{v}(\vec{\mathbf{x}},t,\vec{\boldsymbol{\Omega}}).\end{aligned} \tag{7.42}$$

The total derivation of $\overline{f}(\vec{\mathbf{x}},t,\vec{\boldsymbol{\Omega}})$ with respect to t in direction of $\vec{\boldsymbol{\Omega}}$ leads to

$$\frac{d}{dt}\overline{f}(\vec{\mathbf{x}},t,\vec{\boldsymbol{\Omega}})=\sum_{l=0}^{+\infty}\sum_{m=-l}^{m=+l}\frac{d}{dt}\overline{f}_{lm}(\mathbf{x},t)P_{lm}(\vec{\boldsymbol{\Omega}})=\frac{1}{t_E}\sum_{l=1}^{+\infty}\boldsymbol{\gamma_l}\cdot\sum_{m=-l}^{+l}\overline{f}_{lm}(\vec{\mathbf{x}},t)P_{lm}(\vec{\boldsymbol{\Omega}}). \tag{7.43}$$

The time behavior of the spherical harmonic components is described by the equations

$$\frac{d}{dt}\overline{f}_{lm}(t)=\frac{\gamma_l}{t_E}\overline{f}_{lm} \tag{7.44}$$

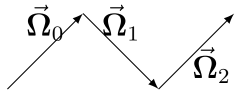
and result in

$$\overline{f}_{lm}(t) \sim \mathbf{exp}(\frac{\gamma_l}{t_E} \cdot t). \tag{7.45}$$

The greater the order l the more powerful is its temporal decay.

7.5. Reconstruction of the transition probabilities $\overline{W}_{\boldsymbol{t_\epsilon}}$

The Transition probability $\widetilde{W}_{\boldsymbol{t_\epsilon}0\to1\to2}$, changing the movement direction $\vec{\mathbf{\Omega}}$ at the times t_0, t_1, t_2 from $\vec{\mathbf{\Omega}}_0$ via $\vec{\mathbf{\Omega}}_1$ to $\vec{\mathbf{\Omega}}_2$



is the product of the single transition probabilities.

$$\widetilde{W}_{\boldsymbol{t_\epsilon},0\to1\to2} = \widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{2}}(\vec{\mathbf{\Omega}}_0 \cdot \vec{\mathbf{\Omega}}_1) \cdot \widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{2}}(\vec{\mathbf{\Omega}}_1 \cdot \vec{\mathbf{\Omega}}_2). \tag{7.46}$$

On the other side

$$\widetilde{W}_{\boldsymbol{t_\epsilon}l} = (1 + \Upsilon_l \boldsymbol{\epsilon})\frac{2l+1}{4\pi} + O(\boldsymbol{\epsilon}^2) \tag{7.47}$$

holds and thus arises

$$\widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{\Omega}} \cdot \vec{\mathbf{\Omega}}') = \sum_{l=0}^{+\infty}(1 + \Upsilon_l \boldsymbol{\epsilon})\frac{2l+1}{4\pi} \cdot \sum_{m=-l}^{+l} P^*_{lm}(\vec{\mathbf{\Omega}})P_{lm}(\vec{\mathbf{\Omega}}') + O(\boldsymbol{\epsilon}^2). \tag{7.48}$$

The probability, that a particle changes the direction after an infinitesimal time interval $\epsilon \cdot t_E$ from $\vec{\mathbf{\Omega}}_0$ to $\vec{\mathbf{\Omega}}_2$ is given by

$$\widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{\Omega}}_0 \cdot \vec{\mathbf{\Omega}}_2) = \int_{4\pi} \widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{2}}(\vec{\mathbf{\Omega}}_0 \cdot \vec{\mathbf{\Omega}}_1) \cdot \widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{2}}(\vec{\mathbf{\Omega}}_1 \cdot \vec{\mathbf{\Omega}}_2)d\vec{\mathbf{\Omega}}_1 \tag{7.49}$$

and

$$\widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{\Omega}}_0 \cdot \vec{\mathbf{\Omega}}_2) = \sum_{l=0}^{+\infty}\left(1 + \Upsilon_l \frac{\boldsymbol{\epsilon}}{\mathbf{2}}\right)^2 \frac{2l+1}{4\pi} \cdot \sum_{m=-l}^{+l} P^*_{lm}(\vec{\mathbf{\Omega}}_0)P_{lm}(\vec{\mathbf{\Omega}}_2) + O(\boldsymbol{\epsilon}^2). \tag{7.50}$$

Using n intermediate steps $\widetilde{W}_{\boldsymbol{t_\epsilon}}$ is expressed by an integral over the product of the single transition probabilities.

$$\widetilde{W}_{\boldsymbol{t_\epsilon},0\to 1\ldots\to n} = \widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{n}}(\vec{\boldsymbol{\Omega}}_0 \cdot \vec{\boldsymbol{\Omega}}_1) \cdot \widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{n}}(\vec{\boldsymbol{\Omega}}_1 \cdot \vec{\boldsymbol{\Omega}}_2) \ldots . \widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{n}}(\vec{\boldsymbol{\Omega}}_{n-1} \cdot \vec{\boldsymbol{\Omega}}_n) \tag{7.51}$$

$$\widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Omega}}_0 \cdot \vec{\boldsymbol{\Omega}}_n) = \int_{4\pi}\int_{4\pi} \ldots . \int_{4\pi} \widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{n}} \cdot \widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{n}} \ldots . \widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{n}} d\vec{\boldsymbol{\Omega}}_1 \ldots . d\vec{\boldsymbol{\Omega}}_{n-1} \tag{7.52}$$

For $n \to \infty$ this results in:

$$\widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Omega}} \cdot \vec{\boldsymbol{\Omega}}') = \lim_{\boldsymbol{n}\to\infty} \sum_{l=0}^{+\infty} \left\{ 1 + \frac{\epsilon \Upsilon_l}{n} \right\}^n \frac{2l+1}{4\pi} \cdot \sum_{m=-l}^{+l} P^*_{lm}(\vec{\boldsymbol{\Omega}}) P_{lm}(\vec{\boldsymbol{\Omega}}') + O(\boldsymbol{\epsilon}^2) \tag{7.53}$$

and finally

$$\boxed{\widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Omega}} \cdot \vec{\boldsymbol{\Omega}}') = \sum_{l=0}^{+\infty} e^{\Upsilon_l \cdot \epsilon} \, \frac{2l+1}{4\pi} \cdot \sum_{m=-l}^{+l} P^*_{lm}(\vec{\boldsymbol{\Omega}}) P_{lm}(\vec{\boldsymbol{\Omega}}') + O(\boldsymbol{\epsilon}^2)}. \tag{7.54}$$

Selecting $\boldsymbol{\varepsilon} = \frac{\boldsymbol{t_\varepsilon}}{\boldsymbol{t_E}(\vec{\boldsymbol{x}},\boldsymbol{t},\vec{\boldsymbol{\Omega}})}$ the exchange function $\widetilde{W}_{\boldsymbol{t_\epsilon}}$ may be understood in the dependencies

$$\widetilde{W}_{\boldsymbol{t_\epsilon}} = \widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Omega}}'). \tag{7.55}$$

Therefore

$$\overline{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Omega}}') \approx \widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Omega}}') \tag{7.56}$$

is calculated, too.$\Longrightarrow$

$$\boxed{\begin{aligned} \overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, \vec{\boldsymbol{\Omega}}, t) &= \int_{4\pi} \overline{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Omega}}') \overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}} - t_\varepsilon \cdot \overline{v}' \vec{\boldsymbol{\Omega}}', \vec{\boldsymbol{\Omega}}', t - t_{\boldsymbol{\epsilon}}) d\vec{\boldsymbol{\Omega}}' \\ \overline{v}' &= \overline{v}'(\vec{\mathbf{x}}, \vec{\boldsymbol{\Omega}}', t) \end{aligned}} \tag{7.57}$$

The Transition probability $\overline{W}_{\boldsymbol{t_\epsilon}}$ is unsymmetrical in the direction quantities on account of $\boldsymbol{\varepsilon} = \frac{\boldsymbol{t_\varepsilon}}{\boldsymbol{t_E}(\vec{\boldsymbol{x}},\boldsymbol{t},\vec{\boldsymbol{\Omega}})}$.

7.6. Approximation formula

In 1st approximation a telegrapher's equations is derived out of the linear Boltzmann Equation leading to the known diffusion equation without taking into account the second time derivation. In this case the diffusion equation is proved to be usefull. Subsequent considerations are displaying which relation exists between the 1st approximation of the particle transport by longitudinal continuum fluctuations and the known diffusion equation.

Assuming the simplification

$$\frac{1}{t_E} = \tau(\vec{\mathbf{x}}, \vec{\boldsymbol{\Omega}}, t) = \tau_0 = const \tag{7.58}$$

the transport equation described in 1st approximation is

$$\frac{\partial \overline{f}}{\partial t} + \overline{v}\vec{\boldsymbol{\Omega}} \cdot \nabla \overline{f} = -\frac{1}{t_E} \cdot \sum_{m=-1}^{+1} \overline{f}_{1m}(\vec{\mathbf{x}}, t) P_{1m}(\vec{\boldsymbol{\Omega}}). \tag{7.59}$$

In cartesian coordinates one gets

$$\frac{\partial \overline{f}}{\partial t} + \overline{v}_x \boldsymbol{\Omega}_x \cdot \frac{\partial \overline{f}}{\partial x} + \overline{v}_y \boldsymbol{\Omega}_y \cdot \frac{\partial \overline{f}}{\partial y} + \overline{v}_z \boldsymbol{\Omega}_z \cdot \frac{\partial \overline{f}}{\partial z} = -\frac{1}{t_E} \cdot \sum_{m=-1}^{+1} \overline{f}_{1m}(\vec{\mathbf{x}}, t) P_{1m}(\vec{\boldsymbol{\Omega}}) \tag{7.60}$$

with

$$\overline{v}_x = \overline{v}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}) \quad \overline{v}_y = \overline{v}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}) \quad \overline{v}_z = \overline{v}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}). \tag{7.61}$$

Subsequently we confine us on

$$\overline{v}_x = \overline{v}_y = \overline{v}_z = \overline{v}(\vec{\mathbf{x}}), \tag{7.62}$$

suggesting an isotropy of fluctuation motions in the statistical ensemble. The conditions are selected such that the further derivations analogous to 6.4 follow until to a telegrapher's equation.

$$t_E \frac{\partial^2 \Phi}{\partial t^2} + \frac{\partial \Phi}{\partial t} = \boldsymbol{t_E} \overline{v} \vec{\nabla} \cdot \frac{\overline{v}}{3} \vec{\nabla} \Phi \tag{7.63}$$

The usual diffusion coefficient normally contained in $\vec{\nabla} \cdot \boldsymbol{D} \vec{\nabla} \Phi$ cannot be found. In the equation above $\boldsymbol{D} = \boldsymbol{t_E} \cdot \frac{\overline{v}^2}{3}$ is contained partly outside partly between the $\vec{\nabla}-$operators. This has consequences in inhomogeneous media. Such problems arise unrecognized using the Bousinesque approach . I.e. in an inhomogenuous medium this approach may be fatal. The term of second derivation by time has nothing to do with relativistic theory. Because of the small size of t_E it may generally be neglected.

8. Stochastic transport by turbulent continuum-fluctuations

$$\boxed{\begin{gathered}\frac{\partial \overline{f}}{\partial t} + \overline{v}\vec{\boldsymbol{\Omega}} \times \vec{\boldsymbol{\Theta}} \cdot \nabla \overline{f} = \frac{-1}{t_E} \sum_{l=0}^{+\infty} \sum_{m=-l}^{+l} \sum_{k=-\infty}^{+\infty} \frac{l(l+1)}{2} \overline{f}_{lmk}(\vec{\mathbf{x}}, t) P_{lm}(\vec{\boldsymbol{\Omega}}) H_k(\vec{\boldsymbol{\Theta}}) \\ \Updownarrow \\ \overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}, t) = \int_{2\pi} \int_{4\pi} \widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}, \vec{\boldsymbol{\Omega}}', \vec{\boldsymbol{\Theta}}') \overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}} - t_\varepsilon \cdot \overline{v}'\vec{\boldsymbol{\Omega}}' \times \vec{\boldsymbol{\Theta}}', \vec{\boldsymbol{\Omega}}', \vec{\boldsymbol{\Theta}}'), t - t_{\boldsymbol{\epsilon}}) d\vec{\boldsymbol{\Omega}}' d\vec{\boldsymbol{\Theta}}'\end{gathered}}$$

8.1. Introduction

The motion of passive particles by turbulent continuum fluctuations is examined. The particles are moved not affecting this field. Their trajectories correspond in every $\varepsilon-$neiborhood of a point to a circle segment passed with the velocity

$$\vec{\mathbf{v}}_{\boldsymbol{t_\epsilon}} = \vec{\boldsymbol{\omega}}_{\boldsymbol{t_\epsilon}} \times \vec{\boldsymbol{r}}_{\boldsymbol{t_\epsilon}}. \tag{8.1}$$

The considered motion quantities $\vec{\boldsymbol{\omega}}_{\boldsymbol{t_\epsilon}}$ and $\vec{\boldsymbol{r}}_{\boldsymbol{t_\epsilon}}$ are determined by successively detecting a single particle originating from a point $\vec{\boldsymbol{x}}_0$ after a time t_ε moving to $\vec{\boldsymbol{x}}_1$ and after a further time t_ε to $\vec{\boldsymbol{x}}_2$. By these 3 points a circle segment is uniquely defined for the point $\vec{\boldsymbol{x}}_1$ with radius vector $\vec{\boldsymbol{r}}_{t_\varepsilon}$ and a rotation speed $\vec{\boldsymbol{\omega}}_{t_\varepsilon}$.

$$\begin{aligned}\vec{\boldsymbol{r}}_{t_\varepsilon} &= \boldsymbol{r}_{t_\varepsilon} \cdot \vec{\boldsymbol{\Theta}}_{t_\varepsilon} \\ \vec{\boldsymbol{\omega}}_{t_\varepsilon} &= \boldsymbol{\omega}_{t_\varepsilon} \cdot \vec{\boldsymbol{\Omega}}_{t_\varepsilon}\end{aligned} \tag{8.2}$$

In the special case $\vec{\boldsymbol{\omega}}_{t_\varepsilon} \to \mathbf{0}$ and $\vec{\boldsymbol{r}} \to +\boldsymbol{\infty}$ the velocity $\vec{\mathbf{v}}_{\boldsymbol{t_\epsilon}}$ is revealed out of its neighborhood.[1] The particle density distributions are received in a thought experiment by an unlimited number of deterministic ensemble-systems (see chapter 4). In every

[1]Applying the deterministic theory this problem must be treated numerically.

point $(\vec{\boldsymbol{x}}, t)$ a continuously differentiable particle density distribution of the motion quantities $\vec{\boldsymbol{\omega}}_{t_\epsilon}$ and $\vec{\boldsymbol{r}}_{t_\epsilon}$ is assigned in accordance with

$$f_{\boldsymbol{t_\epsilon}} = f_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\omega}}, \vec{\boldsymbol{r}}). \tag{8.3}$$

The with $\boldsymbol{t_\epsilon}$ indexed functions are automatically assumed to contain motion quantities of corresponding measurement accuracies. The indexing of the motion quantities can be omitted if the functions are indexed. After execution of a limiting process for example

$$\lim_{\boldsymbol{t_\epsilon} \to 0} f_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\omega}}, \vec{\boldsymbol{r}}) = f(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\omega}}, \vec{\boldsymbol{r}}) \tag{8.4}$$

f and $(\vec{\boldsymbol{\omega}}, \vec{\boldsymbol{r}})$ are understood according to an exact measuring process. Integrating the particle density distribution over the motion quantities one obtains expectation values of a particle density not conforming with the actual particle density ρ.

$$< \boldsymbol{\rho_{t_\epsilon}(\vec{x}, t)} >= \int_{2\pi} \int_{4\pi} \int_0^\infty \int_0^\infty f_{t_\epsilon}(\vec{\boldsymbol{x}}, t, \boldsymbol{\omega} \cdot \vec{\boldsymbol{\Omega}}, \boldsymbol{r} \cdot \vec{\boldsymbol{\Theta}}) d\boldsymbol{\omega} d\boldsymbol{r} d\vec{\boldsymbol{\Omega}} d\vec{\boldsymbol{\Theta}} \neq \boldsymbol{\rho_{t_\epsilon}}(\vec{\boldsymbol{x}}, t) \tag{8.5}$$

A strictly deduced partial differential equation is obtained calculating the development of spatio-temporal particle density distributions. The incipiently unlimited number of unknown coefficients is reduced to a local time-scaling related to the vortex calculation of an associated deterministic theory discussed in further chapters. The initially abstractly formulated transition probabilities get concrete functional dependencies. There are always found suitable Monte-Carlo methods treating them with the help of the deterministic theory described in further chapters.

8.2. The transport as Markov Process with natural causality

A particle at location $\vec{\boldsymbol{x}}$ and time t changing its velocity from $\vec{\boldsymbol{v}}' = (\vec{\boldsymbol{\omega}}' \times \vec{\boldsymbol{r}}')$ to $\vec{\boldsymbol{v}} = (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}})$ is given by the transition probability

$$W_{\boldsymbol{t_\epsilon}} = W_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t; \vec{\boldsymbol{\omega}}, \vec{\boldsymbol{r}}; \vec{\boldsymbol{\omega}}', \vec{\boldsymbol{r}}') \tag{8.6}$$

with

$$\int\limits_0^\infty \int\limits_0^\infty \int\limits_{4\pi} \int\limits_{2\pi} W_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t; \vec{\boldsymbol{\omega}}, \vec{\boldsymbol{r}}; \vec{\boldsymbol{\omega}}', \vec{\boldsymbol{r}}') d\boldsymbol{\omega}' dr' d\boldsymbol{\Omega}' d\boldsymbol{\Theta}' = 1. \tag{8.7}$$

$\Longrightarrow$

$$\boxed{\begin{array}{c} f_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\omega}}, \vec{\boldsymbol{r}}) = \\ \int\limits_0^\infty \int\limits_0^\infty \int\limits_{4\pi} \int\limits_{2\pi} W_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\omega}}, \vec{\boldsymbol{r}}, \vec{\boldsymbol{\omega}}', \vec{\boldsymbol{r}}') f_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{\omega}}' \times \vec{\boldsymbol{r}}' \cdot t_{\boldsymbol{\epsilon}}, \vec{\boldsymbol{\omega}}', \vec{\boldsymbol{r}}', t - t_{\boldsymbol{\epsilon}}) d\boldsymbol{\omega}' dr' d\boldsymbol{\Omega}' d\boldsymbol{\Theta}' \end{array}} \tag{8.8}$$

Continuity is required respectively of all variables of the transition probability $W_{\boldsymbol{t_\epsilon}}$. The sequence of velocities $\vec{\boldsymbol{v}}'_{t_\epsilon}, \vec{\boldsymbol{v}}_{t_\epsilon}$ means a motion from

$$(\vec{\boldsymbol{x}} - \vec{\boldsymbol{\omega}}'_{t_\epsilon} \times \vec{\boldsymbol{r}}'_{t_\epsilon} \cdot t_{\boldsymbol{\epsilon}}, t - t_\epsilon, \vec{\boldsymbol{\omega}}'_{t_\epsilon} \times \vec{\boldsymbol{r}}'_{t_\epsilon}) \quad \text{to} \quad (\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\omega}}_{t_\epsilon} \times \vec{\boldsymbol{r}}_{t_\epsilon}). \tag{8.9}$$

For the limiting process $t_\varepsilon \to 0$ the transition probabilities W_{t_ϵ} prove to be physical realizations of test functions of the distribution theory.

$$\lim_{\boldsymbol{t_\epsilon} \to 0} W_{\boldsymbol{t_\epsilon}} = \delta(\vec{\boldsymbol{\omega}}, \vec{\boldsymbol{r}}; \vec{\boldsymbol{\omega}}', \vec{\boldsymbol{r}}'). \tag{8.10}$$

The passive scalar particles precisely reproduce the motions of the fluctuation field. For the particle density distribution $f_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\omega}}, \vec{\boldsymbol{r}})$ the following separation aproach is used without loss of generality:

$$f_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}} \cdot t_{\boldsymbol{\epsilon}}, t, \vec{\boldsymbol{\omega}}, \vec{\boldsymbol{r}}) = G_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}} \cdot t_{\boldsymbol{\epsilon}}, t, \vec{\boldsymbol{\omega}}, \vec{\boldsymbol{r}}) \overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}} - \overline{v} \vec{\boldsymbol{\Omega}} \times \vec{\boldsymbol{\Theta}} \cdot t_{\boldsymbol{\epsilon}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}) \tag{8.11}$$

with

$$\begin{aligned} &\int_0^\infty \int_0^\infty G_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \boldsymbol{\omega} \vec{\boldsymbol{\Omega}}, r \vec{\boldsymbol{\Theta}}) d\omega dr = 1 \\ &\int_0^\infty \int_0^\infty G_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \boldsymbol{\omega} \vec{\boldsymbol{\Omega}}, r \vec{\boldsymbol{\Theta}}) \omega r d\omega dr = \overline{v}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}) \\ &\overline{v}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}) = \overline{\omega}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}) \cdot \overline{r}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}) \end{aligned} \tag{8.12}$$

$\Longrightarrow$

$$\overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}} - \overline{v} \vec{\boldsymbol{\Omega}} \times \vec{\boldsymbol{\Theta}} \cdot t_\epsilon, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}) = \int_0^\infty \int_0^\infty f_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}} \cdot t_{\boldsymbol{\epsilon}}, t, \boldsymbol{\omega} \cdot \vec{\boldsymbol{\Omega}}, \boldsymbol{r} \cdot \vec{\boldsymbol{\Theta}}) d\omega dr \tag{8.13}$$

One obtains a transition probability $\overline{W}_{\boldsymbol{t_\epsilon}}$ only depending on the directions by integrating $W_{\boldsymbol{t_\epsilon}}$ over the amounts ω', r', ω, r.

$$\boxed{\overline{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}, \vec{\boldsymbol{\Omega}}', \vec{\boldsymbol{\Theta}}') = \int\limits_0^\infty \int\limits_0^\infty \int\limits_0^\infty \int\limits_0^\infty W_{\boldsymbol{t_\epsilon}} G_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{\omega}}' \times \vec{\boldsymbol{r}}' \cdot t_{\boldsymbol{\epsilon}}, t - t_{\boldsymbol{\epsilon}}, \vec{\boldsymbol{\omega}}', \vec{\boldsymbol{r}}') d\omega' dr' d\omega dr} \tag{8.14}$$

The integration

$$\int_0^\infty \int_0^\infty (8.8) d\omega dr \tag{8.15}$$

gives

$$\overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}) = \int\limits_0^\infty \int\limits_0^\infty \int\limits_0^\infty \int\limits_0^\infty \int\limits_{4\pi} \int\limits_{2\pi} W_{\boldsymbol{t_\epsilon}} f_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{\omega}}' \times \vec{\boldsymbol{r}}' \cdot t_{\boldsymbol{\epsilon}}, t - t_{\boldsymbol{\epsilon}}, \vec{\boldsymbol{\omega}}', \vec{\boldsymbol{r}}') d\omega' dr' d\omega dr d\vec{\boldsymbol{\Omega}}' d\vec{\boldsymbol{\Theta}}' \tag{8.16}$$

$$\Longrightarrow \boxed{\overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}) = \int\limits_{4\pi} \int\limits_{2\pi} \overline{W}_{\boldsymbol{t_\epsilon}} \overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}} - \overline{v}' \vec{\boldsymbol{\Omega}}' \times \vec{\boldsymbol{\Theta}}' \cdot t_{\boldsymbol{\epsilon}}, t - t_{\boldsymbol{\epsilon}}, \vec{\boldsymbol{\Omega}}', \vec{\boldsymbol{\Theta}}') d\vec{\boldsymbol{\Omega}}' d\vec{\boldsymbol{\Theta}}'} \tag{8.17}$$

In the integrand $\overline{f}_{\boldsymbol{t_\epsilon}}$ is developed around $\vec{\boldsymbol{x}}$ and t:

$$\overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}} - \Delta\vec{\boldsymbol{x}}, t - t_{\boldsymbol{\epsilon}}, \vec{\boldsymbol{\Omega}}', \vec{\boldsymbol{\Theta}}') = \overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\Omega}}', \vec{\boldsymbol{\Theta}}') - \tau_E \cdot \epsilon \cdot [\frac{\partial \overline{f}'_{\boldsymbol{t_\epsilon}}}{\partial t} + \overline{v}' \vec{\boldsymbol{\Omega}}' \times \vec{\boldsymbol{\Theta}}' \cdot \nabla \overline{f}'_{\boldsymbol{t_\epsilon}} + O(\epsilon^2)] \tag{8.18}$$

This leads to

$$\frac{\int_{4\pi} \int_{2\pi} \overline{W}_{\boldsymbol{t_\epsilon}} \overline{f}'_{\boldsymbol{t_\epsilon}} d\vec{\boldsymbol{\Omega}}' d\vec{\boldsymbol{\Theta}}' - \overline{f}_{\boldsymbol{t_\epsilon}}}{\epsilon} =$$
$$\int\limits_{4\pi} \int\limits_{2\pi} \overline{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\Omega}}', \vec{\boldsymbol{\Theta}}', \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}) \cdot \tau_E [\cdot \frac{\partial \overline{f}'_{\boldsymbol{t_\epsilon}}}{\partial t} + \overline{v}' \vec{\boldsymbol{\Omega}}' \times \vec{\boldsymbol{\Theta}}' \cdot t_{\boldsymbol{\epsilon}} \cdot \nabla \overline{f}'_{\boldsymbol{t_\epsilon}} + O(\epsilon^2)] d\vec{\boldsymbol{\Omega}}' d\vec{\boldsymbol{\Theta}}'. \tag{8.19}$$

As

$$\lim_{\boldsymbol{t_\epsilon} \to 0} \overline{W}_{\boldsymbol{t_\epsilon}} = \delta(\vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}; \vec{\boldsymbol{\Omega}}', \vec{\boldsymbol{\Theta}}') \tag{8.20}$$

$\Longrightarrow$

$$\lim_{\boldsymbol{\epsilon} \to 0} \frac{\int_{4\pi} \int_{2\pi} \overline{W}_{\boldsymbol{t_\epsilon}} \overline{f}'_{\boldsymbol{t_\epsilon}} d\vec{\boldsymbol{\Omega}}' d\vec{\boldsymbol{\Theta}}' - \overline{f}_{\boldsymbol{t_\epsilon}}}{\epsilon \cdot \tau_E} = \frac{\partial \overline{f}}{\partial t} + \overline{v} \vec{\boldsymbol{\Omega}} \times \vec{\boldsymbol{\Theta}} \cdot \nabla \overline{f} \ . \tag{8.21}$$

Furtheron

$$\boxed{\lim_{\epsilon\to 0}\frac{\int_{4\pi}\int_{2\pi}\overline{W}_{\boldsymbol{t_\epsilon}}\overline{f}'_{\boldsymbol{t_\epsilon}}d\vec{\boldsymbol{\Omega}}'d\vec{\boldsymbol{\Theta}}'-\overline{f}_{\boldsymbol{t_\epsilon}}}{\epsilon\cdot\tau_E}} \tag{8.22}$$

is called exchange-term.

8.3. Calculation of the exchange-term

Exchange term dependencies of scalar products $\vec{\boldsymbol{\Omega}}\cdot\vec{\boldsymbol{\Omega}}'$ and $\vec{\boldsymbol{\Theta}}\cdot\vec{\boldsymbol{\Theta}}'$ are taken into account istead of individually depending directions $\vec{\boldsymbol{\Omega}}$, $\vec{\boldsymbol{\Omega}}'$ and $\vec{\boldsymbol{\Theta}}$, $\vec{\boldsymbol{\Theta}}$ demanding the following relation

$$\lim_{\epsilon\to 0}\frac{\int_{2\pi}\int_{4\pi}\overline{W}_{\boldsymbol{t_\epsilon}}\overline{f}'_{\boldsymbol{t_\epsilon}}d\vec{\boldsymbol{\Omega}}'d\vec{\boldsymbol{\Theta}}'-\overline{f}_{\boldsymbol{t_\epsilon}}}{\epsilon\cdot\tau_E}=\lim_{\epsilon\to 0}\frac{\int_{2\pi}\int_{4\pi}\widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Omega}}\cdot\vec{\boldsymbol{\Omega}}',\vec{\boldsymbol{\Theta}}\cdot\vec{\boldsymbol{\Theta}}')\overline{f}'_{\boldsymbol{t_\epsilon}}d\vec{\boldsymbol{\Omega}}'d\vec{\boldsymbol{\Theta}}'-\overline{f}_{\boldsymbol{t_\epsilon}}}{\epsilon\cdot t_E}. \tag{8.23}$$

The following transitions

$$\begin{array}{rcl}\tau_E=const & \longrightarrow & t_E=t_E(\vec{\boldsymbol{x}},t,\vec{\boldsymbol{\Omega}},\vec{\boldsymbol{\Theta}})\\ \overline{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}},t,\vec{\boldsymbol{\Omega}},\vec{\boldsymbol{\Theta}};\vec{\boldsymbol{\Omega}}',\vec{\boldsymbol{\Theta}}') & \longrightarrow & \widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Omega}}\cdot\vec{\boldsymbol{\Omega}}',\vec{\boldsymbol{\Theta}}\cdot\vec{\boldsymbol{\Theta}}')\end{array} \tag{8.24}$$

are regarded. Moreover, a separation of $\vec{\boldsymbol{\Omega}}\cdot\vec{\boldsymbol{\Omega}}'$ and $\vec{\boldsymbol{\Theta}}\cdot\vec{\boldsymbol{\Theta}}'$ is asumed:

$$\widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Omega}}\cdot\vec{\boldsymbol{\Omega}}',\vec{\boldsymbol{\Theta}}\cdot\vec{\boldsymbol{\Theta}}')=V_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Omega}}\cdot\vec{\boldsymbol{\Omega}}')\cdot M_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Theta}}\cdot\vec{\boldsymbol{\Theta}}'). \tag{8.25}$$

Functions of the unit vectors $\vec{\boldsymbol{\Omega}}$ and $\vec{\boldsymbol{\Theta}}$ are presented by a complete orthogonal function system representing an extension of the spherical harmonics called turbulence functions.

$$\begin{aligned}\overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}},t,\vec{\boldsymbol{\Omega}},\vec{\boldsymbol{\Theta}})&=\sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}\sum_{k=-\infty}^{+\infty}\overline{f}_{\boldsymbol{t_\epsilon}lmk}(\vec{\boldsymbol{x}},t)Q_{lmk}(\vec{\boldsymbol{\Omega}},\vec{\boldsymbol{\Theta}})\\&=\sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}\sum_{k=-\infty}^{+\infty}\overline{f}_{\boldsymbol{t_\epsilon}lmk}(\vec{\boldsymbol{x}},t)Q^*_{lmk}(\vec{\boldsymbol{\Omega}},\vec{\boldsymbol{\Theta}})\end{aligned} \tag{8.26}$$

$$\int_{2\pi}\int_{4\pi}Q_{lmk}(\vec{\boldsymbol{\Omega}},\vec{\boldsymbol{\Theta}})Q^*_{lmk}(\vec{\boldsymbol{\Omega}}',\vec{\boldsymbol{\Theta}}')d\vec{\boldsymbol{\Omega}}'d\vec{\boldsymbol{\Theta}}'=\begin{cases}\frac{8\pi^2}{2l+1} & \text{for l = l' and m=m'}\\ 0 & \text{else}\end{cases} \tag{8.27}$$

with

$$
\begin{aligned}
Q_{lmk}(\vec{\mathbf{\Omega}},\vec{\mathbf{\Theta}}) =& P_{lm}(\vec{\mathbf{\Omega}})H_k(\vec{\mathbf{\Theta}}) \\
\int_{2\pi} H_{k'}(\vec{\mathbf{\Theta}})H_k^*(\vec{\mathbf{\Theta}})d\vec{\mathbf{\Theta}} =& \begin{cases} 2\pi & \text{for k'=k} \\ 0 & \text{else} \end{cases} \\
H_k(\vec{\mathbf{\Theta}}) =& e^{ik\theta}
\end{aligned}
\tag{8.28}
$$

The product $\vec{\mathbf{\Omega}}\cdot\vec{\mathbf{\Omega}}'$ in the separated exchange function $V_{\boldsymbol{t_\epsilon}}$ is developed by spherical harmonics.

$$
\begin{gathered}
V_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{\Omega}}'\cdot\vec{\mathbf{\Omega}}) = \sum_{l=0}^{+\infty} V_{\boldsymbol{t_\epsilon} l}P_l(cos(\alpha)) = \sum_{l=0}^{+\infty} V_{\boldsymbol{t_\epsilon} l}\sum_{m=-l}^{m=+l} P_{lm}(\vec{\mathbf{\Omega}}')P_{lm}^*(\vec{\mathbf{\Omega}}) \\
\text{mit} \\
\lim_{\boldsymbol{t\epsilon}\to 0} V_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{\Omega}}'\cdot\vec{\mathbf{\Omega}}) = \delta(\vec{\mathbf{\Omega}},\vec{\mathbf{\Omega}}')
\end{gathered}
\tag{8.29}
$$

The product $\vec{\mathbf{\Theta}}\cdot\vec{\mathbf{\Theta}}'$ in the separated exchange function $M_{\boldsymbol{t_\epsilon}}$ is developed by functions H_k.

$$
M_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{\Theta}}'\cdot\vec{\mathbf{\Theta}}) = \sum_{k=0}^{+\infty} M_{\boldsymbol{t_\epsilon} k}cos(k\beta) = \frac{1}{2}\sum_{k=0}^{k=+\infty} M_{\boldsymbol{t_\epsilon} k}[H_k(\vec{\mathbf{\Theta}}')H_k^*(\vec{\mathbf{\Theta}}) + H_{-k}(\vec{\mathbf{\Theta}}')H_{-k}^*(\vec{\mathbf{\Theta}})]
\tag{8.30}
$$

with

$$
\begin{aligned}
cos(k\beta) =& \frac{1}{2}[H_k(\vec{\mathbf{\Theta}}')H_k^*(\vec{\mathbf{\Theta}}) + H_{-k}(\vec{\mathbf{\Theta}}')H_{-k}^*(\vec{\mathbf{\Theta}})] = \frac{1}{2}[e^{ik(\theta'-\theta)} + e^{-ik(\theta'-\theta)}] \\
\vec{\mathbf{\Theta}}'\cdot\vec{\mathbf{\Theta}} = cos(\beta) = cos(\theta'-\theta) =& \frac{1}{2}[H_1(\vec{\mathbf{\Theta}}')H_1^*(\vec{\mathbf{\Theta}}) + H_{-1}(\vec{\mathbf{\Theta}}')H_{-1}^*(\vec{\mathbf{\Theta}})] = \frac{1}{2}[e^{i(\theta'-\theta)} + e^{-i(\theta'-\theta)}] \\
\lim_{t_\epsilon\to 0} M_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{\Theta}}'\cdot\vec{\mathbf{\Theta}}) =& \delta(\vec{\mathbf{\Theta}},\vec{\mathbf{\Theta}}')
\end{aligned}
\tag{8.31}
$$

$\Longrightarrow$

$$\int_{4\pi}\int_{2\pi}\widetilde{W}_{\boldsymbol{t_\epsilon}}\overline{f}'_{\boldsymbol{t_\epsilon}}d\vec{\boldsymbol{\Omega}}'d\vec{\boldsymbol{\Theta}} = \int_{4\pi}\int_{2\pi}V_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Omega}}'\cdot\vec{\boldsymbol{\Omega}})\cdot M_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Theta}}'\cdot\vec{\boldsymbol{\Theta}})\overline{f}'_{\boldsymbol{t_\epsilon}}d\vec{\boldsymbol{\Omega}}'d\vec{\boldsymbol{\Theta}}'$$
$$= \int_{4\pi}\int_{2\pi}\Big[\{\sum_{l=0}^{+\infty}V_{\boldsymbol{t_\epsilon}l}\sum_{m=-l}^{m=+l}P_{lm}(\vec{\boldsymbol{\Omega}}')P^*_{lm}(\vec{\boldsymbol{\Omega}})\cdot\frac{1}{2}\sum_{k=0}^{k=+\infty}M_{\boldsymbol{t_\epsilon}k}[H_k(\vec{\boldsymbol{\Theta}}')H^*_k(\vec{\boldsymbol{\Theta}})+H_{-k}(\vec{\boldsymbol{\Theta}}')H^*_{-k}(\vec{\boldsymbol{\Theta}})]\}$$
$$\cdot\sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}\sum_{k=-\infty}^{+\infty}\overline{f}_{\boldsymbol{t_\epsilon}lmk}(\vec{\boldsymbol{x}},t)P^*_{lm}(\vec{\boldsymbol{\Omega}}')H^*_k(\vec{\boldsymbol{\Theta}}')\Big]d\vec{\boldsymbol{\Omega}}'d\vec{\boldsymbol{\Theta}}'$$
$$= \sum_{l=0}^{+\infty}V_{\boldsymbol{t_\epsilon}l}\frac{4\pi}{2l+1}\sum_{m=-l}^{+l}P^*_{lm}(\vec{\boldsymbol{\Omega}})\sum_{k=0}^{+\infty}M_{\boldsymbol{t_\epsilon}k}2\pi\overline{f}_{\boldsymbol{t_\epsilon}lmk}(\vec{\boldsymbol{x}},t)H^*_k(\vec{\boldsymbol{\Theta}}). \tag{8.32}$$

Finally the exchange term results in

$$\lim_{\boldsymbol{\epsilon}\to 0}\frac{\int_{4\pi}\int_{2\pi}\widetilde{W}_{\boldsymbol{t_\epsilon}}\overline{f}'_{\boldsymbol{t_\epsilon}}d\vec{\boldsymbol{\Omega}}'d\vec{\boldsymbol{\Theta}}'-\overline{f}_{\boldsymbol{t_\epsilon}}}{\epsilon}$$
$$= \lim_{\boldsymbol{\epsilon}\to 0}\sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}\sum_{k=0}^{+\infty}\frac{(V_{\boldsymbol{t_\epsilon}l}\frac{4\pi}{2l+1}M_{\boldsymbol{t_\epsilon}k}2\pi-1)}{\boldsymbol{\epsilon}}\overline{f}_{\boldsymbol{t_\epsilon}lmk}(\vec{\boldsymbol{x}},t)P^*_{lm}(\vec{\boldsymbol{\Omega}})H^*_k(\vec{\boldsymbol{\Theta}}) \tag{8.33}$$
$$= \sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}\sum_{k=0}^{+\infty}\Upsilon_{lk}\overline{f}_{lmk}(\vec{\boldsymbol{x}},t)P_{lm}(\vec{\boldsymbol{\Omega}})H_k(\vec{\boldsymbol{\Theta}}).$$

With the exchange coefficients

$$\Upsilon_{lk} = \lim_{\boldsymbol{\epsilon}\to 0}\frac{(V_{\boldsymbol{t_\epsilon}l}\frac{4\pi}{2l+1}M_{\boldsymbol{t_\epsilon}k}2\pi-1)}{\boldsymbol{\epsilon}} \tag{8.34}$$

the transport equation

$$\boxed{\frac{\partial\overline{f}}{\partial t}+\overline{v}\vec{\boldsymbol{\Omega}}\times\vec{\boldsymbol{\Theta}}\cdot\nabla\overline{f} = \frac{1}{t_E}\sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}\sum_{k=-\infty}^{+\infty}\Upsilon_{lk}\overline{f}_{lmk}(\vec{\boldsymbol{x}},t)P_{lm}(\vec{\boldsymbol{\Omega}})H_k(\vec{\boldsymbol{\Theta}})} \tag{8.35}$$

is achieved. Further on it is shown that in Υ_{lk} the index k may be skipped.

8.4. Calculation of the exchange-coefficients Υ_l

Considering an overall closed volume range V the particle number in the entire volume remains constant if no absorbtion is assumed.

$$\text{total number of particles} = \int_{\mathbb{V}} \int_{4\pi} \int_{2\pi} \overline{f} d\vec{\boldsymbol{\Omega}} d\vec{\boldsymbol{\Theta}} d\mathbb{V} = const. \tag{8.36}$$

$\Longrightarrow$

$$\frac{d}{dt} \int_{\mathbb{V}} \int_{4\pi} \int_{2\pi} \overline{f} d\vec{\boldsymbol{\Omega}} d\vec{\boldsymbol{\Theta}} d\mathbb{V} = \int_{\mathbb{V}} \int_{4\pi} \int_{2\pi} [\frac{\partial \overline{f}}{\partial t} + \overline{v}\vec{\boldsymbol{\Omega}} \times \vec{\boldsymbol{\Theta}} \cdot \nabla \overline{f}] d\vec{\boldsymbol{\Omega}} d\vec{\boldsymbol{\Theta}} d\mathbb{V} = \Upsilon_{0,0} \cdot \boldsymbol{V} = \boldsymbol{0} \tag{8.37}$$

and thus

$$\boxed{\Upsilon_{0,0} = 0}. \tag{8.38}$$

Getting an overview over the exchange function $M_{\boldsymbol{t}_\epsilon}$ the essential relations are presented again with the following equations:

$$\begin{aligned}
M_{\boldsymbol{t}_\epsilon}(\vec{\boldsymbol{\Theta}}' \cdot \vec{\boldsymbol{\Theta}}) = \sum_{k=0}^{+\infty} M_{\boldsymbol{t}_\epsilon k} cos(k\beta) =& \frac{1}{2} \sum_{k=0}^{k=+\infty} M_{\boldsymbol{t}_\epsilon k} [H_k(\vec{\boldsymbol{\Theta}}') H_k^*(\vec{\boldsymbol{\Theta}}) + H_{-k}(\vec{\boldsymbol{\Theta}}') H_{-k}^*(\vec{\boldsymbol{\Theta}})] \\
cos(k\beta) =& \frac{1}{2} [H_k(\vec{\boldsymbol{\Theta}}') H_k^*(\vec{\boldsymbol{\Theta}}) + H_{-k}(\vec{\boldsymbol{\Theta}}') H_{-k}^*(\vec{\boldsymbol{\Theta}})] = \frac{1}{2} [e^{ik(\theta' - \theta)} + e^{-ik(\theta' - \theta)}] \\
\vec{\boldsymbol{\Theta}}' \cdot \vec{\boldsymbol{\Theta}} = cos(\beta) = cos(\theta' - \theta) =& \frac{1}{2} [H_1(\vec{\boldsymbol{\Theta}}') H_1^*(\vec{\boldsymbol{\Theta}}) + H_{-1}(\vec{\boldsymbol{\Theta}}') H_{-1}^*(\vec{\boldsymbol{\Theta}})] = \frac{1}{2} [e^{i(\theta' - \theta)} + e^{-i(\theta' - \theta)}] \\
& \lim_{\epsilon \to 0} M_{\boldsymbol{t}_\epsilon}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\Theta}}' \cdot \vec{\boldsymbol{\Theta}}) = \delta(\vec{\boldsymbol{\Theta}}, \vec{\boldsymbol{\Theta}}') \\
& \int_{2\pi} H_{k'}(\vec{\boldsymbol{\Theta}}) H_k^*(\vec{\boldsymbol{\Theta}}) d\vec{\boldsymbol{\Theta}} = \begin{cases} 2\pi & \text{für k'=k} \\ 0 & \text{else} \end{cases}
\end{aligned}$$

$M_{\boldsymbol{t}_\epsilon}(\vec{\boldsymbol{\Theta}}' \cdot \vec{\boldsymbol{\Theta}}) = \sum_{k=0}^{+\infty} M_{\boldsymbol{t}_\epsilon k} cos(k\beta)$ only takes values essentially different from 0 in an ε-neighborhood of $\beta = 0$, such that $\vec{\boldsymbol{\Theta}}' \cdot \vec{\boldsymbol{\Theta}} = cos(\beta) = 1 - O(\varepsilon^2)$ is sufficient. $\Longrightarrow$

$$2\pi \cdot M_{\boldsymbol{t}_\epsilon k} = \int_{-\pi}^{+\pi} M_{\boldsymbol{t}_\epsilon} cos(k\beta) d\beta = \int_{-\pi}^{+\pi} M_{\boldsymbol{t}_\epsilon} (1 - O(\epsilon)) d\beta = 2\pi \cdot M_{\boldsymbol{t}_\epsilon 0} - O(\epsilon^2). \tag{8.39}$$

On the other hand

$$\int_{2\pi} M_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Theta}} \cdot \vec{\boldsymbol{\Theta}})' d\vec{\boldsymbol{\Theta}}' =$$
$$\frac{1}{2}\int_{2\pi} \sum_{k=0}^{k=+\infty} M_{\boldsymbol{t_\epsilon} k}[H_k(\vec{\boldsymbol{\Theta}}')H_k^*(\vec{\boldsymbol{\Theta}}) + H_{-k}(\vec{\boldsymbol{\Theta}}')H_{-k}^*(\vec{\boldsymbol{\Theta}})]d\vec{\boldsymbol{\Theta}}' = 2\pi \cdot M_{\boldsymbol{t_\epsilon} 0} = 1. \tag{8.40}$$

is valid. $\Longrightarrow$

$$M_{\boldsymbol{t_\epsilon} k} = M_{\boldsymbol{t_\epsilon} 0} = \frac{1}{2\pi}. \tag{8.41}$$

The calculation of the exchange coefficients is not influenced by $M_{\boldsymbol{t_\epsilon}}$ the Υ-values given by

$$\Upsilon_l = \lim_{t_\epsilon \to 0} \frac{(V_{\boldsymbol{t_\epsilon} l} \frac{4\pi}{2l+1} - 1)}{\boldsymbol{t_\epsilon}}. \tag{8.42}$$

Further calculation of the Υ_l analogously happen to section 7.4 with the result

$$\Upsilon_l = -\frac{l(l+1)}{2}\zeta \qquad \zeta = const. \tag{8.43}$$

Now the equation of turbulent particle transport is written

$$\boxed{\frac{\partial \overline{f}}{\partial t} + \overline{v}\vec{\boldsymbol{\Omega}} \times \vec{\boldsymbol{\Theta}} \cdot \nabla \overline{f} = -\frac{1}{t_E} \sum_{l=0}^{+\infty} \sum_{m=-l}^{+l} \sum_{k=-\infty}^{+\infty} \frac{l(l+1)}{2} \overline{f}_{lmk}(\vec{\boldsymbol{x}}, t) P_{lm}(\vec{\boldsymbol{\Omega}}) H_k(\vec{\boldsymbol{\Theta}})} \tag{8.44}$$

the coefficient $\frac{\zeta}{t_E}$ replaced by $\frac{1}{t_E}$. A more complicated dependency of $t_E = t_E(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\Omega}})$ possibly remains. Maybe, physically justified simplifications lead to practical solutions. The below presented theory of deterministic turbulence enables the calcultion of these coefficients by numerical evaluation.

Die total derivative with respect to time gives

$$\begin{aligned}\frac{d}{dt}\overline{f}(\vec{\boldsymbol{x}},t,\vec{\boldsymbol{\Omega}},\vec{\boldsymbol{\Theta}}) &= \sum_{l=0}^{+\infty}\sum_{m=-l}^{m=+l}\sum_{k=-\infty}^{+\infty}\frac{d}{dt}\overline{f}_{lmk}(\vec{\boldsymbol{x}},t)P_{lm}(\vec{\boldsymbol{\Omega}})H_k(\vec{\boldsymbol{\Theta}}) \\ &= \frac{1}{t_E}\sum_{l=1}^{+\infty}\boldsymbol{\gamma_l}\cdot\sum_{m=-l}^{+l}\sum_{k=-\infty}^{+\infty}\overline{f}_{lmk}(\vec{\boldsymbol{x}},t)P_{lm}(\vec{\boldsymbol{\Omega}})H_k(\vec{\boldsymbol{\Theta}}).\end{aligned} \tag{8.45}$$

The time behavior of the single modes are obtained by

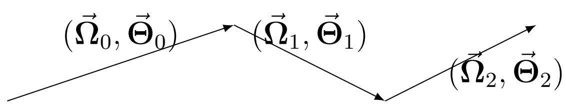
$$\frac{d}{dt}\overline{f}_{lmk}(t) = \frac{\gamma_l}{t_E}\overline{f}_{lmk} \tag{8.46}$$

$$\overline{f}_{lmk}(t) \sim \boldsymbol{exp}(\frac{\gamma_l}{t_E}\cdot t). \tag{8.47}$$

The greater the order l the more powerful is its temporal decay.

8.5. Reconstruction of the transition probabilities $\overline{W}_{\boldsymbol{t_\epsilon}}$

The transition probability $\widetilde{W}_{\boldsymbol{t_\epsilon},0\to 1\to 2}$, a particle changing its motion pair of directions $(\vec{\boldsymbol{\Omega}},\vec{\boldsymbol{\Theta}})$ at the times t_0, t_1, t_2 from $(\vec{\boldsymbol{\Omega}}_0,\vec{\boldsymbol{\Theta}}_0)$ via $(\vec{\boldsymbol{\Omega}}_1,\vec{\boldsymbol{\Theta}}_1)$ to $(\vec{\boldsymbol{\Omega}}_2,\vec{\boldsymbol{\Theta}}_2)$,



results out of the product of the single probabilities of the pairs of directions (vortex vector and radius vector direction of motion in a circle segment). The grafical presentation is meant symbolically because such a pair of directions does not compose to an overall direction. $\vec{\boldsymbol{\Omega}}_i$ is always orthogonal to $\vec{\boldsymbol{\Theta}}_i$. A vectorial overall direction of $\vec{\boldsymbol{\Omega}}_i$ and $\vec{\boldsymbol{\Theta}}_i$ has no physical meaning in the 3 dimensional space. [2]

$$\widetilde{W}_{\boldsymbol{t_\epsilon},0\to 1\to 2} = \widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{2}}(\vec{\boldsymbol{\Omega}}_0\cdot\vec{\boldsymbol{\Omega}}_1,\vec{\boldsymbol{\Theta}}_0\cdot\vec{\boldsymbol{\Theta}}_1)\cdot\widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{2}}(\vec{\boldsymbol{\Omega}}_1\cdot\vec{\boldsymbol{\Omega}}_2,\vec{\boldsymbol{\Theta}}_1\cdot\vec{\boldsymbol{\Theta}}_2) \tag{8.48}$$

[2] $\vec{\boldsymbol{\Omega}},\vec{\boldsymbol{\Theta}}$ would make a single direction vector in a 4-dimensional space. The longitudinal fluctuations in the 4-dimensional space should accord to turbulence in the 3-dimensional space.

The probability , that a particle changes its pair of directions within a time $t_\varepsilon = \varepsilon \cdot t_E$ from $(\vec{\mathbf{\Omega}}_0, \vec{\mathbf{\Theta}}_0)$ to $(\vec{\mathbf{\Omega}}_2, \vec{\mathbf{\Theta}}_2)$, is obtained by

$$\widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{\Omega}}_0 \cdot \vec{\mathbf{\Omega}}_2, \vec{\mathbf{\Theta}}_0 \cdot \vec{\mathbf{\Theta}}_2) = \int_{2\pi}\int_{4\pi} \widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{2}}(\vec{\mathbf{\Omega}}_0 \cdot \vec{\mathbf{\Omega}}_1, \vec{\mathbf{\Theta}}_0 \cdot \vec{\mathbf{\Theta}}_1) \boldsymbol{\cdot} \widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{2}}(\vec{\mathbf{\Omega}}_1 \cdot \vec{\mathbf{\Omega}}_2, \vec{\mathbf{\Theta}}_1 \cdot \vec{\mathbf{\Theta}}_2) d\vec{\mathbf{\Omega}}_1 d\vec{\mathbf{\Theta}}_1. \tag{8.49}$$

The evolution coefficients of the transition probability are

$$\widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{\mathbf{2}} l} = \left\{1 + \frac{\epsilon \cdot \Upsilon_l}{2}\right\} \frac{2l+1}{4\pi} \cdot \frac{1}{2\pi} \tag{8.50}$$

and therefore

$$\begin{aligned} \widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{2}}(\vec{\mathbf{\Omega}}_0 \cdot \vec{\mathbf{\Omega}}_1, \vec{\mathbf{\Theta}}_0 \cdot \vec{\mathbf{\Theta}}_1) = & \sum_{l=0}^{+\infty} \left\{1 + \frac{\epsilon \cdot \Upsilon_l}{2}\right\} \frac{2l+1}{4\pi} \cdot \frac{1}{2\pi} \sum_{m=-l}^{+l} P_{lm}(\vec{\mathbf{\Omega}}_1) P^*_{lm}(\vec{\mathbf{\Omega}}_0) \\ & \cdot \frac{1}{2} \sum_{k=-\infty}^{+\infty} [H_k(\vec{\mathbf{\Theta}}_1) H^*_k(\vec{\mathbf{\Theta}}_0) + H_{-k}(\vec{\mathbf{\Theta}}_1) H^*_{-k}(\vec{\mathbf{\Theta}}_0)]. \end{aligned} \tag{8.51}$$

respectively

$$\begin{aligned} \widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{2}}(\vec{\mathbf{\Omega}}_1 \cdot \vec{\mathbf{\Omega}}_2, \vec{\mathbf{\Theta}}_1 \cdot \vec{\mathbf{\Theta}}_2) = & \sum_{l=0}^{+\infty} \left\{1 + \frac{\epsilon \cdot \Upsilon_l}{2}\right\} \frac{2l+1}{4\pi} \cdot \frac{1}{2\pi} \sum_{m=-l}^{+l} P_{lm}(\vec{\mathbf{\Omega}}_2) P^*_{lm}(\vec{\mathbf{\Omega}}_1) \\ & \cdot \frac{1}{2} \sum_{k=-\infty}^{+\infty} [H_k(\vec{\mathbf{\Theta}}_2) H^*_k(\vec{\mathbf{\Theta}}_1) + H_{-k}(\vec{\mathbf{\Theta}}_2) H^*_{-k}(\vec{\mathbf{\Theta}}_1)]. \end{aligned} \tag{8.52}$$

Integrating (8.49) one obtains

$$\begin{aligned} \widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{\Omega}}_0 \cdot \vec{\mathbf{\Omega}}_2, \vec{\mathbf{\Theta}}_0 \cdot \vec{\mathbf{\Theta}}_2) = & \sum_{l=0}^{+\infty} \left\{1 + \frac{\epsilon \cdot \Upsilon_l}{2}\right\}^2 \frac{2l+1}{4\pi} \cdot \frac{1}{2\pi} \sum_{m=-l}^{+l} P^*_{lm}(\vec{\mathbf{\Omega}}_0) P_{lm}(\vec{\mathbf{\Omega}}_2) \\ & \cdot \frac{1}{2} \sum_{k=-\infty}^{+\infty} [H_k(\vec{\mathbf{\Theta}}_0) H^*_k(\vec{\mathbf{\Theta}}_2) + H_{-k}(\vec{\mathbf{\Theta}}_0) H^*_{-k}(\vec{\mathbf{\Theta}}_2)]. \end{aligned} \tag{8.53}$$

Using n intermediate stages $\widetilde{W}_{\boldsymbol{t_\epsilon}}$ is expressed by an integral over the product of the single transition probabilities.

$$\widetilde{W}_{\boldsymbol{t_\epsilon},0\to1\ldots\to n} = \widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{n}}(\vec{\boldsymbol{\Omega}}_0\cdot\vec{\boldsymbol{\Omega}}_1,\vec{\boldsymbol{\Theta}}_0\cdot\vec{\boldsymbol{\Theta}}_1)\cdot\widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{n}}(\vec{\boldsymbol{\Omega}}_1\cdot\vec{\boldsymbol{\Omega}}_2,\vec{\boldsymbol{\Theta}}_1\cdot\vec{\boldsymbol{\Theta}}_2)\ldots.\widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{n}}(\vec{\boldsymbol{\Omega}}_{n-1}\cdot\vec{\boldsymbol{\Omega}}_n,\vec{\boldsymbol{\Theta}}_{n-1}\cdot\vec{\boldsymbol{\Theta}}_n) \tag{8.54}$$

$$\widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Omega}}_0\cdot\vec{\boldsymbol{\Omega}}_n,\vec{\boldsymbol{\Theta}}_0\cdot\vec{\boldsymbol{\Theta}}_n) = \int_{2\pi}\int_{4\pi}\int_{2\pi}\int_{4\pi}\ldots.\int_{2\pi}\int_{4\pi}\widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{n}}\cdot\widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{n}}\ldots.\widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{n}}\,d\vec{\boldsymbol{\Omega}}_1 d\vec{\boldsymbol{\Theta}}_1\ldots.d\vec{\boldsymbol{\Omega}}_{n-1}d\vec{\boldsymbol{\Theta}}_{n-1} \tag{8.55}$$

$$\begin{aligned}&\widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Omega}}\cdot\vec{\boldsymbol{\Omega}}',\vec{\boldsymbol{\Theta}}\cdot\vec{\boldsymbol{\Theta}}') = \lim_{\boldsymbol{n}\to\infty}\sum_{l=0}^{+\infty}\left\{1+\frac{\epsilon\cdot\Upsilon_l}{2}\right\}^n\frac{2l+1}{4\pi}\\ &\cdot\frac{1}{2\pi}\cdot\sum_{m=-l}^{+l}P^*_{lm}(\vec{\boldsymbol{\Omega}})P_{lm}(\vec{\boldsymbol{\Omega}}')\cdot\frac{1}{2}\sum_{k=-\infty}^{+\infty}[H_k(\vec{\boldsymbol{\Theta}}')H^*_k(\vec{\boldsymbol{\Theta}})+H_{-k}(\vec{\boldsymbol{\Theta}}')H^*_{-k}(\vec{\boldsymbol{\Theta}})]\end{aligned} \tag{8.56}$$

For $n\to\infty$ arises

$$\lim_{\boldsymbol{n}\to\infty}\left\{1+\frac{\epsilon\cdot\Upsilon_l}{2}\right\}^n = e^{\Upsilon_l\cdot\epsilon} \tag{8.57}$$

and using (8.55)

$\Longrightarrow$

$$\boxed{\begin{gathered}\widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Omega}}\cdot\vec{\boldsymbol{\Omega}}',\vec{\boldsymbol{\Theta}}\cdot\vec{\boldsymbol{\Theta}}') =\\ \sum_{l=0}^{+\infty}e^{\Upsilon_l\cdot\epsilon}\frac{2l+1}{4\pi}\cdot\frac{1}{2\pi}\cdot\sum_{m=-l}^{+l}P^*_{lm}(\vec{\boldsymbol{\Omega}})P_{lm}(\vec{\boldsymbol{\Omega}}')\cdot\frac{1}{2}\sum_{k=-\infty}^{+\infty}[H_k(\vec{\boldsymbol{\Theta}}')H^*_k(\vec{\boldsymbol{\Theta}})+H_{-k}(\vec{\boldsymbol{\Theta}}')H^*_{-k}(\vec{\boldsymbol{\Theta}})]\end{gathered}}\,. \tag{8.58}$$

Choosing $\boldsymbol{\varepsilon} = \frac{\boldsymbol{t_\varepsilon}}{\boldsymbol{t_E(\vec{x},t,\vec{\Omega})}}$ the exchange function $\widetilde{W}_{\boldsymbol{t_\epsilon}}$ may be understood in the dependencies

$$\widetilde{W}_{\boldsymbol{t_\epsilon}} = \widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}},t,\vec{\boldsymbol{\Omega}}\cdot\vec{\boldsymbol{\Omega}}',\vec{\boldsymbol{\Theta}}\cdot\vec{\boldsymbol{\Theta}}') \tag{8.59}$$

and

$$\overline{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}},t,\vec{\boldsymbol{\Omega}},\vec{\boldsymbol{\Theta}},\vec{\boldsymbol{\Omega}}',\vec{\boldsymbol{\Theta}}') \approx \widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}},t,\vec{\boldsymbol{\Omega}}\cdot\vec{\boldsymbol{\Omega}}',\vec{\boldsymbol{\Theta}}\cdot\vec{\boldsymbol{\Theta}}') \tag{8.60}$$

is given, too. $\Longrightarrow$

$$\boxed{\begin{gathered}\overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}},\vec{\boldsymbol{\Omega}},\vec{\boldsymbol{\Theta}},t)=\int_{2\pi}\int_{4\pi}\widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}},t,\vec{\boldsymbol{\Omega}},\vec{\boldsymbol{\Theta}},\vec{\boldsymbol{\Omega}}',\vec{\boldsymbol{\Theta}}')\overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}-t_\varepsilon\cdot\overline{v}'\vec{\boldsymbol{\Omega}}'\times\vec{\boldsymbol{\Theta}}',\vec{\boldsymbol{\Omega}}',\vec{\boldsymbol{\Theta}}'),t-t_{\boldsymbol{\epsilon}})d\vec{\boldsymbol{\Omega}}'d\vec{\boldsymbol{\Theta}}'\\ \overline{v}'=\overline{v}'(\vec{\mathbf{x}},\vec{\boldsymbol{\Omega}}',\vec{\boldsymbol{\Theta}}',t)=\overline{\omega}'(\vec{\mathbf{x}},\vec{\boldsymbol{\Omega}}',\vec{\boldsymbol{\Theta}}',t)\cdot\overline{r}'(\vec{\mathbf{x}},\vec{\boldsymbol{\Omega}}',\vec{\boldsymbol{\Theta}}',t)\end{gathered}} \tag{8.61}$$

9. Appendix

9.1. Legendre-Polynomials

The Legendre-polynomials are defined within the interval $[-1, +1]$ by

$$P_n = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n, n \in N. \tag{9.1}$$

They represent a complete orthogonal function system with

$$\int_{-1}^{+1} P_n(x) P_m(x) dx = \begin{cases} \frac{2}{2m+1} & \text{for m} = n \\ 0 & \text{else.} \end{cases} \tag{9.2}$$

Every continuously differentiable function f(x) defined within [-1,+1] can be developed by Legendre-polynomials according to

$$f(x) = \sum_{l=0}^{\infty} f_l P_l(x). \tag{9.3}$$

The f_l are the evolution coefficients. A presentation of the $\delta - function$ by Legendre-polynomials is obtained by

$$\delta(x, x') = \sum_{l=0}^{\infty} \frac{2m+1}{2} P_l(x) P_l(x') \ . \tag{9.4}$$

Important recurrence equations are [42]

$$\begin{aligned} (n+1)P_{n+1} &= (2n+1)xP_n(x) - nP_{n-1}(x) \\ P'_{n+1}(x) - xP'_n(x) &= (n+1)P_n(x) \ , n = 0, 1, 2, ... \\ (1 - x^2)P'_n(x) &= nP_{n-1}(x) - nxP_n(x). \end{aligned} \tag{9.5}$$

An integral representation of the Legendre-polynomials is obtained by

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi}(x + \sqrt{x^2-1}cos(\varphi))^n d\varphi. \tag{9.6}$$

Owing to $|x + \sqrt{x^2-1}cos(\theta)| = |cos(\theta) + isin(\theta)cos(\theta)| \leqslant 1$

$$|P_n(x)| \leqslant 1 \tag{9.7}$$

follows. These polynomials have their maximum for $x = 1$, particularly

$$P_n(1) = 1. \tag{9.8}$$

$$\boxed{\frac{dP_l(x)}{dx}|_1 = \frac{l(l+1)}{2}} \tag{9.9}$$

is proved by complete induction.

Proof :

1.$P_0'(1) = 0$

Assumption:
2.$P_n'(1) = \frac{n(n+1)}{2}$

$\Longrightarrow$

3.$P_{n+1}'(1) = \frac{(n+2)(n+1)}{2}$ wegen (9.5) $P_{n+1}'(1) - P_n'(1) = (n+1)P_n(1)$
q.e.d.

9.2. Spherical harmonics

The Spherical harmonics [[43] page 224] represent a complete orthogonal, complex function system on the spherical surface

$$\begin{aligned} P_{lm}(\vec{\boldsymbol{\Omega}}) =& e^{im\varphi}\frac{(-sin(\vartheta))^m}{l!2^l} \cdot \left(\frac{(l-m)!}{(l+m)!}\right)^{\frac{1}{2}} \frac{d^{l+m}(cos^2\vartheta-1)^l}{(dcos\vartheta)^{l+m}} \\ =& e^{im\varphi}\frac{(sin(\vartheta))^{-m}}{l!2^l} \cdot \left(\frac{(l+m)!}{(l-m)!}\right)^{\frac{1}{2}} \frac{d^{l-m}(cos^2\vartheta-1)^l}{(dcos\vartheta)^{l-m}} \end{aligned} \tag{9.10}$$

with

$$P_{l,-m}(\vec{\boldsymbol{\Omega}}) = (-)^m P^*_{lm}(\vec{\boldsymbol{\Omega}}) \tag{9.11}$$

and

$$\int_{4\pi} d\vec{\boldsymbol{\Omega}} P_{l'm'}(\vec{\boldsymbol{\Omega}}) P^*_{lm}(\vec{\boldsymbol{\Omega}}) = \delta_{l'l}\delta_{m'm}\frac{4\pi}{2l+1} \ . \tag{9.12}$$

All continuously differentiable functions on the spherical surface $f(\Omega) = f(\theta, \phi)$ can be developed according to

$$f(\vec{\boldsymbol{\Omega}}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{m=+l} f_{lm}P_{lm}(\vec{\boldsymbol{\Omega}}) \tag{9.13}$$

the f_{lm} representing the evolution coefficients. The $P^*_{lm}(\vec{\boldsymbol{\Omega}})$ being complex to $P_{lm}(\vec{\boldsymbol{\Omega}})$ $f(\vec{\boldsymbol{\Omega}})$ can be alternatively considered

$$f(\vec{\boldsymbol{\Omega}}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{m=+l} f_{lm}P^*_{lm}(\vec{\boldsymbol{\Omega}}). \tag{9.14}$$

The spherical harmonics for $l = 0, 1$ are

$$\begin{aligned} P_{00} =& P^*_{00} = 1 \\ P_{1,-1}(\vec{\boldsymbol{\Omega}}) =& 2^{-\frac{1}{2}}e^{-i\varphi}sin(\vartheta),\ P^*_{1,-1} = 2^{-\frac{1}{2}}e^{i\varphi}sin(\vartheta) \\ P_{1,0}(\vec{\boldsymbol{\Omega}}) =& P^*_{1,0}(\vec{\boldsymbol{\Omega}}) = cos(\vartheta) = P_1(\vec{\boldsymbol{\Omega}}) \\ P_{1,1}(\vec{\boldsymbol{\Omega}}) =& -2^{-\frac{1}{2}}e^{i\varphi}sin(\vartheta),\ P^*_{1,1}(\vec{\boldsymbol{\Omega}}) = -2^{-\frac{1}{2}}e^{-i\varphi}sin(\vartheta) \ . \end{aligned} \tag{9.15}$$

The connection of spherical harmonics and Legendre-polynomials is obtained by

$$P_{l0} = P_{l0}^{*} = P_l. \tag{9.16}$$

Furthermore the addition theorem

$$P_l(cos(\vartheta)) = \sum_{m=-l}^{m=+l} P_{lm}(\vec{\mathbf{\Omega}}^{'}) P_{lm}^{*}(\vec{\mathbf{\Omega}}) \tag{9.17}$$

matters with

$$cos(\vartheta) = \vec{\mathbf{\Omega}}^{'} \cdot \vec{\mathbf{\Omega}}. \tag{9.18}$$

The $\delta-$function depending on the spherical harmonics may be stated by

$$\delta(\vec{\mathbf{\Omega}}, \vec{\mathbf{\Omega}}^{'}) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} \sum_{m=-l}^{m=+l} P_{lm}(\vec{\mathbf{\Omega}}) P_{lm}^{*}(\vec{\mathbf{\Omega}}^{'}) \tag{9.19}$$

and

$$\delta(\vec{\mathbf{\Omega}}, \vec{\mathbf{\Omega}}^{'}) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} P_l(\vec{\mathbf{\Omega}} \cdot \vec{\mathbf{\Omega}}^{'}). \tag{9.20}$$

9.3. Turbulence-functions

Functions of the unit direction vectors $\vec{\mathbf{\Omega}}$ and $\vec{\mathbf{\Theta}}$ are represented by a complete orthogonal function system meaning an extension of the spherical harmonics. We call them turbulence functions.

$$\begin{aligned} Q_{lmk}(\vec{\mathbf{\Omega}}, \vec{\mathbf{\Theta}}) =& P_{lm}(\vec{\mathbf{\Omega}}) H_k(\vec{\mathbf{\Theta}}) \\ & P_{lm}(\vec{\mathbf{\Omega}}) \qquad \text{spherical harmonics} \\ \int_{2\pi} H_{k'}(\vec{\mathbf{\Theta}}) H_k^{*}(\vec{\mathbf{\Theta}}) d\vec{\mathbf{\Theta}} =& \begin{cases} 2\pi & \text{for k'=k} \\ 0 & \text{else} \end{cases} \\ H_k(\vec{\mathbf{\Theta}}) =& e^{ik\theta} \end{aligned} \tag{9.21}$$

$$cos(\vartheta) = \vec{\mathbf{\Omega}}^{'} \cdot \vec{\mathbf{\Omega}}. \tag{9.22}$$

with

$$\int_{2\pi}\int_{4\pi} Q_{lmk}(\vec{\mathbf{\Omega}},\vec{\mathbf{\Theta}})Q^{*}_{lmk}(\vec{\mathbf{\Omega}}',\vec{\mathbf{\Theta}}')d\vec{\mathbf{\Omega}}'d\vec{\mathbf{\Theta}}' = \begin{cases} \frac{8\pi^2}{2l+1} & \text{for } l=l'; m=m'; k=k' \\ 0 & \text{else} \end{cases} \tag{9.23}$$

Such, suitable distribution functions are described by

$$\begin{aligned} f_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}},t,\vec{\mathbf{\Omega}},\vec{\mathbf{\Theta}}) &= \sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}\sum_{k=-\infty}^{+\infty} f_{lmk}(\vec{\boldsymbol{x}},t)Q_{lmk}(\vec{\mathbf{\Omega}},\vec{\mathbf{\Theta}}) \\ f(\vec{\boldsymbol{x}},t,\vec{\mathbf{\Omega}},\mathbf{\Theta}) &= \sum_{l=0}^{+\infty}\sum_{m=-l}^{+l} P_{lm}(\vec{\mathbf{\Omega}})\sum_{k=-\infty}^{+\infty} f_{lmk}(\vec{\boldsymbol{x}},t)H_k(\vec{\mathbf{\Theta}}). \end{aligned} \tag{9.24}$$

Die $\delta-$function depending on the turbulence functions is expressed

$$\delta(\vec{\mathbf{\Omega}},\vec{\mathbf{\Omega}}^{'};\vec{\mathbf{\Theta}},\vec{\mathbf{\Theta}}^{'}) = \sum_{l=0}^{\infty}\frac{2l+1}{4\pi}\sum_{m=-l}^{m=+l} P_{lm}(\vec{\mathbf{\Omega}})P^{*}_{lm}(\vec{\mathbf{\Omega}}^{'})\sum_{k=-\infty}^{+\infty}\frac{1}{2\pi}H_k(\vec{\mathbf{\Theta}})H^{*}_k(\vec{\mathbf{\Theta}}^{'}) \tag{9.25}$$

and such

$$\delta(\vec{\mathbf{\Omega}},\vec{\mathbf{\Omega}}^{'};\vec{\mathbf{\Theta}},\vec{\mathbf{\Theta}}^{'}) = \frac{1}{8\pi^2}\sum_{l=0}^{\infty}(2l+1)P_l(\vec{\mathbf{\Omega}}\cdot\vec{\mathbf{\Omega}}^{'})\sum_{k=-\infty}^{+\infty} exp(ik(\Theta-\Theta^{'})). \tag{9.26}$$

9.4. Euler-angles as fluctuation properties of the turbulent particle transport

The angles respectively unit direction vectors $\vec{\mathbf{\Omega}}$ and $\vec{\mathbf{\Theta}}$ of turbulent motions are applied using the turbulence functions. The unit vector $\vec{\mathbf{\Omega}}\times\vec{\mathbf{\Theta}}$ with $\vec{\mathbf{\Omega}}\perp\vec{\mathbf{\Theta}}$ depending on the angles θ, φ and ϑ is determined. Initially, the direction vector $\vec{\mathbf{\Omega}}$

$$\vec{\mathbf{\Omega}}^{0} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{9.27}$$

may be given before a rotation. The orthogonal direction vector $\vec{\mathbf{\Theta}}^{0}$ may be descripted in this starting situation by

$$\vec{\mathbf{\Theta}}^0 = \begin{pmatrix} sin\theta \\ cos\theta \\ 0 \end{pmatrix} \tag{9.28}$$

The rotation $\mathbf{T} = \mathbf{T}_2 \cdot \mathbf{T}_1$ with

$$\mathbf{T}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\vartheta & sin\vartheta \\ 0 & -sin\vartheta & cos\vartheta \end{pmatrix} \tag{9.29}$$

and

$$\mathbf{T}_2 = \begin{pmatrix} cos\varphi & sin\varphi & 0 \\ -sin\varphi & cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{9.30}$$

results in

$$\mathbf{T} = \mathbf{T}_2 \cdot \mathbf{T}_1 = \begin{pmatrix} cos\varphi & sin\varphi cos\vartheta & sin\varphi sin\vartheta \\ -sin\varphi & cos\varphi cos\vartheta & cos\varphi sin\vartheta \\ 0 & -sin\vartheta & cos\vartheta \end{pmatrix} \tag{9.31}$$

with the unit vectors

$$\begin{aligned} \vec{\mathbf{\Theta}} = \mathbf{T} \cdot \vec{\mathbf{\Theta}}^0 &= \begin{pmatrix} cos\varphi sin\theta + sin\varphi cos\vartheta cos\theta \\ -sin\varphi sin\theta + cos\varphi cos\vartheta cos\theta \\ -sin\vartheta cos\theta \end{pmatrix} \\ \vec{\mathbf{\Omega}} = \mathbf{T} \cdot \vec{\mathbf{\Omega}}^0 &= \begin{pmatrix} sin\varphi sin\vartheta \\ cos\varphi sin\vartheta \\ cos\vartheta \end{pmatrix} \end{aligned} \tag{9.32}$$

and

$$\begin{aligned} \vec{\mathbf{\Omega}} \times \vec{\mathbf{\Theta}} &= \begin{pmatrix} sin\varphi sin\vartheta \\ cos\varphi sin\vartheta \\ cos\vartheta \end{pmatrix} \times \begin{pmatrix} cos\varphi sin\theta + sin\varphi cos\vartheta cos\theta \\ -sin\varphi sin\theta + cos\varphi cos\vartheta cos\theta \\ -sin\vartheta cos\theta \end{pmatrix} \\ &= \begin{pmatrix} -cos\varphi cos\theta + sin\varphi cos\vartheta sin\theta \\ cos\varphi cos\vartheta sin\theta + sin\varphi cos\theta \\ -sin\vartheta sin\theta \end{pmatrix}. \end{aligned} \tag{9.33}$$

Part III.

Deterministic continuum-fluctuations and their stochastic view within the meaning of an ensemble-theory

10. Introduction

In part II a probabilistic theory of turbulent particle transport is developed from a stochastic ensemble consideration of an unlimited number of parallelly existent, deterministic continuum fluctuations. In this part III the relation of partial differential equations of deterministic continuum fluctuations to the stochastic ensemble-counterpart is established. The causal Markov Process (section 4.5) matters, essentially. Its local description leads to two vector fields with a dual pair of coupled partial, quasilinear differential vector equations distinguishing between mass transport and transport of pure motion quantities $\partial\vec{\mathbf{A}}/\partial t$ and $\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}}$ of fluctuating vector fields $\vec{\mathbf{A}}$.

Chapter 11: Turbulent motions have the local velocities $\vec{\mathbf{v}} = \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}$ resulting in a dual equation system of a vortex field $\vec{\boldsymbol{\omega}}$ and a curvature vector field $\vec{\mathbf{b}}$. Including the underlying momentum equations (not Navier-Stokes-equations) this system is not yet complete.

Chapter 12: The deterministic transport of pure motion quantities of sufficiently often continuously differentiable fields $\partial\vec{\mathbf{A}}/\partial t$ and $\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}}$ is examined leading to a pair of dual coupled vector equations. Depending on interpretation they may be viewed as deformation fluctuation-equations, as generalisations of the Maxwell Equations of vacuum or applied as equations of $\partial\vec{\mathbf{v}}/\partial t$ and $\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{v}}$ of the turbulent velocity field $\vec{\mathbf{v}}$.

Chapter 13: After a discussion of possible momentum equations as foundation for turbulence-calculations the results of chapter 11 and 12 are combined to a complete turbulence-equation system. This system consists of 12 equations with 12 unknowns.[1] From an initial velocity field $\vec{\mathbf{v}}(\vec{\boldsymbol{x}}, t_0)$ and its partial, temporal derivation $\frac{\partial}{\partial t}\vec{\mathbf{v}}(\vec{\boldsymbol{x}}, t)|_{t_0}$ the further evolution of the velocity field, its related vortex- and curvature fields as well as the accelleration field $\vec{\mathbf{q}}$ operating in the turbulence field may be calculated.[2] The accelleration field generally is not conservative meaning $\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}} \neq \mathbf{0}$. Matter density distributions may be determined via the continuum-equation in the frame

[1] The Einstein Equations of General Relativity consist of 10 equations. Suitable evolution equations with initial- and possibly boundary conditions remain troublesome in a 3+1-geometry.

[2] As there are only motion quantities in this equation system, it is successfully used for evolution problems in General Relativity.

of a subsequent evaluation in consequence of thermodynamic state quantities beeing computable (as far as a local thermodynamic equilibrium is existent).

11. Deterministic turbulent mass-transport and its stochastic formulation

$$\boxed{\begin{gathered} f_{t_\epsilon}(t,\vec{\mathbf{x}},\vec{\boldsymbol{\omega}},\vec{\mathbf{r}}) = \int\limits_{\vec{\boldsymbol{\omega}}'}\int\limits_{\vec{\boldsymbol{r}}'} W_{t_\epsilon}(t,\vec{\mathbf{x}},\vec{\boldsymbol{\omega}},\vec{\mathbf{r}},\vec{\boldsymbol{\omega}'},\vec{\boldsymbol{r}'})\cdot f_{t_\epsilon}(\vec{\mathbf{x}}-\Delta\vec{\mathbf{x}},t-t_\epsilon,\vec{\boldsymbol{\omega}}',\vec{\mathbf{r}}')d\vec{\boldsymbol{\omega}}'d\vec{\mathbf{r}'} \\ \Updownarrow \\ \frac{\partial}{\partial t}\vec{\boldsymbol{\omega}} - \vec{\boldsymbol{\nabla}}\times\vec{\mathbf{a}} - \frac{1}{2}\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{q}} = 0 \\ \frac{\partial}{\partial t}\vec{\mathbf{b}} + \vec{\boldsymbol{\nabla}}\times\vec{\omega} - \frac{1}{2}\vec{\mathbf{b}}\left[\frac{\vec{\boldsymbol{\omega}}}{\omega^2}\cdot\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{q}}\right] = 0 \end{gathered}}$$

11.1. Introduction

A stochastical ensemble-consideration of deterministic fields is understood as the examination of an unlimited number of comparable, parallelly existent systems, analogously to chapter 4. In this case turbulently moved fluids are examined considering statistical deliberations and its deterministic counterparts. That a linking of deterministic and stochastic theory may be available and further more that out of this connection additionally important (sometimes otherwise not known) relations arise for deterministic formulations, is shown in the following. This is discussed for a turbulent mass transport.

11.2. The transition: stochastic theory $\longleftrightarrow$ deterministic theory

Every space-time-point $(\vec{\boldsymbol{x}},t)$ a continuously differentiable fluid element distribution over the motion quantities $\vec{\omega}_{t_\epsilon}$ and $\vec{\mathbf{r}}_{t_\epsilon}$ is assigned according to

$$f_{\boldsymbol{t_\epsilon}} = f_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\omega}}, \vec{\mathbf{r}}). \tag{11.1}$$

Indexing functions with $\boldsymbol{t_\epsilon}$ it is automatically assumed that the included motion quantities $(\vec{\boldsymbol{\omega}}, \vec{\mathbf{r}})$ are assigned to a $\boldsymbol{t_\epsilon}$-measurement accuracy. The indexing of the motion quantities may be omitted in the functions if the functions are accordingly indexed.

After an execution of a $\lim \boldsymbol{t_\epsilon} \to 0$ process, such as

$$\lim_{\boldsymbol{t_\epsilon} \to 0} f_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\omega}}, \vec{\boldsymbol{r}}) = f(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\omega}}, \vec{\boldsymbol{r}}) \tag{11.2}$$

f and $(\vec{\boldsymbol{\omega}}, \vec{\mathbf{r}})$ are understood as results of an exact measuring process.

The change of motion quantities in point $(\vec{\boldsymbol{x}}, t)$

$$\Big(\vec{\boldsymbol{\omega}}'_{t_\epsilon}(\vec{\boldsymbol{x}} - \Delta\vec{\boldsymbol{x}}, t - t_\epsilon), \vec{\boldsymbol{r}}'_{t_\epsilon}(\vec{\boldsymbol{x}} - \Delta\vec{\boldsymbol{x}}, t - t_\epsilon)\Big) \longrightarrow \Big(\vec{\boldsymbol{\omega}}_{t_\epsilon}(\vec{\boldsymbol{x}}, t), \vec{\boldsymbol{r}}_{t_\epsilon}(\vec{\boldsymbol{x}}, t)\Big)$$

is controlled by the transition probability density $W_{t_\epsilon} = W_{t_\epsilon}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\omega}}, \vec{\boldsymbol{r}}, \vec{\boldsymbol{\omega}}', \vec{\boldsymbol{r}}')$. [1] with

$$\boxed{\begin{aligned} \lim_{t_\epsilon \to 0} W_{t_\epsilon} &= \delta(\vec{\boldsymbol{\omega}}, \vec{\mathbf{r}}; \vec{\boldsymbol{\omega}}', \vec{\mathbf{r}}') \\ f_{t_\epsilon}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\omega}}, \vec{\mathbf{r}}) &= \int\limits_{\vec{\mathbf{r}}} \int\limits_{\vec{\boldsymbol{\omega}}} W_{t_\epsilon}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\omega}}, \vec{\mathbf{r}}, \vec{\boldsymbol{\omega}}', \vec{\mathbf{r}}') \cdot f_{t_\epsilon}(\vec{\mathbf{x}} - \Delta\vec{\mathbf{x}}, t - t_\epsilon, \vec{\boldsymbol{\omega}}', \vec{\mathbf{r}}') d\vec{\boldsymbol{\omega}}' d\vec{\mathbf{r}}' \\ \Delta\vec{\mathbf{x}} &= t_\epsilon \cdot \vec{\boldsymbol{\omega}}' \times \vec{\mathbf{r}}' \end{aligned}} . \tag{11.3}$$

These equations characterize stochastic turbulence of the continuum in the frame of an ensemble theory and represent a Markov Process with natural causality.

f_{t_ϵ} is developed in (11.3) until the 1st order around $(\vec{\mathbf{x}}, t) \Longrightarrow$

$$f_{t_\epsilon}(\vec{\mathbf{x}} - \triangle\vec{\mathbf{x}}, t - t_\epsilon, \vec{\boldsymbol{\omega}}', \vec{\mathbf{r}}') = f_{t_\epsilon}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\omega}}', \vec{\mathbf{r}}') - \frac{\partial f'_{t_\epsilon}}{\partial t} \cdot t_\epsilon - \triangle\vec{\mathbf{x}} \cdot \vec{\boldsymbol{\nabla}} f_{t_\epsilon}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\omega}}', \vec{\mathbf{r}}') + \boldsymbol{O}({t_\epsilon}^2) \tag{11.4}$$

[1] The otherwise in distribution theory used test functions in this connection have an immediate physical meaning with the formulation of the transition probability density.

with $f'_{t_\epsilon} = f_{t_\epsilon}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\omega}}', \vec{\mathbf{r}}')$ and one obtains

$$\int_{\vec{\mathbf{r}}}\int_{\vec{\boldsymbol{\omega}}} W_{t_\epsilon}[\frac{\partial f'_{t_\epsilon}}{\partial t} + \vec{\boldsymbol{\omega}}' \times \vec{\mathbf{r}}' \cdot \vec{\boldsymbol{\nabla}} f'_{t_\epsilon}] d\vec{\boldsymbol{\omega}}' d\vec{\mathbf{r}}' + \boldsymbol{O}({t_\epsilon}^2) = \frac{\int_{\vec{\mathbf{r}}}\int_{\vec{\boldsymbol{\omega}}} W_{t_\epsilon} f'_{t_\epsilon} d\vec{\boldsymbol{\omega}}' d\vec{\mathbf{r}}' - f_{t_\epsilon}}{t_\epsilon}. \quad (11.5)$$

$\lim \boldsymbol{t_\epsilon} \to 0$ applied to (11.5) leads to

$$\frac{\partial f}{\partial t} + \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}} \cdot \vec{\boldsymbol{\nabla}} f = \lim_{t_\epsilon \to 0} \frac{\int_{\vec{\mathbf{r}}}\int_{\vec{\boldsymbol{\omega}}} W_{t_\epsilon} f'_{t_\epsilon} d\vec{\boldsymbol{\omega}}' d\vec{\mathbf{r}}' - f_{t_\epsilon}}{t_\epsilon}. \quad (11.6)$$

The right side must contain the characteristics of the turbulent fluid.

$$\lim_{t_\epsilon \to 0} \frac{\int_{\vec{\mathbf{r}}}\int_{\vec{\boldsymbol{\omega}}} W_{t_\epsilon} f'_{t_\epsilon} d\vec{\boldsymbol{\omega}}' d\vec{\mathbf{r}}' - f_{t_\epsilon}}{t_\epsilon} = F \quad (11.7)$$

F has to be chosen such, that the deterministic vortex equations result under the influence of the assumed acceleration field. Further on the ansatz

$$F = \frac{1}{2}\left[\frac{\vec{\boldsymbol{\omega}}}{\omega^2} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}}\right] f \quad (11.8)$$

is shown precisely fulfilling this condition. Thus one obtains

$$\frac{\partial f}{\partial t} + \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}} \cdot \vec{\boldsymbol{\nabla}} f = \frac{1}{2}\left[\frac{\vec{\boldsymbol{\omega}}}{\omega^2} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}}\right] f. \quad (11.9)$$

Limiting ourselves to one system of the ensemble the distribution function f degenerates to a δ-function.

$$f \longrightarrow \delta(\vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)}, \vec{\mathbf{r}}_{(\vec{\boldsymbol{x}},t)}; \vec{\boldsymbol{\omega}}, \vec{\mathbf{r}}) \quad (11.10)$$

The indexing of quantities like $\vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)}$ by $(\vec{\boldsymbol{x}}, t)$ means the vector $\vec{\boldsymbol{\omega}}$ in the space-time point $(\vec{\boldsymbol{x}}, t)$ [2] whereas $\vec{\boldsymbol{\omega}}(\vec{\boldsymbol{x}}, t)$ represents the spatiotemporal field $\vec{\boldsymbol{\omega}}$ in dependence on $(\vec{\boldsymbol{x}}, t)$.

It results in the key equation for the transition stochastic-deterministic

$$\boxed{\frac{\partial}{\partial t}\delta + \vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)} \times \vec{\mathbf{r}}_{(\vec{\boldsymbol{x}},t)} \cdot \vec{\boldsymbol{\nabla}}\delta = \frac{1}{2}\left[\frac{\vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)}}{\omega^2_{(\vec{\boldsymbol{x}},t)}} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}}_{(\vec{\boldsymbol{x}},t)}\right]\delta}. \quad (11.11)$$

[2]That is the situation considering stochastically.

Definition of the operator $\Xi[...]$:
From the vector $\vec{\mathbf{A}}_{(\vec{x},t)}$ respectively the scalar function value $\mathbf{f}_{(\vec{x},t)}$ existing in the space-time-point $(\vec{x},t)$ of the system a vector function respectively a scalar function arises by the operator Ξ

$$\Xi\left[\vec{\mathbf{A}}_{(\vec{x},t)}\right] = \vec{\mathbf{A}}(\vec{x},t) \tag{11.12}$$

respectively

$$\Xi\left[\mathbf{f}_{(\vec{x},t)}\right] = \mathbf{f}(\vec{x},t) \tag{11.13}$$

an appropriate field existing around the point $(\vec{x},t)$. The Operator $\Xi[...]$ evokes this functionality to "life".
Accordingly the following relationships are noted:

$$\begin{aligned}
&\Xi\left[\int_{\vec{\mathbf{r}}}\int_{\vec{\omega}}\delta(\vec{\omega}_{(\vec{x},t)},\vec{\mathbf{r}}_{(\vec{x},t)};\vec{\omega},\vec{\mathbf{r}})d\vec{\omega}d\vec{\mathbf{r}}\right] = \mathbf{1}\\
&\Xi\left[\int_{\vec{\mathbf{r}}}\int_{\vec{\omega}}\delta(\vec{\omega}_{(\vec{x},t)},\vec{\mathbf{r}}_{(\vec{x},t)};\vec{\omega},\vec{\mathbf{r}})\vec{\omega}d\vec{\omega}d\vec{\mathbf{r}}\right] = \Xi\left[\vec{\omega}_{(\vec{x},t)}\right] = \vec{\omega}(\vec{x},t)\\
&\Xi\left[\int_{\vec{\mathbf{r}}}\int_{\vec{\omega}}\delta(\vec{\omega}_{(\vec{x},t)},\vec{\mathbf{r}}_{(\vec{x},t)};\vec{\omega},\vec{\mathbf{r}})\vec{\mathbf{r}}d\vec{\omega}d\vec{\mathbf{r}}\right] = \Xi\left[\vec{\mathbf{r}}_{(\vec{x},t)}\right] = \vec{\mathbf{r}}(\vec{x},t)
\end{aligned} \tag{11.14}$$

or

$$\Xi\left[\int_{\vec{\mathbf{r}}}\int_{\vec{\omega}}\delta(\vec{\omega}_{(\vec{x},t)},\vec{\mathbf{r}}_{(\vec{x},t)};\vec{\omega},\vec{\mathbf{r}})\boldsymbol{\omega}^2\vec{\mathbf{r}}d\vec{\omega}d\vec{\mathbf{r}}\right] = \Xi\left[\boldsymbol{\omega}^2_{(\vec{\mathbf{x}},t)}\vec{\mathbf{r}}_{(\vec{\mathbf{x}},t)}\right] = \boldsymbol{\omega}^2(\vec{\mathbf{x}},t)\vec{\mathbf{r}}(\vec{\mathbf{x}},t). \tag{11.15}$$

11.3. The deterministic equations of turbulence

From the general momentum equation

$$\frac{\partial\vec{\mathbf{v}}}{\partial t} + (\vec{\mathbf{v}}\cdot\vec{\boldsymbol{\nabla}})\vec{\mathbf{v}} = \vec{\boldsymbol{q}} \tag{11.16}$$

the vortex equation may be developed using the $\vec{\boldsymbol{\nabla}}\times$-operator

$$\frac{\partial}{\partial t}\vec{\omega} - \vec{\boldsymbol{\nabla}}\times(\vec{\mathbf{v}}\times\vec{\omega}) - \frac{1}{2}\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{q}} = 0. \tag{11.17}$$

The relations of deterministic and stochastic description are established the same vortex equation opening up from the above key equation. In the following the method is presented designing the dual pair of deterministic vector equations from the key equation

$$\frac{\partial}{\partial t}\delta + \vec{\boldsymbol{\omega}}_{(\vec{x},t)} \times \vec{\mathbf{r}}_{(\vec{x},t)} \cdot \vec{\boldsymbol{\nabla}}\delta = \frac{1}{2}\left[\frac{\vec{\boldsymbol{\omega}}_{(\vec{x},t)}}{\omega^2_{(\vec{x},t)}} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}}_{(\vec{x},t)}\right]\delta. \tag{11.18}$$

In this situation the vectors of the motion quantities may be pushed before and after the differential operators. The Term

$$\frac{1}{2}\left[\frac{\vec{\boldsymbol{\omega}}_{(\vec{x},t)}}{\omega^2_{(\vec{x},t)}} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}}_{(\vec{x},t)}\right]\delta \tag{11.19}$$

guarantees the finding of equation (11.17) and its dual one. It is

$$\vec{\mathbf{v}} \perp \vec{\boldsymbol{\omega}} \perp \vec{\mathbf{r}}. \tag{11.20}$$

and setting

$$\vec{\mathbf{a}} = \vec{\mathbf{v}} \times \vec{\boldsymbol{\omega}} \tag{11.21}$$

this results in

$$\vec{\mathbf{r}} \parallel \vec{\mathbf{a}}. \tag{11.22}$$

Such $\vec{\mathbf{a}}$ and $\vec{\mathbf{r}}$ are linked as follows[3]

$$\vec{\mathbf{r}} = \frac{\vec{\mathbf{a}}}{\omega^2}. \tag{11.23}$$

$\Longrightarrow$

$$\text{with} \qquad \delta = \delta(\vec{\boldsymbol{\omega}}_{(\vec{x},t)}, \vec{\mathbf{r}}_{(\vec{x},t)}; \vec{\boldsymbol{\omega}}, \vec{\mathbf{r}})$$

$$\begin{aligned}
\vec{\boldsymbol{\omega}}_{(\vec{x},t)} \times \vec{\mathbf{r}}_{(\vec{x},t)} \cdot \vec{\boldsymbol{\nabla}}\delta &= -\vec{\mathbf{r}}_{(\vec{x},t)} \times \vec{\boldsymbol{\omega}}_{(\vec{x},t)} \cdot \vec{\boldsymbol{\nabla}}\delta \\
&= -\vec{\boldsymbol{\omega}}_{(\vec{x},t)} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{r}}_{(\vec{x},t)}\delta \\
&= -\frac{\vec{\boldsymbol{\omega}}_{(\vec{x},t)}}{\omega^2_{(\vec{x},t)}} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{a}}_{(\vec{x},t)}\delta.
\end{aligned}$$

[3]Symbols as ω, r, a, v etc. always mean amounts of the corresponding vectors.

Inserting in (11.18) gives

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big(\frac{\vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)}\cdot\vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)}}{\omega^2_{(\vec{\boldsymbol{x}},t)}}\delta\Big)-\frac{\vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)}}{\omega^2_{(\vec{\boldsymbol{x}},t)}}\cdot\vec{\boldsymbol{\nabla}}\times(\vec{\mathbf{a}}_{(\vec{\boldsymbol{x}},t)}\delta)-\frac{1}{2}\left[\frac{\vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)}}{\omega^2_{(\vec{\boldsymbol{x}},t)}}\cdot\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{q}}_{(\vec{\boldsymbol{x}},t)}\right]\delta=0\\
&\implies\frac{\vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)}}{\omega^2_{(\vec{\boldsymbol{x}},t)}}\cdot\left[\frac{\partial}{\partial t}(\vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)}\delta)-\vec{\boldsymbol{\nabla}}\times(\vec{\mathbf{a}}_{(\vec{\boldsymbol{x}},t)}\delta)-\frac{1}{2}\left[\cdot\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{q}}_{(\vec{\boldsymbol{x}},t)}\right]\delta\right]=0\\
&\implies\frac{\partial}{\partial t}(\vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)}\delta)-\vec{\boldsymbol{\nabla}}\times(\vec{\mathbf{a}}_{(\vec{\boldsymbol{x}},t)}\delta)-\frac{1}{2}\left[\cdot\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{q}}_{(\vec{\boldsymbol{x}},t)}\right]\delta=0
\end{aligned}\tag{11.24}$$

One obtains

$$\boldsymbol{\Xi}\left[\int_{\vec{\mathbf{r}}}\int_{\vec{\boldsymbol{\omega}}}\left[\frac{\partial}{\partial t}(\vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)}\delta)-\vec{\boldsymbol{\nabla}}\times(\vec{\mathbf{a}}_{(\vec{\boldsymbol{x}},t)}\delta)-\frac{1}{2}\left[\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{q}}_{(\vec{\boldsymbol{x}},t)}\right]\delta=0\right]d\vec{\boldsymbol{\omega}}d\vec{\mathbf{r}}\right]\tag{11.25}$$

because integration and differentiation beeing exchangeable follows

$$\left[\frac{\partial}{\partial t}\boldsymbol{\Xi}\left[\vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)}\right]-\vec{\boldsymbol{\nabla}}\times\boldsymbol{\Xi}\left[\vec{\mathbf{a}}_{(\vec{\boldsymbol{x}},t)}\right]-\frac{1}{2}\vec{\boldsymbol{\nabla}}\times\boldsymbol{\Xi}\left[\vec{\mathbf{q}}_{(\vec{\boldsymbol{x}},t)}\right]=0\tag{11.26}$$

and we have the first of the dual turbulence equations

$$\frac{\partial}{\partial t}\vec{\boldsymbol{\omega}}-\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{a}}-\frac{1}{2}\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{q}}=0\tag{11.27}$$

accordingly

$$\frac{\partial}{\partial t}\vec{\boldsymbol{\omega}}-\vec{\boldsymbol{\nabla}}\times(\vec{\mathbf{v}}\times\vec{\boldsymbol{\omega}})-\frac{1}{2}\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{q}}=0.$$

Hereby the connection of stochastics and deterministics is achieved. From the key-equation above a second equation, the dual one, may be derived.

Back to the initial equation (11.18)

$$\frac{\partial}{\partial t}\delta+\vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)}\times\vec{\mathbf{r}}_{(\vec{\boldsymbol{x}},t)}\cdot\vec{\boldsymbol{\nabla}}\delta=\frac{1}{2}\left[\frac{\vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)}}{\omega^2_{(\vec{\boldsymbol{x}},t)}}\cdot\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{q}}_{(\vec{\boldsymbol{x}},t)}\right]\delta$$

Simple conversions give

$$\frac{\partial}{\partial t}\left(\vec{\mathbf{r}}_{(\vec{x},t)} \cdot \frac{\vec{\mathbf{r}}_{(\vec{x},t)}}{r^2_{(\vec{x},t)}}\delta\right) + \vec{\mathbf{r}}_{(\vec{x},t)} \cdot \vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{\omega}}_{(\vec{x},t)}\delta) - \frac{\vec{\mathbf{r}}_{(\vec{x},t)} \cdot \vec{\mathbf{r}}_{(\vec{x},t)}}{r^2_{(\vec{x},t)}} \frac{1}{2}\left[\frac{\vec{\boldsymbol{\omega}}_{(\vec{x},t)}}{\omega^2_{(\vec{x},t)}} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}}_{(\vec{x},t)}\right]\delta = 0$$
$$\longrightarrow \vec{\boldsymbol{r}}_{(\vec{x},t)}\left[\frac{\partial}{\partial t}\frac{\vec{\mathbf{r}}_{(\vec{x},t)}}{r^2_{(\vec{x},t)}}\delta + \vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{\omega}}_{(\vec{x},t)}\delta) - \frac{\vec{\mathbf{r}}_{(\vec{x},t)}}{r^2_{(\vec{x},t)}}\frac{1}{2}\left[\frac{\vec{\boldsymbol{\omega}}_{(\vec{x},t)}}{\omega^2_{(\vec{x},t)}} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}}_{(\vec{x},t)}\right]\delta\right] = 0 \tag{11.28}$$

Using the curvature vector field of the fluid trajectories $\vec{\mathbf{b}} = \frac{\vec{\mathbf{r}}}{r^2}$ the equation is written

$$\frac{\partial}{\partial t}(\vec{\mathbf{b}}_{(\vec{x},t)}\delta) + \vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{\omega}}_{(\vec{x},t)}\delta) - \frac{1}{2}\vec{\mathbf{b}}_{(\vec{x},t)}\frac{\vec{\boldsymbol{\omega}}_{(\vec{x},t)}}{\omega^2_{(\vec{x},t)}} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}}_{(\vec{x},t)}\delta = 0 \tag{11.29}$$

and applying the operators $\boldsymbol{\Xi}$ arises

$$\boldsymbol{\Xi}\left[\int_{\vec{\mathbf{r}}}\int_{\vec{\boldsymbol{\omega}}}\left[\frac{\partial}{\partial t}(\vec{\mathbf{b}}_{(\vec{x},t)}\delta) + \vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{\omega}}_{(\vec{x},t)}\delta) - \frac{1}{2}\vec{\mathbf{b}}_{(\vec{x},t)}\frac{\vec{\boldsymbol{\omega}}_{(\vec{x},t)}}{\omega^2_{(\vec{x},t)}} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}}_{(\vec{x},t)}\delta = 0\right] d\vec{\boldsymbol{\omega}} d\vec{\mathbf{r}}\right] \tag{11.30}$$

respectively

$$\frac{\partial}{\partial t}\boldsymbol{\Xi}[\vec{\mathbf{b}}_{(\vec{x},t)}] + \vec{\boldsymbol{\nabla}} \times \boldsymbol{\Xi}[\vec{\boldsymbol{\omega}}_{(\vec{x},t)}] - \frac{1}{2}\boldsymbol{\Xi}\left[\left(\vec{\mathbf{b}}\frac{\vec{\boldsymbol{\omega}}}{\omega^2} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}}\right)_{(\vec{x},t)}\right] = 0. \tag{11.31}$$

Such the second of the dual turbulence equations is approached

$$\frac{\partial}{\partial t}\vec{\mathbf{b}} + \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{\omega}} - \frac{1}{2}\vec{\mathbf{b}}\left[\frac{\vec{\boldsymbol{\omega}}}{\omega^2} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}}\right] = 0. \tag{11.32}$$

Closing this dual equation system

$$\boxed{\begin{array}{l}\frac{\partial}{\partial t}\vec{\boldsymbol{\omega}} - \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{a}} - \frac{1}{2}\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}} = 0 \\ \frac{\partial}{\partial t}\vec{\mathbf{b}} + \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{\omega}} - \frac{1}{2}\vec{\mathbf{b}}\left[\frac{\vec{\boldsymbol{\omega}}}{\omega^2} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}}\right] = 0 \\ \vec{\mathbf{v}} = \vec{\boldsymbol{\omega}} \times \frac{\vec{\mathbf{b}}}{b^2},\ \vec{\mathbf{a}} = \vec{\mathbf{v}} \times \vec{\boldsymbol{\omega}}\end{array}} \tag{11.33}$$

further equations are necessary besides the momentum equations. In the case of the Navier-Stokes-equations

$$\frac{\partial \vec{\mathbf{v}}}{\partial t} + (\vec{\mathbf{v}} \cdot \vec{\boldsymbol{\nabla}})\vec{\mathbf{v}} = -\frac{1}{\rho}\vec{\boldsymbol{\nabla}}\mathbf{p} + \vec{\mathbf{g}} + \nu\Delta\vec{\mathbf{v}} + (\zeta + \frac{\nu}{3})\vec{\boldsymbol{\nabla}}\ (\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{v}})$$

i.e.

$$\vec{\mathbf{q}} = -\frac{1}{\rho}\vec{\boldsymbol{\nabla}}\mathbf{p} + \vec{\mathbf{g}} + \nu\Delta\vec{\mathbf{v}} + (\zeta + \frac{\nu}{3})\vec{\boldsymbol{\nabla}}\ (\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{v}})$$

this could happen by simultaneously using the known continuity, energy as well as state equation. But this proves not to be expedient. In chapter 13 the complete equation system is presented and it is shown that the usual Navier-Stokes-equations are not warranting the correct momentum balancing in turbulence.
The term

$$-\frac{1}{2}\vec{\mathbf{b}}\left[\frac{\vec{\boldsymbol{\omega}}}{\omega^2} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}}\right]$$

may lead to removable singularities in space-time-points $(\vec{\mathbf{x}}, t)$ when turning points occur in the fluid element trajectories $\vec{\boldsymbol{\omega}} = 0$ and $\vec{\mathbf{b}} = \mathbf{0}$ arising simultaneously. In this case the whole term is calculated from its surroundings. The same shall apply for the calculation of the velocity $\vec{\mathbf{v}}$. In such cases there is an alternative way shown in chapter 13, too.

12. Stochastic and deterministic general vector fields

$$\boxed{\begin{array}{c} f_{t_\epsilon}(\vec{\mathbf{x}},t,\vec{\mathbf{E}},\vec{\mathbf{B}}) = \int\limits_{\vec{\mathbf{B}}}\int\limits_{\vec{\mathbf{E}}} W_{t_\epsilon}(\vec{\mathbf{x}},t,\vec{\mathbf{E}},\vec{\mathbf{B}},\vec{\mathbf{E}}',\vec{\mathbf{B}}')\cdot f_{t_\epsilon}(\vec{\mathbf{x}}-\Delta\vec{\mathbf{x}},t-t_\epsilon,\vec{\mathbf{E}}',\vec{\mathbf{B}}')d\vec{\mathbf{E}}'d\vec{\mathbf{B}}' \\ \Updownarrow \\ \frac{\partial}{\partial t}\vec{\mathbf{B}} - \vec{\boldsymbol{\nabla}}\times\vec{\mathbf{E}} = 0 \\ \frac{\partial}{\partial t}\left(\frac{B^2}{E^2}\cdot\vec{\mathbf{E}}\right) + \vec{\boldsymbol{\nabla}}\times\vec{\mathbf{B}} = 0 \end{array}}$$

12.1. Introduction

Subsequently continuum fluctuations of general 3 dimensional vector fields $\vec{\mathbf{A}}(\vec{\mathbf{x}},t)$ with $\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{A}}\neq\mathbf{0}$ are analysed. They have to be sufficiently often continuously differentiable. Defining the vector fields $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ by

$$\begin{aligned} \vec{\mathbf{E}} &= \partial\vec{\mathbf{A}}/\partial t \neq 0 \\ \vec{\mathbf{B}} &= \vec{\boldsymbol{\nabla}}\times\vec{\mathbf{A}} \neq 0 \end{aligned} \tag{12.1}$$

and owing to the exchangeability of the operators $\partial/\partial t$ and $\vec{\boldsymbol{\nabla}}\times$

$$\frac{\partial\vec{\mathbf{B}}}{\partial t} = \vec{\boldsymbol{\nabla}}\times\vec{\mathbf{E}} \tag{12.2}$$

follows. This is a necessary consequence of the condition of the continuous differentiability of $\vec{\mathbf{A}}(\vec{\mathbf{x}},t)$. This relation is known according to the Maxwell Equations. The for this purpose dual equation is subsequently beeing looked for. In an analogous approach derivating the turbulence equations a stochastic continuum process in the frame of an ensemble theory is formulated such that according to a deterministic theory the already known as well as the related dual equation arise with fluctuating quantities $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$.

12.2. The Transition: stochastic theory $\longleftrightarrow$ deterministic theory

This transition takes place in the same way as the derivation of the dual turbulence equation pair. Every space-time-point$(\vec{\mathbf{x}}, t)$ a continuously differentiable distribution density $f_{\boldsymbol{t_\epsilon}}$ is assigned to the motion quantities $\vec{\mathbf{E}}_{t_\epsilon} = \partial\vec{\mathbf{A}}_{t_\epsilon}/\partial t$ and $\vec{\mathbf{B}}_{t_\epsilon} = \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}}_{t_\epsilon}$ with

$$f_{\boldsymbol{t_\epsilon}} = f_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, t, \vec{\mathbf{E}}, \vec{\mathbf{B}}). \tag{12.3}$$

In the with $\boldsymbol{t_\epsilon}$ or $\boldsymbol{\epsilon}$ indexed functions $f_{\boldsymbol{t_\epsilon}}$ it is automatically assumed that the included motion quantities $(\vec{\mathbf{E}}, \vec{\mathbf{B}})$ are assigned to a $\boldsymbol{t_\epsilon}$-measurement accuracy. The indexing of the motion quantities may be omitted in functions appropriately indexed themselves.

After the execution of a $\lim \boldsymbol{t_\epsilon} \rightarrow 0$-process

$$\lim_{\boldsymbol{t_\epsilon}\rightarrow 0} f_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, t, \vec{\mathbf{E}}, \vec{\mathbf{B}}) = f(\vec{\mathbf{x}}, t, \vec{\mathbf{E}}, \vec{\mathbf{B}}) \tag{12.4}$$

f and $(\vec{\boldsymbol{E}}, \vec{\boldsymbol{B}})$ are understood in the sense of an exact measurement process.

The stochastic transport of the fluctuation quantities

$$\left(\vec{\mathbf{E}}'_{t_\epsilon}(\vec{\boldsymbol{x}} - \Delta\vec{\boldsymbol{x}}, t - t_\epsilon), \vec{\mathbf{B}}'_{t_\epsilon}(\vec{\boldsymbol{x}} - \Delta\vec{\boldsymbol{x}}, t - t_\epsilon)\right) \longrightarrow \left(\vec{\mathbf{E}}_{t_\epsilon}(\vec{\boldsymbol{x}}, t), \vec{\mathbf{B}}_{t_\epsilon}(\vec{\boldsymbol{x}}, t)\right)$$

happens by the transition probability density $W_{t_\epsilon} = W_{t_\epsilon}(\vec{\boldsymbol{x}}, t, \vec{\mathbf{E}}, \vec{\mathbf{B}}, \vec{\mathbf{E}}', \vec{\mathbf{B}}')$ with

$$\boxed{\begin{aligned} \lim_{t_\epsilon \to 0} W_{t_\epsilon} =& \delta(\vec{\mathbf{E}}, \vec{\mathbf{B}}; \vec{\mathbf{E}}', \vec{\mathbf{B}}') \\ f_{t_\epsilon}(\vec{\mathbf{x}}, t, \vec{\mathbf{E}}, \vec{\mathbf{B}}) =& \int\limits_{\vec{\mathbf{B}}'} \int\limits_{\vec{\mathbf{E}}'} W_{t_\epsilon}(\vec{\mathbf{x}}, t, \vec{\mathbf{E}}, \vec{\mathbf{B}}, \vec{\mathbf{E}}', \vec{\mathbf{B}}') \cdot f_{t_\epsilon}(\vec{\mathbf{x}} - \Delta\vec{\mathbf{x}}, t - t_\epsilon, \vec{\mathbf{E}}', \vec{\mathbf{B}}') d\vec{\mathbf{E}}' d\vec{\mathbf{B}}' \\ \Delta\vec{\mathbf{x}} =& t_\epsilon \cdot \vec{\mathbf{E}}' \times \frac{\vec{\mathbf{B}}'}{B'^2} \text{ and } \vec{\mathbf{E}}' \times \frac{\vec{\mathbf{B}}'}{B'^2} = \text{velocity of propagation.} \end{aligned}} \tag{12.5}$$

These equations define stochastic continuum fluctuations of the quantities $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ in the sense of an ensemble-theory and represent a Markov Process of natural causality. The test-functions of distribution theory obtain by this formulation of a transition probability density W_{t_ϵ} an immediate physical meaning.

f_{t_ϵ} is developed until the 1st order about $(\vec{\mathbf{x}}, t) \Longrightarrow$

$$\begin{aligned} f_{t_\epsilon}(t - t_\epsilon, \vec{\mathbf{x}} - \triangle\vec{\mathbf{x}}, \vec{\mathbf{E}}', \vec{\mathbf{B}}') = f'_{t_\epsilon} - \frac{\partial f'_{t_\epsilon}}{\partial t} \cdot t_\epsilon - \triangle\vec{\mathbf{x}} \cdot \vec{\boldsymbol{\nabla}} f'_{t_\epsilon} + \boldsymbol{O}({t_\epsilon}^2) \\ f'_{t_\epsilon} = f_{t_\epsilon}(\vec{\mathbf{x}}, t, \vec{\mathbf{E}}', \vec{\mathbf{B}}') \end{aligned} \tag{12.6}$$

und one gets

$$\int_{\vec{\mathbf{E}}} \int_{\vec{\mathbf{B}}} W_{t_\epsilon} \left[\frac{\partial f'_{t_\epsilon}}{\partial t} + \vec{\mathbf{E}}' \times \frac{\vec{\mathbf{B}}'}{B'^2} \cdot \vec{\boldsymbol{\nabla}} f'_{t_\epsilon} \right] d\vec{\mathbf{E}}' d\vec{\mathbf{B}}' + \boldsymbol{O}({t_\epsilon}^2) = \frac{\int_{\vec{\mathbf{B}}} \int_{\vec{\mathbf{E}}} W_{t_\epsilon} f'_{t_\epsilon} d\vec{\mathbf{E}}' d\vec{\mathbf{B}}' - f_{t_\epsilon}}{t_\epsilon}. \tag{12.7}$$

By the process $t_\epsilon \to 0$ W_{t_ϵ} degenerates to a δ-function:

$$\lim_{\boldsymbol{t_\epsilon \to 0}} W_{t_\epsilon} = \delta(\vec{\mathbf{E}}, \vec{\mathbf{B}}; \vec{\mathbf{E}}', \vec{\mathbf{B}}') \tag{12.8}$$

$\lim \boldsymbol{t_\epsilon} \to 0$ applied leads to

$$\frac{\partial f}{\partial t} + \vec{\mathbf{E}} \times \frac{\vec{\mathbf{B}}}{B^2} \cdot \vec{\boldsymbol{\nabla}} f = \lim_{t_\epsilon \to 0} \frac{\int_{\vec{\mathbf{E}}} \int_{\vec{\mathbf{B}}} W_{t_\epsilon} f'_{t_\epsilon} d\vec{\mathbf{E}}' d\vec{\mathbf{B}}' - f_{t_\epsilon}}{t_\epsilon}. \tag{12.9}$$

Recovering equation (12.2) after the transition to deterministic consideration the exchange term has to vanish, in this case.

$$\lim_{t_\epsilon \to 0} \frac{\int_{\vec{\mathbf{B}}} \int_{\vec{\mathbf{E}}} W_{t_\epsilon} f'_{t_\epsilon} d\vec{\mathbf{E}}' d\vec{\mathbf{B}}' - f_{t_\epsilon}}{t_\epsilon} = \mathbf{0}. \tag{12.10}$$

This link is an integral part of the considered stochastic process.

Limiting ourselves to one system of the ensemble the function $f(\vec{\mathbf{x}}, t, \vec{\mathbf{E}}, \vec{\mathbf{B}})$ in the space-time-point $(\vec{\boldsymbol{x}}, t)$ degenerates to a $\delta-$function

$$f(\vec{\mathbf{x}}, t, \vec{\mathbf{E}}, \vec{\mathbf{B}}) \longrightarrow \delta(\vec{\mathbf{E}}_{(\vec{\mathbf{x}},t)}, \vec{\mathbf{B}}_{(\vec{\mathbf{x}},t)}; \vec{\mathbf{E}}, \vec{\mathbf{B}})\text{-function}. \tag{12.11}$$

From equation (12.9) arises the key-equation

$$\boxed{\frac{\partial}{\partial t}\delta + \vec{\mathbf{E}}_{(\vec{\boldsymbol{x}},t)} \times \frac{\vec{\mathbf{B}}_{(\vec{\boldsymbol{x}},t)}}{B^2_{(\vec{\boldsymbol{x}},t)}} \cdot \vec{\boldsymbol{\nabla}}\delta = \mathbf{0}}. \tag{12.12}$$

Respectively section 11.2 the $\Xi[...]$-operator is inserted as follows

$$\begin{aligned} \Xi\left[\int_{\vec{\mathbf{E}}} \int_{\vec{\mathbf{B}}} \delta(\vec{\mathbf{B}}_{(\vec{\boldsymbol{x}},t)}, \vec{\mathbf{E}}_{(\vec{\boldsymbol{x}},t)}; \vec{\mathbf{B}}, \vec{\mathbf{E}}) \vec{\mathbf{B}} d\vec{\mathbf{B}} d\vec{\mathbf{E}}\right] &= \Xi[\vec{\mathbf{B}}_{(\vec{\boldsymbol{x}},t)}] = \vec{\mathbf{B}}(\vec{\boldsymbol{x}}, t) \\ \Xi\left[\int_{\vec{\mathbf{E}}} \int_{\vec{\mathbf{B}}} \delta(\vec{\mathbf{B}}_{(\vec{\boldsymbol{x}},t)}, \vec{\mathbf{E}}_{(\vec{\boldsymbol{x}},t)}; \vec{\mathbf{B}}, \vec{\mathbf{E}}) \vec{\mathbf{E}} d\vec{\mathbf{B}} d\vec{\mathbf{E}}\right] &= \Xi\left[\vec{\mathbf{E}}_{(\vec{\boldsymbol{x}},t)}\right] = \vec{\mathbf{E}}(\vec{\boldsymbol{x}}, t) \end{aligned} \tag{12.13}$$

or

$$\Xi\left[\int_{\vec{\mathbf{E}}} \int_{\vec{\mathbf{B}}} \delta(\vec{\mathbf{B}}_{(\vec{\boldsymbol{x}},t)}, \vec{\mathbf{E}}_{(\vec{\boldsymbol{x}},t)}; \vec{\mathbf{b}}, \vec{\mathbf{E}}) \left(\frac{B^2}{E^2} \cdot \vec{\mathbf{E}}\right) d\vec{\mathbf{B}} d\vec{\mathbf{E}}\right] = \Xi\left[\frac{B^2_{(\vec{\boldsymbol{x}},t)}}{E^2_{(\vec{\boldsymbol{x}},t)}} \cdot \vec{\mathbf{E}}_{(\vec{\boldsymbol{x}},t)}\right] = \frac{B^2(\vec{\boldsymbol{x}}, t)}{E^2(\vec{\boldsymbol{x}}, t)} \cdot \vec{\mathbf{E}}(\vec{\boldsymbol{x}}, t), \tag{12.14}$$

developing the deterministic equations from the key equation.

12.3. The deterministic fluctuation-equations

The key-equation (12.12) represents the interface for the transition of stochastic to deterministic consideration. From the perspective of statistics over the states of movement of the parallelly assumed deterministic processes in the respective point $(\vec{\mathbf{x}}, t)$ one is confined to a single system and such to a single state of motion $(\vec{\mathbf{E}}_{(\vec{\mathbf{x}},t)}, \vec{\mathbf{B}}_{(\vec{\mathbf{x}},t)})$.

In this situation the vectors of the motion quantities may be pushed before and behind the differential operators

$$\vec{\mathbf{E}}_{(\vec{x},t)} \times \frac{\vec{\mathbf{B}}_{(\vec{x},t)}}{B^2_{(\vec{x},t)}} \cdot \vec{\nabla}\delta = -\frac{\vec{\mathbf{B}}_{(\vec{x},t)}}{B^2_{(\vec{x},t)}} \times \vec{\mathbf{E}}_{(\vec{x},t)} \cdot \vec{\nabla}\delta$$
$$= -\frac{\vec{\mathbf{B}}_{(\vec{x},t)}}{B^2_{(\vec{x},t)}} \cdot \vec{\nabla} \times \vec{\mathbf{E}}_{(\vec{x},t)}\delta$$

Further more there is

$$\begin{aligned}
&\frac{\partial}{\partial t}(\frac{\vec{\mathbf{B}}_{(\vec{x},t)} \cdot \vec{\mathbf{B}}_{(\vec{x},t)}}{B^2_{(\vec{x},t)}}\delta) - \frac{\vec{\mathbf{B}}_{(\vec{x},t)}}{B^2_{(\vec{x},t)}} \cdot \vec{\nabla} \times (\vec{\mathbf{E}}_{(\vec{x},t)}\delta) = 0 \\
\Longrightarrow\ &\frac{\vec{\mathbf{B}}_{(\vec{x},t)}}{B^2_{(\vec{x},t)}} \cdot [\frac{\partial}{\partial t}(\vec{\mathbf{B}}_{(\vec{x},t)}\delta) - \vec{\nabla} \times (\vec{\mathbf{E}}_{(\vec{x},t)}\delta)] = 0 \\
\Longrightarrow\ &\frac{\partial}{\partial t}(\vec{\mathbf{B}}_{(\vec{x},t)}\delta) - \vec{\nabla} \times (\vec{\mathbf{E}}_{(\vec{x},t)}\delta) = 0.
\end{aligned} \tag{12.15}$$

Now the vector fields of the motion quantities $(\vec{\mathbf{E}}_{(\vec{x},t)}, \vec{\mathbf{B}}_{(\vec{x},t)})$ of the one determinstic system are created about the point $(\vec{x}, t)$ and such the transition to the deterministic equations of the one system has succeeded.

One obtains

$$\Xi\left[\int_{\vec{\mathbf{B}}}\int_{\vec{\mathbf{E}}}\left[\frac{\partial}{\partial t}(\vec{\mathbf{B}}_{(\vec{x},t)}\delta) - \vec{\nabla} \times (\vec{\mathbf{E}}_{(\vec{x},t)}\delta) = 0\right] d\vec{\mathbf{E}}d\vec{\mathbf{B}}\right]. \tag{12.16}$$

As integration and differentiation are exchangeable $\Longrightarrow$

$$\frac{\partial}{\partial t}\Xi[\vec{\mathbf{B}}_{(\vec{x},t)}] - \vec{\nabla} \times \Xi[\vec{\mathbf{E}}_{(\vec{x},t)}] = 0 \tag{12.17}$$

and it results in the 1.st of the dual fluctuation equations

$$\frac{\partial}{\partial t}\vec{\mathbf{B}} - \vec{\nabla} \times \vec{\mathbf{E}} = 0. \tag{12.18}$$

Hereby the stochastic-deterministic connection is established.

Back to the key-equation (12.12)

$$\frac{\partial}{\partial t}\delta + \vec{\mathbf{E}}_{(\vec{x},t)} \times \frac{\vec{\mathbf{B}}_{(\vec{x},t)}}{B^2_{(\vec{x},t)}} \cdot \vec{\nabla}\delta = \mathbf{0}$$

one obtains by simple conversion

$$\begin{aligned}\frac{\partial}{\partial t}\left(\vec{\mathbf{E}}_{(\vec{x},t)}\cdot\frac{\vec{\mathbf{E}}_{(\vec{x},t)}}{E^2_{(\vec{x},t)}}\delta\right)+\vec{\mathbf{E}}_{(\vec{x},t)}\cdot\vec{\boldsymbol{\nabla}}\times\left(\frac{\vec{\mathbf{B}}_{(\vec{x},t)}}{B^2_{(\vec{x},t)}}\delta\right)=0\\ \frac{\partial}{\partial t}\left(\frac{B^2_{(\vec{x},t)}}{E^2_{(\vec{x},t)}}\cdot\vec{\mathbf{E}}_{(\vec{x},t)}\delta\right)+\vec{\boldsymbol{\nabla}}\times(\vec{\mathbf{B}}_{(\vec{x},t)}\delta)=0\end{aligned} \tag{12.19}$$

and

$$\boldsymbol{\Xi}\left[\int_{\vec{\mathbf{B}}}\int_{\vec{\mathbf{E}}}\left[\frac{\partial}{\partial t}\left(\frac{B^2_{(\vec{x},t)}}{E^2_{(\vec{x},t)}}\cdot\vec{\mathbf{E}}_{(\vec{x},t)}\delta\right)+\vec{\boldsymbol{\nabla}}\times(\vec{\mathbf{B}}_{(\vec{x},t)}\delta)=0\right]d\vec{\mathbf{E}}d\vec{\mathbf{B}}\right] \tag{12.20}$$

respectively

$$\frac{\partial}{\partial t}\boldsymbol{\Xi}\left[\frac{B^2_{(\vec{x},t)}}{E^2_{(\vec{x},t)}}\cdot\vec{\mathbf{E}}_{(\vec{x},t)}\right]+\vec{\boldsymbol{\nabla}}\times\boldsymbol{\Xi}[\vec{\mathbf{B}}_{(\vec{x},t)}]=0. \tag{12.21}$$

So we have the second of the two dual equations

$$\frac{\partial}{\partial t}(\frac{B^2}{E^2}\cdot\vec{\mathbf{E}})+\vec{\boldsymbol{\nabla}}\times(\vec{\mathbf{B}})=0. \tag{12.22}$$

The result is recapitulated by the following equation system:

$$\boxed{\begin{aligned}&\frac{\partial}{\partial t}\vec{\mathbf{B}}-\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{E}}=0\\ &\frac{\partial}{\partial t}\left(\frac{B^2}{E^2}\cdot\vec{\mathbf{E}}\right)+\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{B}}=0\\ &\vec{\mathbf{E}}\times\frac{\vec{\mathbf{B}}}{B^2}=\text{propagation speed}\end{aligned}} \tag{12.23}$$

with $|\vec{\mathbf{E}}\times\frac{\vec{\mathbf{B}}}{B^2}|\leq|\vec{\mathbf{E}}|\cdot|\frac{\vec{\mathbf{B}}}{B^2}|$. I.e. $\frac{E^2}{B^2}$ is not the quadratic propagation speed. Interestingly, this only becomes clear after the involvement of the stochastic ensemble theory.

The equation system (12.23) is in such general terms that the physical significance depends on the interpretation of the starting field $\vec{\mathbf{A}}$, the boundary conditions as well as on the initial conditions. Hereunder, a deformation vector field, the velocity vector field of turbulence motions or the fluctuations of any other continuously differentiable vector field may be understood. These equations possess with boundary- and suitable initial conditions exactly one solution after the theorem of Cauchy-Kowalewskaja [7]. This statement is at first restricted to the calculation of the fields $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$. Calculating the field $\vec{\mathbf{A}}$ with the mere knowledge of

$$\frac{\partial \vec{\mathbf{A}}}{\partial \boldsymbol{t}} = \vec{\mathbf{E}} \tag{12.24}$$

is not possible in all cases. A negative example is the calculation of $\vec{\mathbf{v}}$ with the knowledge of $\frac{\partial \vec{\mathbf{v}}}{\partial \boldsymbol{t}}$ related to turbulent velocity fluctuations as shown in chapter 13. However, in this case these relations are applied completing the turbulence equations. The particular definition of turbulence fluctuation elements (chapter 3) makes this problem almost vividly comprehensible.
Considering turbulent motions this can be done from a different perspective. With the equation system (12.23) the motion quantities

$$\vec{\mathbf{E}} = \frac{\partial}{\partial t}\vec{\mathbf{v}} \qquad \text{and} \qquad \vec{\mathbf{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{v}}$$

are transported with the propagation speed

$$\vec{\mathbf{v}} = \vec{\mathbf{E}} \times \frac{\vec{\mathbf{B}}}{B^2}.$$

The equation system (11.33) describes the mass transport by the velocity $\vec{\mathbf{v}}$. In consideration of $\vec{\mathbf{b}} = \frac{\vec{\mathbf{a}}}{\vec{\mathbf{v}}^2}$ (11.33) may be formulated omitting the viscosity and assuming $\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}} = 0$ as follows:

$$\begin{aligned} &\frac{\partial}{\partial t}\vec{\boldsymbol{\omega}} - \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{a}} = 0 \\ &\frac{\partial}{\partial t}\left(\frac{\vec{\mathbf{a}}}{v^2}\right) + \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{\omega}} = 0 \\ &\vec{\mathbf{v}} = \vec{\boldsymbol{\omega}} \times \frac{\vec{\mathbf{b}}}{b^2},\ \vec{\mathbf{a}} = \vec{\mathbf{v}} \times \vec{\boldsymbol{\omega}},\ \vec{\mathbf{v}} = \text{propagation speed} \end{aligned} \tag{12.25}$$

In doing so $\vec{\mathbf{v}} \perp \vec{\boldsymbol{\omega}} \perp \vec{\mathbf{a}}$ holds. The equations (12.23) and (12.25) do not formally differ apart from orthogonality conditions. But it is not expected, that the fluctuations generated by a conservative accelleration field ($\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{q}} = 0$) may describe hydrodynamic turbulences. This is discussed in chapter 13.

12.3.1. The vacuum Maxwell Equations

The propagation speed having the constant amount of light velocity one obtains the known equations of vacuum-electrodynamics in the coordinate system of the observer:

$$\boxed{\begin{array}{l} \frac{\partial}{\partial t}\vec{\mathbf{B}} - \vec{\mathbf{\nabla}} \times \vec{\mathbf{E}} = 0 \\ \frac{1}{c^2}\frac{\partial}{\partial t}\vec{\mathbf{E}} + \vec{\mathbf{\nabla}} \times \vec{\mathbf{B}} = 0 \qquad \text{mit} \qquad \vec{\mathbf{E}} \perp \vec{\mathbf{B}} \\ \vec{\mathbf{E}} \times \frac{\vec{\mathbf{B}}}{B^2} = \vec{\mathbf{c}} = \text{propagation speed of light.} \end{array}} \tag{12.26}$$

Hereby a formal analogy is established between electrodynamics and turbulent fluid dynamics. It is only based on the analogy of the propagation of the motion quantities $(\vec{\mathbf{E}}, \vec{\mathbf{B}})$ and $(\frac{\partial}{\partial t}\vec{\mathbf{v}}, \vec{\mathbf{\nabla}} \times \vec{\mathbf{v}})$. But a turbulent mass transport with the local velocity $\vec{\mathbf{v}}$ cannot be sufficiently described in this way as stated in chapter 13.

So the electrodynamic equations of vacuum are generally derived, too. Usually, they are seen in the above equations with $-\vec{\mathbf{E}}$. It is more than pure supposition, that they describe properties of space-time without a unification of General Relativity and electromagnetic field in vacuum having succeeded, though many physicists not least Einstein [11], Jordan [23] and many others having endeavoured.

There is still the explanation of the associated initial field $\vec{\mathbf{A}}$ it generally happening in the frame of vector potential considerations, without recognizing $\vec{\mathbf{A}}$ as definite physical object. Only by a direct comprehension of the vector potential the electromagnetic field may be explained without means of mechanical quantities.[1]

[1] Electrodynamics is introduced in physics via mechanical effects.

12.4. Surfacelike deformation-fluctuations in 3-dimensional space

Let $\vec{\mathbf{d}}$ be a continuously differentiable deformation vector field defining an area and $\vec{\mathbf{b}}$ and $\vec{\mathbf{e}}$ the derived fields

$$\vec{\mathbf{e}} = \frac{\partial}{\partial t}\vec{\mathbf{d}}, \qquad \vec{\mathbf{b}} = \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{d}} \tag{12.27}$$

with

$$\begin{aligned}
\vec{\mathbf{d}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}) &= \Big(\mathrm{d_x}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}), \mathrm{d_y}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}), \mathrm{d_z}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t})\Big)\\
\vec{\mathbf{e}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}) &= \Big(\mathrm{e_x}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}), \mathrm{e_y}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}), \mathrm{e_z}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t})\Big)\\
\vec{\mathbf{b}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}) &= \Big(\mathrm{b_x}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}), \mathrm{b_y}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}), \mathrm{b_z}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t})\Big).
\end{aligned} \tag{12.28}$$

Then the deformation is without loss of generality seen as deformation of the $\mathbf{x}-\mathbf{y}$-area. The equations of motion formally equal the equations of 3-dimensional fluctuations

$$\begin{aligned}
&\frac{\partial}{\partial t}\vec{\mathbf{b}} - \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{e}} = 0\\
&\frac{\partial}{\partial t}\left(\frac{b^2}{e^2}\cdot\vec{\mathbf{e}}\right) + \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{b}} = 0\\
&\vec{\mathbf{e}} \times \frac{\vec{\mathbf{b}}}{b^2} = \text{propagation speed,}
\end{aligned} \tag{12.29}$$

only, the operator $\vec{\boldsymbol{\nabla}}\times$ corresponds to

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{d}} = \begin{pmatrix} \partial d_z/\partial y \\ -\partial d_z/\partial x \\ \partial d_y/\partial x - \partial d_x \partial y \end{pmatrix}. \tag{12.30}$$

The solution uniquely succeeds by the initial conditions $\vec{\mathbf{b}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t_0})$ and $\vec{\mathbf{e}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t_0})$ according to the theorem of Cauchy-Kowalewskaya [7]. The solution for this area corresponds to a partial solution of a 3-dimensional complete solution. Physical material properties are not explicitly included in these equations. They have to be implicitly considered by initial and boundary conditions. Sole precondition is that the appropriate materials act continuously. It also means that the physical process has to be clarified enabling the corresponding initial and border conditions.

12.5. 1-dimensional deformation-fluctuations in 3-dimensional space

Let $\vec{\mathbf{d}}$ be a continuously differentiable deformation vector field defining a trajectory and $\vec{\mathbf{b}}$ and $\vec{\mathbf{e}}$ the derived fields

$$\vec{\mathbf{e}} = \frac{\partial}{\partial t}\vec{\mathbf{d}}, \qquad \vec{\mathbf{b}} = \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{d}} \tag{12.31}$$

with

$$\begin{aligned} \vec{\mathbf{d}}(\boldsymbol{x},\boldsymbol{t}) =& (\mathrm{d_x}(\boldsymbol{x},\boldsymbol{t}), \mathbf{d_y}(\boldsymbol{x},\boldsymbol{t}), \mathbf{d_z}(\boldsymbol{x},\boldsymbol{t})) \\ \vec{\mathbf{e}}(\boldsymbol{x},\boldsymbol{t}) =& (\mathrm{e_x}(\boldsymbol{x},\boldsymbol{t}), \mathrm{e_y}(\boldsymbol{x},\boldsymbol{t}), \mathrm{e_z}(\boldsymbol{x},\boldsymbol{t})) \\ \vec{\mathbf{b}}(\boldsymbol{x},\boldsymbol{t}) =& (\mathrm{b_x}(\boldsymbol{x},\boldsymbol{t}), \mathrm{b_y}(\boldsymbol{x},\boldsymbol{t}), \mathrm{b_z}(\boldsymbol{x},\boldsymbol{t})). \end{aligned} \tag{12.32}$$

Then the deformation is without loss of generality seen as deformation of the **x**-coordinate. The equations of motion formally equal the equations of 3-dimensional fluctuations

$$\begin{aligned} &\frac{\partial}{\partial t}\vec{\mathbf{b}} - \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{e}} = 0 \\ &\frac{\partial}{\partial t}\left(\frac{b^2}{e^2}\cdot\vec{\mathbf{e}}\right) + \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{b}} = 0 \\ &\vec{\mathbf{e}} \times \frac{\vec{\mathbf{b}}}{b^2} = \text{propagation speed,} \end{aligned} \tag{12.33}$$

only, the operator $\vec{\boldsymbol{\nabla}}\times$ corresponds to

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{d}} = \begin{pmatrix} & 0 \\ - & \partial d_z/\partial x \\ & \partial d_y/\partial x \end{pmatrix}. \tag{12.34}$$

This leads to the component equations

$$
\begin{aligned}
\partial b_y/\partial t =& - \partial e_z/\partial x \\
\partial b_z/\partial x =& \partial e_y/\partial x
\end{aligned}
$$

$$
\begin{aligned}
\partial[(b^2/e^2) \cdot e_y)]\partial t =& - \partial b_z/\partial x \\
\partial[(b^2/e^2) \cdot e_z)]\partial t =& \partial b_y/\partial x
\end{aligned}
\tag{12.35}
$$

$$\vec{\mathbf{e}} \times \vec{\mathbf{b}}/b^2 = \text{propagation speed.}$$

The x-component remains constant. The solution uniquely results from the initial conditions $\vec{\mathbf{b}}\boldsymbol{(x, t_0)}$ and $\vec{\mathbf{e}}\boldsymbol{(x, t_0)}$ according to the theorem of Cauchy-Kowalewskaya [7]. The solution for this 1-dimensional trajectory corresponds to a partial solution of a 3-dimensional complete solution. Physical material properties are not explicitly included in these equations. They have to be implicitly considered by initial and boundary conditions. Sole precondition is that the appropriate materials act continuously. It also means that the physical process has to be clarified enabling the corresponding initial and border conditions.

13. Geometrodynamics of turbulence

$$\boxed{\begin{array}{l} \vec{\mathbf{E}} + \frac{1}{2}\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}^2 - 2\vec{\mathbf{v}} \times \vec{\boldsymbol{\omega}} = \vec{\mathbf{q}} \\ \hline \frac{\partial}{\partial t}\vec{\boldsymbol{\omega}} - \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{a}} = \frac{1}{2}\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}} \\ \frac{\partial}{\partial t}\vec{\mathbf{b}} + \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{\omega}} = \frac{1}{2}\vec{\mathbf{b}}\left[\frac{\vec{\boldsymbol{\omega}}}{\boldsymbol{\omega}^2} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}}\right] \\ \frac{\partial}{\partial t}\vec{\mathbf{F}} = -2\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{\omega}} \text{ mit } \vec{\mathbf{F}} = \frac{4\omega^2}{E^2} \cdot \vec{\mathbf{E}} \end{array}}$$

13.1. Introduction

For a fluctuating continuum field

$$\frac{d}{dt}\vec{\mathbf{v}}(\vec{\mathbf{x}}, t) = \frac{\partial \vec{\mathbf{v}}}{\partial t} + (\vec{\mathbf{v}} \cdot \vec{\boldsymbol{\nabla}})\vec{\mathbf{v}} = \vec{\mathbf{q}}(\vec{\mathbf{x}}, t) \tag{13.1}$$

may be formally comprehended as a momentum equation. As soon as hydrodynamics is involved where a local thermodynamic balance is assumed, the Eulerian equations

$$\vec{\mathbf{q}} \stackrel{?}{=} -\frac{1}{\rho}\vec{\boldsymbol{\nabla}}\mathbf{p} \tag{13.2}$$

are noted with the indication of the 2nd Newtonian law. They are only justified under restrictive rules like incompressibility of fluids or $\frac{1}{\rho}\vec{\boldsymbol{\nabla}}\mathbf{p} = \vec{\boldsymbol{\nabla}}\mathbf{h}$ (h=spec. enthalpy) and or negligible rubbing viscosity. So only limiting cases of fluid dynamics are characterized.

But generally, $\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}} \neq \mathbf{0}$ is to be presumed. $\vec{\mathbf{q}}$ is in contrast to Newtonian mechanics a non-conservative acceleration field. $\vec{\mathbf{q}}$ has transversal and longitudinal parts

$$\vec{\mathbf{q}} = \vec{\mathbf{q}}_{\perp} + \vec{\mathbf{q}}_{||}. \tag{13.3}$$

The same applies for the velocity field $\vec{\mathbf{v}}$

$$\vec{\mathbf{v}} = \vec{\mathbf{v}}_{\perp} + \vec{\mathbf{v}}_{||} = \vec{\boldsymbol{\omega}} \times \vec{\mathbf{R}}. \tag{13.4}$$

The disassembly of the velocity field is adequately taken into account by the development of the dual turbulence equation system. In the momentum equation (13.1) 12 unknowns are “hiddenly“ contained and with the turbulence equation only 9 coupled equations are available. For the field $\boldsymbol{\rho}\vec{\mathbf{q}}$ a disassembly in longitudinal and transversal part has to be considered, too.

$$\boldsymbol{\rho}\frac{d}{dt}\vec{\mathbf{v}}(\vec{\mathbf{x}}, t) = \boldsymbol{\rho}\vec{\mathbf{q}} = (\boldsymbol{\rho}\vec{\mathbf{q}})_{\perp} + (\boldsymbol{\rho}\vec{\mathbf{q}})_{||} \tag{13.5}$$

Using the Navier-Stokes-equations (Appendix A) leads to

$$\boldsymbol{\rho}\vec{\mathbf{q}} = (\boldsymbol{\rho}\vec{\mathbf{q}})_{\perp} + (\boldsymbol{\rho}\vec{\mathbf{q}})_{||} \stackrel{?}{=} -\vec{\boldsymbol{\nabla}}\mathbf{p} + \boldsymbol{\rho}\cdot\vec{\mathbf{g}} + \eta\Delta\vec{\mathbf{v}} + (\xi + \frac{\eta}{3})\vec{\boldsymbol{\nabla}}\ (\vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{v}})$$

$\Longrightarrow$ [1]

$$(\boldsymbol{\rho}\vec{\mathbf{q}})_{\perp} \stackrel{?}{=} -\eta\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{v}} \tag{13.6}$$

and

$$\begin{aligned}(\boldsymbol{\rho}\vec{\mathbf{q}})_{||} &\stackrel{?}{=} -\ \vec{\boldsymbol{\nabla}}\mathbf{p} + \boldsymbol{\rho}\cdot\vec{\mathbf{g}} + (\xi + \eta\frac{4}{3})\vec{\boldsymbol{\nabla}}\ (\vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{v}}).\\ \vec{\mathbf{g}} &= \text{earth acceleration}\end{aligned} \tag{13.7}$$

As turbulent motions of sufficiently high reynolds number create negligible viscosity effects and on the other hand $\vec{\mathbf{q}}_{\perp}$ represents the decisive propulsion of the vortex motions turbulences are not correctly calculated by the usual equation system consisting of Navier-Stokes-equations, equation of continuity and energy equation. Equation (13.6) can not be correct. $\vec{\mathbf{q}}_{||}$ contributes nothing to the propulsion of the vortex motions. The turbulent dissipation can not be attributed to viscosity but to the matter exchange of the fluid elements and involved thermodynamic changes of state, if a local thermodynamic state is possible. Then the turbulent dissipation decisively decomposes the kinetic energy. $\Longrightarrow$

$$\boldsymbol{\rho}\vec{\mathbf{q}} = (\boldsymbol{\rho}\vec{\mathbf{q}})_{\perp} + (\boldsymbol{\rho}\vec{\mathbf{q}})_{||} \neq -\vec{\boldsymbol{\nabla}}\mathbf{p} + \boldsymbol{\rho}\cdot\vec{\mathbf{g}} + \eta\Delta\vec{\mathbf{v}} + (\xi + \frac{\eta}{3})\vec{\boldsymbol{\nabla}}\ (\vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{v}}) \tag{13.8}$$

[1] $\Delta\vec{\mathbf{v}} = \vec{\boldsymbol{\nabla}}\ (\vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{v}}) - \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{v}}$

The equations, often called conservation laws [3](Navier-Stokes-equations, equation of continuity and energy equation), do not meet these requirements for turbulence with the exception of the equation of continuity.

13.2. The complete set of turbulence-equations

In the turbulence equations (11.33) the viscous terms according to high reynolds numbers may be omitted whereas for sufficienly small reynolds numbers (laminar motions) they obtain significance.

The equation system

$$\frac{\partial \vec{\mathbf{v}}}{\partial t} + \frac{1}{2}\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}^2 - 2\vec{\mathbf{v}} \times \vec{\boldsymbol{\omega}} = \vec{\mathbf{q}} \tag{13.9}$$

$$\frac{\partial}{\partial t}\vec{\boldsymbol{\omega}} - \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{a}} = \frac{1}{2}\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}} \tag{13.10}$$

$$\frac{\partial}{\partial t}\vec{\mathbf{b}} + \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{\omega}} = \frac{1}{2}\vec{\mathbf{b}}\left[\frac{\vec{\boldsymbol{\omega}}}{\boldsymbol{\omega}^2} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}}\right] \tag{13.11}$$

with

$$\vec{\mathbf{v}} = \vec{\boldsymbol{\omega}} \times \frac{\vec{\mathbf{b}}}{\mathbf{b}^2},\ \vec{\mathbf{a}} = \vec{\mathbf{v}} \times \vec{\boldsymbol{\omega}},\ \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{v}} \perp \vec{\mathbf{v}} \tag{13.12}$$

is not complete and as the Navier-Stokes-equations as momentum balance are refuted, the usual energy equation, derived from Navier-Stokes-equations and equation of continuity, is rejected, too. So the customarily for completion used energy equation, equation of continuity and state equation can not fill this gap.

There is the possibility observing the evolution of the velocity field not only by mass transport via the equations (13.9), (13.10) and (13.11) but via the progress of their fluctuation quantities $\frac{\partial \vec{\mathbf{v}}}{\partial t}$ and $\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{v}}$, too. Assuming the equation system (12.25)

$$\frac{\partial}{\partial t}\vec{\mathbf{B}} - \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}} = 0 \tag{13.13}$$

$$\frac{\partial}{\partial t}\left(\frac{B^2}{E^2} \cdot \vec{\mathbf{E}}\right) + \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}} = 0 \tag{13.14}$$

$$\vec{\mathbf{E}} \times \frac{\vec{\mathbf{B}}}{B^2} = propagationspeed \tag{13.15}$$

with

$$|\vec{\mathbf{E}} \times \frac{\vec{\mathbf{B}}}{B^2}| \leq |\vec{\mathbf{E}}| \cdot |\frac{\vec{\mathbf{B}}}{B^2}|$$

and

$$\vec{\mathbf{E}} = \frac{\partial \vec{\mathbf{v}}}{\partial t} \text{ and } \vec{\mathbf{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{v}}, \text{as well as} \vec{\mathbf{F}} = \frac{B^2}{E^2} \cdot \vec{\mathbf{E}},$$

one obtains the further equation

$$\boxed{\frac{\partial}{\partial t}\vec{\mathbf{F}} + 2\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{\omega}} = 0}. \tag{13.16}$$

Equation 13.13 with $\vec{\mathbf{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{v}} = 2\vec{\boldsymbol{\omega}}$ results in

$$\frac{\partial}{\partial t} 2\vec{\boldsymbol{\omega}} - \vec{\boldsymbol{\nabla}} \times \frac{\partial \vec{\mathbf{v}}}{\partial t} = 0.$$

It corresponds to (13.10) on account of

$$\vec{\boldsymbol{\nabla}} \times \frac{\partial \vec{\mathbf{v}}}{\partial t} = 2 \cdot \left(\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{a}} + \frac{1}{2} \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}} \right) = 2 \cdot \frac{\partial \vec{\boldsymbol{\omega}}}{\partial t}$$

with

$$\begin{aligned}
\vec{\mathbf{v}} &= \vec{\boldsymbol{\omega}} \times \frac{\vec{\mathbf{b}}}{b^2}, \\
\vec{\mathbf{a}} &= \vec{\mathbf{v}} \times \vec{\boldsymbol{\omega}}, \\
\vec{\mathbf{v}} &\perp \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{v}} \\
\vec{\mathbf{E}} &= \frac{\partial \vec{\mathbf{v}}}{\partial t} \\
\vec{\mathbf{E}} &= 4\omega^2 \vec{\mathbf{F}}^{-1}.
\end{aligned}$$

The invers vector respectively the scalar product means $\vec{\mathbf{F}}^{-1} = \vec{\mathbf{F}}/\vec{\mathbf{F}}^2 \Longrightarrow \vec{\mathbf{F}}^{-1} \cdot \vec{\mathbf{F}} = \mathbf{1}$. This corresponds to the relation of a curvature vector $\vec{\mathbf{b}}$ and its associated radius vector $\vec{\mathbf{r}}$ of a continuously differentiable trajectory in one point $(\vec{\mathbf{x}}, t)$ with $\vec{\mathbf{b}} \cdot \vec{\mathbf{r}} = \mathbf{1}$.

The motion of a turbulence field is characterised by a vortex field $\vec{\boldsymbol{\omega}}(\vec{\mathbf{x}}, t)$ and a curvature vector field[2] $\vec{\mathbf{b}}(\vec{\mathbf{x}}, t)$.

[2]Generally, one meets in physics curvature tensor fields at least of 2nd degree as in deformation theory or General Relativity.

So one obtains the complete equation system

$$\boxed{\begin{array}{l} \vec{\mathbf{E}} + \frac{1}{2}\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}^2 - 2\vec{\mathbf{v}} \times \vec{\boldsymbol{\omega}} = \vec{\mathbf{q}} \\ \hline \frac{\partial}{\partial t}\vec{\boldsymbol{\omega}} - \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{a}} = \frac{1}{2}\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}} \\ \frac{\partial}{\partial t}\vec{\mathbf{b}} + \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{\omega}} = \frac{1}{2}\vec{\mathbf{b}}\left[\frac{\vec{\boldsymbol{\omega}}}{\boldsymbol{\omega}^2} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}}\right] \\ \frac{\partial}{\partial t}\vec{\mathbf{F}} = -2\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{\omega}} \text{ with } \vec{\mathbf{F}} = \frac{4\omega^2}{E^2} \cdot \vec{\mathbf{E}} \end{array}} . \tag{13.17}$$

At this a pairwise orthogonality of the vectors $(\vec{\mathbf{v}}, \vec{\boldsymbol{\omega}}, \vec{\mathbf{b}})$ i.e.: $\vec{\mathbf{v}} \perp \vec{\boldsymbol{\omega}}$, $\vec{\mathbf{v}} \perp \vec{\mathbf{b}}$, $\vec{\mathbf{b}} \perp \vec{\boldsymbol{\omega}}$ exists. Pursueing the trajectory of a fluid element beeing possible only after the calculation of the deterministic turbulence field the trajectory is accompanied by a frame of $\vec{\mathbf{v}}, \vec{\boldsymbol{\omega}}$ and $\vec{\mathbf{b}}$ except in points where $\vec{\boldsymbol{\omega}} = \mathbf{0}$ and $\vec{\mathbf{b}} = \mathbf{0}$ (turning points). Nevertheless, in this case $\vec{\mathbf{v}} \neq \mathbf{0}$ has to be otherwise the turbulence has come to an end.

13.3. Comments on the application of the complete equation system

On account of the theorem of Cauchy-Kowalewskaja [7] a unique solution is existing. The equation system may be numerically solved for the fields $\vec{\boldsymbol{\omega}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{q}}$ and $\vec{\mathbf{E}} = \frac{\partial \vec{\mathbf{v}}}{\partial t}$ (this is treated as an independent field as well as $\vec{\boldsymbol{\omega}}$,$\mathbf{b}$ and $\vec{\mathbf{q}}$) simultaneously obtaining the fields $\vec{\mathbf{a}}$ and $\vec{\mathbf{v}}$. The special approach of [40] enables 2 times continuously differentiable solutions not meaning analytical results. The order of differentiability may be principally driven forward. This particularly goes at the expense of the calculation effort.
Numerically solving this equation system [40] inflexible difference schemes are forbidden as beeing usual according to DNS-calculations (Direct Numerical Simulations related to Navier-Stokes-, continuum- and energy equation), as in the above equation system from the field environment removable singularities of $\vec{\mathbf{v}} = \vec{\boldsymbol{\omega}} \times \frac{\vec{\mathbf{b}}}{\mathbf{b}^2}$, $\frac{1}{2}\vec{\mathbf{b}}\left[\frac{\vec{\boldsymbol{\omega}}}{\boldsymbol{\omega}^2} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}}\right]$ and $(2\vec{\omega})^2\vec{\mathbf{F}}^{-1} = \frac{\partial \vec{\mathbf{v}}}{\partial t}$ in different space-time-points $(\vec{\mathbf{x}}, t)$ are to be recognized. This outcome is a result of possible turning points of the fluid element trajectories leading to simultaneous values of $\vec{\boldsymbol{\omega}} = 0$ and $\vec{\boldsymbol{b}} = 0$. Die fineness of the time discretisations is determined by the vortex field $\vec{\boldsymbol{\omega}}$.
The in some turbulence models mentioned space- and time-scaling in this theory is

led back to the fluctuations of curvature fields $\vec{\mathbf{b}}$ and vortex fields $\vec{\boldsymbol{\omega}}$. Quantitative dependencies become accessible through numerical calculations.
Though friction losses according to heavy turbulent motions (high reynolds numbers) may be omitted the kinetic energy density may significantly decrease. Thus a part has to be converted into inner energy of thermodynamics if a local thermodynamic balance is existent. It is recalled (chapter 3.4), that turbulent fluid motions are characterized the surroundings of fluid elements continuously exchanging their matter and thus their thermodynamic state quantities, too.
The equation system (13.17) stands out only consisting of motion quantities, i.e. velocities and their temporal and spatial differentiations, a vector curvature field, its assigned vortex field and an abstract accelleration field $\vec{\mathbf{q}}$. Mass distributions respectively densities and thermodynamic quantities as pressure and inner energy do not appear. This fact finds its application in the general-relativistic considerations, too. The density distributions may be calculated by subsequent evaluation via the known velocity fields and the equation of continuity

$$\frac{\partial}{\partial t}\boldsymbol{\rho} = -\vec{\boldsymbol{\nabla}} \cdot (\boldsymbol{\rho}\vec{\mathbf{v}}). \tag{13.18}$$

The complete turbulence equation system may be solved even if no local thermodynamics is existent. Then the subsequent evaluation is limited to density calculations. One obtains the thermodynamic pressure distribution if existent by the subsequently calculated density field $\boldsymbol{\rho}$ and the accelleration field $\vec{\mathbf{q}}$ assuming

$$(\boldsymbol{\rho}\vec{\mathbf{q}})_{\parallel} = -\vec{\boldsymbol{\nabla}}\mathbf{p} + \boldsymbol{\rho}\vec{\mathbf{g}} + (\xi + \eta\frac{4}{3})\vec{\boldsymbol{\nabla}}\ (\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{v}}). \tag{13.19}$$

via Poisson-equation [3] :

$$\Delta\mathbf{p} = -\vec{\boldsymbol{\nabla}} \cdot (\boldsymbol{\rho}\vec{\mathbf{q}}) + \vec{\boldsymbol{\nabla}} \cdot \boldsymbol{\rho}\vec{\mathbf{g}} + \vec{\boldsymbol{\nabla}} \cdot (\xi + \eta\frac{4}{3})\vec{\boldsymbol{\nabla}}\ (\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{v}}). \tag{13.20}$$

At high reynolds numbers

$$\Delta\mathbf{p} = -\vec{\boldsymbol{\nabla}} \cdot (\boldsymbol{\rho}\vec{\mathbf{q}}) + \vec{\boldsymbol{\nabla}} \cdot \boldsymbol{\rho}\vec{\mathbf{g}} \tag{13.21}$$

is certainly sufficient. But it is not obvious, whether $(\boldsymbol{\rho}\vec{\mathbf{q}})_{\parallel}$ may be represented this way. Upon positive comparison density- and pressure evolution are determined without knowledge of a related state equation. Knowing the state equation all desired thermodynamic state quantities of a single-phase system result. On the other hand a physical process is to be found to create the used inital conditions.

[3] The transversal part $(\boldsymbol{\rho}\vec{\mathbf{q}})_{\perp}$ disappears with divergence formation

The Turbulence depends on an initially assumed motion field

$$\left(\vec{\boldsymbol{\omega}}(\vec{\mathbf{x}},t_0),\vec{\mathbf{b}}(\vec{\mathbf{x}},t_0),\frac{\partial\vec{\mathbf{v}}}{\partial t}|_{t_0}\right)\Longrightarrow\vec{\mathbf{q}}(\vec{\mathbf{x}},t_0),^{4} \tag{13.22}$$

determining the further course, alone. Evaluating $\vec{\mathbf{q}}(\vec{\mathbf{x}},t_0)$ happens by summation of the terms in the momentum equation. An interaction of geometrodynamics and thermodynamics, maybe assumed in accordance with the Navier-Stokes-equations, does not apply.

13.4. The impossibility of calculating turbulence fields only knowing $\frac{\partial\vec{\mathbf{v}}}{\partial t}(\vec{\mathbf{x}},t)$

The impression may arise applying turbulence calculations that it is sufficient to use the equation system

$$\begin{aligned}&\frac{\partial}{\partial t}\vec{\mathbf{B}}-\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{E}}=0\\&\frac{\partial}{\partial t}\left(\frac{B^2}{E^2}\cdot\vec{\mathbf{E}}\right)+\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{B}}=0\\&\vec{\mathbf{E}}\times\frac{\vec{\mathbf{B}}}{B^2}=\text{propagation speed,}\end{aligned}$$

$$|\vec{\mathbf{E}}\times\frac{\vec{\mathbf{B}}}{B^2}|\leq|\vec{\mathbf{E}}|\cdot|\frac{\vec{\mathbf{B}}}{B^2}|$$

and

$$\vec{\mathbf{E}}=\frac{\partial\vec{\mathbf{v}}}{\partial t}\text{ and }\vec{\mathbf{B}}=\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{v}},\text{ sowie }\vec{\mathbf{F}}=\frac{B^2}{E^2}\cdot\vec{\mathbf{E}}$$

to determinate the velocity field numerically from the knowledge of $\frac{\partial\vec{\mathbf{v}}}{\partial t}|_i$ by

$$\vec{\mathbf{v}}(\vec{\mathbf{x}},t)_{i+1}=\frac{\partial\vec{\mathbf{v}}(\vec{\mathbf{x}},t)}{\partial t}|_i\Delta t_i+\vec{\mathbf{v}}(\vec{\mathbf{x}},t_0)_i. \tag{13.23}$$

Usually numerical time-integrations via $\Delta\vec{\mathbf{v}}=\frac{\partial\vec{\mathbf{v}}}{\partial t}\cdot\Delta t$ lead in relation to turbulence calculations firstly to error accumulation for a $\vec{\mathbf{v}}(\vec{\mathbf{x}},t)$ evaluation (refined methods of numerical mathematics integrating such vector functions do not help) and secondly achieve $\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{v}}\not\perp\vec{\mathbf{v}}$ with progressing time evolution. 1st is one reason why weather

[4] Inserting in equation (13.9)

forecasts at meteorology are difficult (besides the principally faults of the used momentum equations). The forcasts are limited to few days. The choice of shorter time steps does not help. This difficulty does not exist regarding laminar fluid dynamics. The reason for this fundamental problem of turbulence is explained as follows:

Solving the equation system (13.17) numerically the pairwise orthogonality of the vectors $\vec{\mathbf{v}}, \vec{\boldsymbol{\omega}}, \vec{\mathbf{b}}$ ($\vec{\mathbf{v}} \perp \vec{\boldsymbol{\omega}}$, $\vec{\mathbf{v}} \perp \vec{\mathbf{b}}$, $\vec{\mathbf{b}} \perp \vec{\boldsymbol{\omega}}$) has to be considered as constraint. For analytic solutions, which cannot be formulated, these conditions should be fulfilled by the initial values, alone.
Calculating $\vec{\mathbf{v}}(\vec{\mathbf{x}}, t)$ by $\vec{\boldsymbol{\omega}}$ and $\vec{\mathbf{r}}$

$$\vec{\mathbf{v}}(\vec{\mathbf{x}}, t) = \vec{\boldsymbol{\omega}}(\vec{\mathbf{x}}, t) \times \vec{\mathbf{r}}(\vec{\mathbf{x}}, t) \text{ with } \vec{\mathbf{r}}(\vec{\mathbf{x}}, t) = \frac{\vec{\mathbf{b}}(\vec{\mathbf{x}}, t)}{b^2}$$

there is a time integration of the velocity field of higher accuracy. It holds

$$\frac{\partial \vec{\mathbf{v}}}{\partial t} = \frac{\partial \vec{\boldsymbol{\omega}}}{\partial t} \times \vec{\mathbf{r}} + \vec{\boldsymbol{\omega}} \times \frac{\partial \vec{\mathbf{r}}}{\partial t}.$$

The numerical time evolution of $\vec{\mathbf{v}}_{\boldsymbol{i}} \Longrightarrow \vec{\mathbf{v}}_{\boldsymbol{i+1}}$ arises calculating $\vec{\mathbf{v}}_{\boldsymbol{i}} = \vec{\boldsymbol{\omega}}_i \times \vec{\mathbf{r}}_i$ by means of

$$\vec{\boldsymbol{\omega}}_{i+1} = \frac{\partial \vec{\boldsymbol{\omega}}}{\partial t}|_i \cdot \Delta t_i + \vec{\boldsymbol{\omega}}_i + ...$$

and

$$\vec{\mathbf{r}}_{i+1} = \frac{\partial \vec{\mathbf{r}}}{\partial t}|_i \cdot \Delta t_i + \vec{\mathbf{r}}_i + ...$$

to

$$\vec{\mathbf{v}}_{\boldsymbol{i+1}} = \left(\frac{\partial \vec{\boldsymbol{\omega}}}{\partial t}|_i \cdot \Delta t_i + \vec{\boldsymbol{\omega}}_i \right) \times \left(\frac{\partial \vec{\mathbf{r}}}{\partial t}|_i \cdot \Delta t_i + \vec{\mathbf{r}}_i \right) + ...$$

i.e.

$$\vec{\mathbf{v}}_{\boldsymbol{i+1}} = \left(\vec{\boldsymbol{\omega}}_i \times \vec{\mathbf{r}}_i \right) + \left(\frac{\partial \vec{\boldsymbol{\omega}}}{\partial t}|_i \cdot \Delta t_i \times \vec{\mathbf{r}}_i + \vec{\boldsymbol{\omega}}_i \times \frac{\partial \vec{\mathbf{r}}}{\partial t}|_i \cdot \Delta t_i \right) + \left(\frac{\partial \vec{\boldsymbol{\omega}}}{\partial t}|_i \cdot \Delta t_i \times \frac{\partial \vec{\mathbf{r}}}{\partial t}|_i \cdot \Delta t_i \right) + ...$$

respectively

$$\boxed{\vec{\mathbf{v}}_{\boldsymbol{i+1}} = \vec{\mathbf{v}}_i + \frac{\partial \vec{\mathbf{v}}}{\partial t}|_i \cdot \Delta t_i + \left(\frac{\partial \vec{\boldsymbol{\omega}}}{\partial t}|_i \times \frac{\partial \vec{\mathbf{r}}}{\partial t}|_i \right) \cdot (\Delta t_i)^2 + \boldsymbol{...}} \tag{13.24}$$

$\frac{\partial \vec{\mathbf{r}}}{\partial t}$ is derived as follows:

$$\vec{\mathbf{b}} = \vec{\mathbf{r}} \cdot (\vec{\mathbf{b}} \cdot \vec{\mathbf{b}}) \tag{13.25}$$

$\Longrightarrow$

$$\frac{\partial \vec{\mathbf{b}}}{\partial t} = \mathbf{b^2} \frac{\partial \vec{\mathbf{r}}}{\partial t} + 2\vec{\mathbf{r}} \left(\frac{\partial \vec{\mathbf{b}}}{\partial t} \cdot \vec{\mathbf{b}} \right)$$

$\Longrightarrow$

$$\frac{\partial \vec{\mathbf{r}}}{\partial t} = \left[\frac{\partial \vec{\mathbf{b}}}{\partial t} - 2 \frac{\vec{\mathbf{b}}}{b^2} \left(\frac{\partial \vec{\mathbf{b}}}{\partial t} \cdot \vec{\mathbf{b}} \right) \right] / b^2.$$

In particular space-time points $(\vec{\mathbf{x}}, t)$ fluid elements may be in the proximity or direct in a turning point, in which $\vec{\boldsymbol{\omega}}(\vec{\mathbf{x}}, t) = \mathbf{0}$ as well as $\vec{\mathbf{b}}(\vec{\mathbf{x}}, t) = \mathbf{0}$ and such $\vec{\mathbf{r}}(\vec{\mathbf{x}}, t) = \vec{\mathbf{b}}/b^2 = \boldsymbol{\infty}$ holds. So the temporal evolution term of 2nd order is vital for turbulence calculations. That is why the with (13.23) mentioned velocity integration is not expedient. Considering the complete turbulence equation set the temporal velocity integration automatically results in the desired order.

The described link leads to the fact, too: Navier-Stokes equations cannot describe turbulent fluiddynamics.

13.5. Summary

With the installation of the equation system (13.17) a geometrodynamics of turbulence is expressed only obtaining motion quantities i.e. it only consists of velocities and their time and space derivatives. A corresponding statement is made for their initial- and boundary conditions. Special material properties of a fluid may only influence solutions via initial- and boundary conditions. Therefore it may be important to formulate the suitable process of the genesis of such conditions.
In the case of fluid turbulence there is no requirement for establishing chaos theories!

13.6. Appendix A: The basic equations of laminar fluid-dynamics

In the following, the known and established fundamental equations of laminar fluid-dynamics are derived and presented.

13.6.1. The equation of continuity

Beeing $\boldsymbol{\rho} = \boldsymbol{\rho}(\vec{\mathbf{x}}, t)$ the mass density and $\vec{\mathbf{v}} = \vec{\mathbf{v}}(\vec{\mathbf{x}}, t)$ the velocity of the fluid elements the mass density passing through the surface of a volume per unit time is

$$\oint \boldsymbol{\rho} \cdot \vec{\mathbf{v}} \cdot d\vec{\mathbf{f}}.$$

On the other hand this corresponds to the mass density decrease of the volume per time of

$$-\frac{\partial}{\partial t} \int_V \boldsymbol{\rho} \cdot d\mathbf{V}.$$

So one obtains

$$\frac{\partial}{\partial t} \int_V \boldsymbol{\rho} \cdot d\mathbf{V} = -\oint \boldsymbol{\rho} \cdot \vec{\mathbf{v}} \cdot d\vec{\mathbf{f}}.$$

Using Green's formula

$$\oint \boldsymbol{\rho} \cdot \vec{\mathbf{v}} \cdot d\vec{\mathbf{f}} = \int_{\boldsymbol{V}} \vec{\nabla} \cdot (\boldsymbol{\rho} \cdot \vec{\mathbf{v}}) \cdot d\mathbf{V}$$

there is

$$\int_V \left(\frac{\partial}{\partial t} \boldsymbol{\rho} + \vec{\nabla} \cdot (\boldsymbol{\rho} \cdot \vec{\mathbf{v}}) \cdot d\mathbf{V} \right) = \mathbf{0}$$

respectively

$$\boxed{\frac{\partial}{\partial t} \boldsymbol{\rho} + \vec{\nabla} \cdot (\boldsymbol{\rho} \cdot \vec{\mathbf{v}}) = \mathbf{0}}. \tag{13.26}$$

Originating from a constant density $\boldsymbol{\rho} = const$ one obtains

$$\vec{\nabla} \cdot \vec{\mathbf{v}} = \mathbf{0}. \tag{13.27}$$

That means, a divergence-free velocity field is the necessary condition of a constant mass-distribution, but this condition is not sufficient! The equation of continuity proves to be the only correct conservation equation of the three fundamental fluid-dynamics equations: equation of continuity, Navier-Stokes-equations and energy equation. The structure of this equation of continuity may be the formal standard for the categorisation of scalar conservation quantities at least in continuum physics. In the following it is shown, that this can not be achieved for the energy equation.

13.6.2. The Navier-Stokes-equations

Considering the influence of viscose friction one usually [25] takes as its starting point the momentum density conservation. Expediently, the Ricci Calculus is used as formulation. $\vec{\mathbf{g}}$ means an additional accelleration field, such as the earth acceleration. These equations finding their empirical confirmation in the laminar or stationary case are based directly on

$$\rho \cdot \left(\frac{\partial v_i}{\partial t} + v_k \cdot \frac{\partial v_i}{\partial x_k} \right) = -\frac{\partial}{\partial x_k}(p\delta_{ik} - \sigma_{ik}) + \rho \cdot g_i. \tag{13.28}$$

The stress tensor$\boldsymbol{\sigma}_{ik}$ must consist of a symmetrical tensor whose trace vanishes in the situation $\vec{\mathbf{v}} = \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}$ with $\vec{\boldsymbol{\omega}} \neq \mathbf{0}$ plus a diagonal tensor. Such a linear composition of the tensors

$$\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \text{ and } \delta_{ik}\frac{\partial v_l}{\partial x_l} \tag{13.29}$$

is needed with coefficients determined experimentally [25]. $\boldsymbol{\sigma}_{ik}$ is the most general tensor of linear, spatial velocity derivations of 1st order within these conditions:

$$\sigma_{ik} = \eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l} \right) + \xi\delta_{ik}\frac{\partial v_l}{\partial x_l}. \tag{13.30}$$

The coefficients η and ξ not essentially changing in the fluid one obtains

$$\begin{aligned} \frac{\partial}{\partial x_k}\sigma_{ik} &= \eta\left(\frac{\partial^2 v_i}{\partial x_k^2} + \frac{\partial}{\partial x_i}\frac{\partial v_k}{\partial x_k} - \frac{2}{3}\frac{\partial}{\partial x_i}\frac{\partial v_l}{\partial x_l} \right) + \xi\frac{\partial}{\partial x_i}\frac{\partial v_l}{\partial x_l} \\ &= \eta\frac{\partial^2 v_i}{\partial x_k^2} + (\xi + \frac{1}{3}\eta)\frac{\partial}{\partial x_i}\frac{\partial v_l}{\partial x_l}. \end{aligned} \tag{13.31}$$

Because of

$$\sum_{k=1}^{3} \frac{\partial^2 v_i}{\partial x_k^2} = \Delta v_i \tag{13.32}$$

and

$$\sum_{l=1}^{3} \frac{\partial v_l}{\partial x_l} = \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{v}} \tag{13.33}$$

the known Navier-Stokes equations

$$\frac{\partial \vec{\mathbf{v}}}{\partial t} + (\vec{\mathbf{v}} \cdot \vec{\boldsymbol{\nabla}})\vec{\mathbf{v}} = -\frac{1}{\rho}\vec{\boldsymbol{\nabla}}p + \vec{\mathbf{g}} + \nu\Delta\vec{\mathbf{v}} + (\zeta + \frac{\nu}{3})\vec{\boldsymbol{\nabla}}\ (\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{v}}) \tag{13.34}$$

are finally achieved with $\nu = \frac{\eta}{\rho}$ and $\zeta = \frac{\xi}{\rho}$. This demonstrates the coefficients ν and ζ not beeing constant coefficients at heavy density fluctuations. On the other hand the question arises, if the the coefficients η and ξ may be seen as essentialy constant (prudently not!). Whereby the above derivation of the viscose terms of the Navier-Stokes-equations for turbulence problems of sufficiently high reynolds numbers appears dubious. On the other side, numerical examinations [40] show that the viscous friction in case of heavy turbulence problems is irrelevant. The known considerations of similarity by the Reynolds law of similarity do not have a sufficient sustainability.

13.6.3. The equation of energy

With the energy-equation of laminar hydrodynamics (not the hydrodynamics of turbulent motion) the kinetic energy of the fluid and its inner thermodynamic ernergy density are balanced. The derivation essentially follows the presentation of [27]. Here non adiabatic fluid motions and the friction viscosity are additionally considered. It is

$$\mathbf{e_{sum}} = \mathbf{e_{kin}} + \mathbf{e_{in}} \tag{13.35}$$

defined with

$$\boxed{\frac{\partial}{\partial t}\mathbf{e_{sum}} = -\vec{\boldsymbol{\nabla}} \cdot (\vec{\mathbf{v}}(\mathbf{e_{sum}} + \mathbf{p})) + \rho \cdot \vec{\mathbf{v}} \cdot \left(\vec{\mathbf{g}} + \nu \Delta \vec{\mathbf{v}} + (\zeta + \frac{\nu}{3})\vec{\boldsymbol{\nabla}}\ (\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{v}})\right) + \vec{\boldsymbol{\nabla}} \lambda \vec{\boldsymbol{\nabla}} T}\ . \tag{13.36}$$

In the following this connection is derived. Heat radiation is ignored significantly arising above $500°C$. The following specifications are made:

Symbol	Physical Quantity
$\mathbf{e_{kin}} = \frac{\boldsymbol{\rho}}{\mathbf{2}}\boldsymbol{v}^{\mathbf{2}}$	kinetic energy density
$\boldsymbol{\varepsilon}$	spec. inner energy
$\mathbf{e_{in}} = \boldsymbol{\rho\varepsilon}$	inner energy density
$\mathbf{e_{sum}} = \mathbf{e_{kin}} + \mathbf{e_{in}}$	sum of kinetic energy density and inner energy density
$\boldsymbol{h} = \boldsymbol{\varepsilon} + \frac{p}{\rho}$	spec. enthalpy
$\rho\boldsymbol{h}$	enthalpy density
$\boldsymbol{s}$	spec. entropy
$\rho\boldsymbol{s}$	entropy density
$\boldsymbol{p}$	pressure
$\boldsymbol{T}$	temperature
$\boldsymbol{\rho}$	density
$\boldsymbol{Q}$	spec. heat
$\boldsymbol{\lambda}$	heat conduction coefficient
$\vec{\mathbf{g}}$	external mass acceller-ation field

Eamined is:

$$\frac{\partial}{\partial t}\mathbf{e_{sum}} = \frac{\partial}{\partial t}\left(\frac{\boldsymbol{\rho}}{\mathbf{2}}\boldsymbol{v}^{\mathbf{2}}\right) + \frac{\partial}{\partial t}(\boldsymbol{\rho\varepsilon}) \tag{13.37}$$

First $\frac{\partial}{\partial t}\left(\frac{\boldsymbol{\rho}}{\mathbf{2}}\boldsymbol{v}^{\mathbf{2}}\right)$ **:**

$$\frac{\partial}{\partial t}\left(\frac{\boldsymbol{\rho}}{\mathbf{2}}\boldsymbol{v}^{\mathbf{2}}\right) = \frac{v^2}{2}\frac{\partial}{\partial t}\boldsymbol{\rho} + \boldsymbol{\rho}\vec{\mathbf{v}}\cdot\frac{\partial}{\partial t}\vec{\mathbf{v}} \tag{13.38}$$

$$\frac{\partial}{\partial t}\boldsymbol{\rho} = -\vec{\boldsymbol{\nabla}}\cdot(\boldsymbol{\rho}\vec{\mathbf{v}}) \quad \text{equation of continuity} \tag{13.39}$$

Navier-Stokes-equations:

$$\frac{\partial\vec{\mathbf{v}}}{\partial t} = -(\vec{\mathbf{v}}\cdot\vec{\boldsymbol{\nabla}})\vec{\mathbf{v}} - \frac{1}{\rho}\vec{\boldsymbol{\nabla}}\,\mathbf{p} + \left(\vec{\mathbf{g}} + \nu\Delta\vec{\mathbf{v}} + (\zeta + \frac{\nu}{3})\vec{\boldsymbol{\nabla}}\,(\vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{v}})\right) \qquad \Longrightarrow \tag{13.40}$$

$$\frac{\partial}{\partial t}\left(\frac{\boldsymbol{\rho}}{\mathbf{2}}\boldsymbol{v}^{\mathbf{2}}\right) = -\frac{v^2}{2}\vec{\boldsymbol{\nabla}}\cdot(\boldsymbol{\rho}\vec{\mathbf{v}}) - \boldsymbol{\rho}\vec{\mathbf{v}}\cdot(\vec{\mathbf{v}}\cdot\vec{\boldsymbol{\nabla}})\vec{\mathbf{v}} - \vec{\mathbf{v}}\cdot\vec{\boldsymbol{\nabla}}\mathbf{p} + \boldsymbol{\rho}\vec{\mathbf{v}}\cdot\left(\vec{\mathbf{g}} + \nu\Delta\vec{\mathbf{v}} + (\zeta + \frac{\nu}{3})\vec{\boldsymbol{\nabla}}\,(\vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{v}})\right) \tag{13.41}$$

$$\begin{aligned}\boldsymbol{\rho}\vec{\mathbf{v}}\cdot(\vec{\mathbf{v}}\cdot\vec{\boldsymbol{\nabla}})\vec{\mathbf{v}} &= \boldsymbol{\rho}\vec{\mathbf{v}}\cdot\left[\frac{1}{2}\vec{\boldsymbol{\nabla}}v^2 - \vec{\mathbf{v}}\times\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{v}}\right] \\ &= \frac{1}{2}\boldsymbol{\rho}\vec{\mathbf{v}}\cdot\vec{\boldsymbol{\nabla}}v^2 \qquad \text{on account of} \qquad \vec{\mathbf{v}}\cdot\vec{\mathbf{v}}\times\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{v}} = 0\end{aligned} \tag{13.42}$$

$$\boldsymbol{dh} = \boldsymbol{Tds} + \frac{1}{\rho}\boldsymbol{dp} \;\text{ from thermodynamics} \tag{13.43}$$

$$\vec{\boldsymbol{\nabla}}\boldsymbol{p} = \rho\vec{\boldsymbol{\nabla}}\boldsymbol{h} - \boldsymbol{\rho}T\vec{\boldsymbol{\nabla}}\boldsymbol{s} \tag{13.44}$$

$$\vec{\boldsymbol{\nabla}}\cdot(\boldsymbol{\rho}\vec{\mathbf{v}}\cdot\frac{v^2}{2}) = \frac{v^2}{2}\vec{\boldsymbol{\nabla}}\cdot(\boldsymbol{\rho}\vec{\mathbf{v}}) + \boldsymbol{\rho}\vec{\mathbf{v}}\cdot\frac{1}{2}\vec{\boldsymbol{\nabla}}\mathbf{v^2} \tag{13.45}$$

$$\boxed{\begin{aligned}&\frac{\partial}{\partial t}\left(\frac{\boldsymbol{\rho}}{\mathbf{2}}\boldsymbol{v^2}\right) = \\ &-\vec{\boldsymbol{\nabla}}\cdot(\boldsymbol{\rho}\vec{\mathbf{v}}\cdot\frac{v^2}{2}) - \rho\vec{\mathbf{v}}\cdot[\vec{\boldsymbol{\nabla}}\boldsymbol{h} - T\vec{\boldsymbol{\nabla}}\boldsymbol{s}] + \boldsymbol{\rho}\vec{\mathbf{v}}\cdot\left(\vec{\mathbf{g}} + \nu\Delta\vec{\mathbf{v}} + (\zeta + \frac{\nu}{3})\vec{\boldsymbol{\nabla}}\;(\vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{v}})\right)\end{aligned}} \tag{13.46}$$

Now $\frac{\partial}{\partial t}(\boldsymbol{\rho\varepsilon})$ **:**

$$\frac{\partial}{\partial t}(\boldsymbol{\rho\varepsilon}) = \boldsymbol{\varepsilon}\cdot\frac{\partial\rho}{\partial t} + \boldsymbol{\rho}\frac{\partial\boldsymbol{\varepsilon}}{\partial t} \tag{13.47}$$

on account of

$$\begin{aligned}\boldsymbol{d\varepsilon} =& \boldsymbol{Tds} + \frac{p}{\rho^2}\boldsymbol{d\rho} \;\text{ from thermodynamics and} \\ \frac{\partial}{\partial t}\boldsymbol{\rho} =& -\vec{\boldsymbol{\nabla}}\cdot(\boldsymbol{\rho}\vec{\mathbf{v}}) \;\text{ equation of continuity}\end{aligned}$$

$$\frac{\partial}{\partial t}(\boldsymbol{\rho\varepsilon}) = \boldsymbol{h}\cdot\frac{\partial\rho}{\partial t} + \boldsymbol{\rho T}\frac{\partial\boldsymbol{s}}{\partial t}. \tag{13.48}$$

Adiabatic transport means

$$\frac{\boldsymbol{ds}}{\boldsymbol{dt}} = \frac{\partial s}{\partial t} + \vec{\mathbf{v}}\cdot\vec{\boldsymbol{\nabla}}\boldsymbol{s} = \mathbf{0} \tag{13.49}$$

Considering non adiabatic transport heat is withdrawn from the Fluid by heat conduction. I.e.

$$\boldsymbol{\rho T}\frac{\boldsymbol{ds}}{\boldsymbol{dt}} = \boldsymbol{\rho T}\frac{\partial s}{\partial t} + \boldsymbol{\rho T}\vec{\mathrm{v}} \cdot \vec{\boldsymbol{\nabla}}\boldsymbol{s} = \vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \boldsymbol{\lambda}\vec{\boldsymbol{\nabla}}\boldsymbol{T} \tag{13.50}$$

$$\frac{\partial s}{\partial t} = -\vec{\mathrm{v}} \cdot \vec{\boldsymbol{\nabla}}\boldsymbol{s} + \frac{1}{\boldsymbol{\rho T}}\vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \boldsymbol{\lambda}\vec{\boldsymbol{\nabla}}\boldsymbol{T} \tag{13.51}$$

$$\boxed{\frac{\partial}{\partial t}(\boldsymbol{\rho\varepsilon}) = -\boldsymbol{h} \cdot \vec{\boldsymbol{\nabla}} \cdot (\boldsymbol{\rho}\vec{\mathrm{v}}) - \boldsymbol{\rho}\mathbf{T}\vec{\mathrm{v}} \cdot \vec{\boldsymbol{\nabla}}\boldsymbol{s} + \vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \boldsymbol{\lambda}\vec{\boldsymbol{\nabla}}\boldsymbol{T}} \tag{13.52}$$

On account of

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\boldsymbol{\rho}}{\mathbf{2}}\boldsymbol{v}^{\mathbf{2}}\right) =& -\vec{\boldsymbol{\nabla}} \cdot (\boldsymbol{\rho}\vec{\mathrm{v}} \cdot \frac{v^2}{2}) - \rho\vec{\mathrm{v}} \cdot [\vec{\boldsymbol{\nabla}}\boldsymbol{h} - T\vec{\boldsymbol{\nabla}}\boldsymbol{s}] \\ & + \boldsymbol{\rho}\vec{\mathrm{v}} \cdot \left(\vec{\mathbf{g}} + \nu\Delta\vec{\mathrm{v}} + (\zeta + \frac{\nu}{3})\vec{\boldsymbol{\nabla}}\ (\vec{\boldsymbol{\nabla}} \cdot \vec{\mathrm{v}})\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\boldsymbol{\rho}}{\mathbf{2}}\boldsymbol{v}^{\mathbf{2}} + \boldsymbol{\rho\varepsilon}\right) =& -\vec{\boldsymbol{\nabla}} \cdot (\boldsymbol{\rho}\vec{\mathrm{v}} \cdot \frac{v^2}{2}) - \boldsymbol{h} \cdot \vec{\boldsymbol{\nabla}} \cdot (\boldsymbol{\rho}\vec{\mathrm{v}}) - \rho\vec{\mathrm{v}} \cdot \vec{\boldsymbol{\nabla}}\boldsymbol{h} \\ & + \boldsymbol{\rho}\vec{\mathrm{v}} \cdot \left(\vec{\mathbf{g}} + \nu\Delta\vec{\mathrm{v}} + (\zeta + \frac{\nu}{3})\vec{\boldsymbol{\nabla}}\ (\vec{\boldsymbol{\nabla}} \cdot \vec{\mathrm{v}})\right) + \vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \boldsymbol{\lambda}\vec{\boldsymbol{\nabla}}\boldsymbol{T}.\end{aligned} \tag{13.53}$$

follows. Respectively

$$\vec{\boldsymbol{\nabla}} \cdot (\boldsymbol{\rho}\vec{\mathrm{v}}\boldsymbol{h}) = \boldsymbol{h} \cdot \vec{\boldsymbol{\nabla}} \cdot (\boldsymbol{\rho}\vec{\mathrm{v}}) + \rho\vec{\mathrm{v}} \cdot \vec{\boldsymbol{\nabla}}\boldsymbol{h} \tag{13.54}$$

follows

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\boldsymbol{\rho}}{\mathbf{2}}\boldsymbol{v}^{\mathbf{2}} + \boldsymbol{\rho\varepsilon}\right) =& -\vec{\boldsymbol{\nabla}} \cdot (\boldsymbol{\rho}\vec{\mathrm{v}} \cdot \frac{v^2}{2}) - \vec{\boldsymbol{\nabla}} \cdot (\boldsymbol{\rho}\vec{\mathrm{v}}\boldsymbol{h}) \\ & + \boldsymbol{\rho}\vec{\mathrm{v}} \cdot \left(\vec{\mathbf{g}} + \nu\Delta\vec{\mathrm{v}} + (\zeta + \frac{\nu}{3})\vec{\boldsymbol{\nabla}}\ (\vec{\boldsymbol{\nabla}} \cdot \vec{\mathrm{v}})\right) + \vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \boldsymbol{\lambda}\vec{\boldsymbol{\nabla}}\boldsymbol{T}\end{aligned} \tag{13.55}$$

and

$$\frac{\partial}{\partial t}\left(\frac{\boldsymbol{\rho}}{\mathbf{2}}\boldsymbol{v}^{\mathbf{2}} + \boldsymbol{\rho\varepsilon}\right) = -\vec{\boldsymbol{\nabla}} \cdot \boldsymbol{\rho}\vec{\mathrm{v}} \cdot (\frac{v^2}{2} + \boldsymbol{h}) + \boldsymbol{\rho}\vec{\mathrm{v}} \cdot \left(\vec{\mathbf{g}} + \nu\Delta\vec{\mathrm{v}} + (\zeta + \frac{\nu}{3})\vec{\boldsymbol{\nabla}}\ (\vec{\boldsymbol{\nabla}} \cdot \vec{\mathrm{v}})\right) + \vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \boldsymbol{\lambda}\vec{\boldsymbol{\nabla}}\boldsymbol{T} \tag{13.56}$$

and finally

$$\boxed{\frac{\partial}{\partial t}\mathbf{e_{sum}} = -\vec{\boldsymbol{\nabla}} \cdot (\vec{\mathrm{v}}(\mathbf{e_{sum}} + \mathbf{p})) + \rho \cdot \vec{\mathrm{v}} \cdot \left(\vec{\mathbf{g}} + \nu\Delta\vec{\mathrm{v}} + (\zeta + \frac{\nu}{3})\vec{\boldsymbol{\nabla}}\ (\vec{\boldsymbol{\nabla}} \cdot \vec{\mathrm{v}})\right) + \vec{\boldsymbol{\nabla}}\lambda\vec{\boldsymbol{\nabla}}T}. \tag{13.57}$$

Even without external field $\vec{\mathbf{g}}$ and without consideration of heat conduction there is no equation of conservation for the sum of kinetic and inner energy density $\mathbf{e_{sum}}$!

I.e. $\mathbf{e_{sum}}$ is not a conservation quantity. Energy is lost by fluid motions against non conservative local forces. A potential representing the seemingly lost energy density does not exist! But this is neccessary to formulate a real conservation equation. Non conservative accelleration fields do not permit energy-density conservation. Apart from this the energy in a point $(\vec{\boldsymbol{x}}, t)$ is zero in pure field theories, anyway.

$$\frac{\partial}{\partial t}\mathbf{e_{sum}} + \vec{\boldsymbol{\nabla}} \cdot (\mathbf{e_{sum}}\vec{\mathbf{v}}) \neq \mathbf{0} \tag{13.58}$$

A conservation of the sum of kinetic and inner energy density had to be represented by a suitable equation of continuity. In many cases the energy equation is described in sufficient approximation by

$$\frac{\partial}{\partial t}\mathbf{e_{sum}} = -\vec{\boldsymbol{\nabla}} \cdot (\vec{\mathbf{v}}(\mathbf{e_{sum}} + \mathbf{p})). \tag{13.59}$$

Generally, heat conduction can usually be neglected contrary to the energy convection (adiabatic convection), unless the velocities are small enough.
(Example: thermics in the atmosphere)

13.7. Appendix B

13.7.1. The Helmholtz-decomposition-theorem

It is shown, that every continuously differentiable vector field may be uniquely disassembled in a longitudinal and a transversal part, i.e.

$$\vec{\boldsymbol{q}}(\vec{\boldsymbol{x}}) = \vec{\nabla}\boldsymbol{\Phi}(\vec{\boldsymbol{x}}) + \vec{\nabla} \times \vec{\boldsymbol{A}}(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{q}}(\vec{\boldsymbol{x}})_{\parallel} + \vec{\boldsymbol{q}}(\vec{\boldsymbol{x}})_{\perp}. \tag{13.60}$$

Proof:

Without loss of generality

$$\begin{aligned} \vec{\boldsymbol{q}}(\vec{\boldsymbol{x}}) &= \int_V \vec{\boldsymbol{q}}(\vec{\boldsymbol{x}}')\boldsymbol{\delta}(\vec{\boldsymbol{x}}, \vec{\boldsymbol{x}}')dV' \\ &= \int_V \vec{\boldsymbol{q}}(\vec{\boldsymbol{x}}')\Delta\boldsymbol{G}(\vec{\boldsymbol{x}}, \vec{\boldsymbol{x}}')dV' \qquad \text{with the Green function } \boldsymbol{G}, \qquad \Delta\boldsymbol{G} = \boldsymbol{\delta}(\vec{\boldsymbol{x}}, \vec{\boldsymbol{x}}') \\ &= \boldsymbol{\Delta}\int_V \vec{\boldsymbol{q}}(\vec{\boldsymbol{x}}')\boldsymbol{G}(\vec{\boldsymbol{x}}, \vec{\boldsymbol{x}}')dV' \qquad \boldsymbol{\Delta} \text{ only differentiated by the unslashed } \vec{\boldsymbol{x}}. \end{aligned} \tag{13.61}$$

can be written. Because of

$$\Delta\vec{\boldsymbol{Q}} = \vec{\nabla}(\vec{\nabla}\cdot\vec{\boldsymbol{Q}}) - \vec{\nabla}\times(\vec{\nabla}\times\vec{\boldsymbol{Q}}) \tag{13.62}$$

follows

$$\vec{\boldsymbol{q}}(\vec{\boldsymbol{x}}) = \vec{\nabla}\left(\vec{\nabla}\cdot\int_V \vec{\boldsymbol{q}}(\vec{\boldsymbol{x}}')\boldsymbol{G}(\vec{\boldsymbol{x}},\vec{\boldsymbol{x}}')dV'\right) - \vec{\nabla}\times\left(\vec{\nabla}\times\int_V \vec{\boldsymbol{q}}(\vec{\boldsymbol{x}}')\boldsymbol{G}(\vec{\boldsymbol{x}},\vec{\boldsymbol{x}}')dV'\right). \tag{13.63}$$

and

$$\vec{\boldsymbol{q}}(\vec{\boldsymbol{x}}) = \vec{\nabla}\left(\int_V \vec{\boldsymbol{q}}(\vec{\boldsymbol{x}}')\vec{\nabla}\cdot\boldsymbol{G}(\vec{\boldsymbol{x}},\vec{\boldsymbol{x}}')dV'\right) + \vec{\nabla}\times\left(\int_V \vec{\boldsymbol{q}}(\vec{\boldsymbol{x}}')\times\vec{\nabla}\boldsymbol{G}(\vec{\boldsymbol{x}},\vec{\boldsymbol{x}}')dV'\right). \tag{13.64}$$

As[5]

$$\vec{\nabla}\boldsymbol{G}(\vec{\boldsymbol{x}},\vec{\boldsymbol{x}}') = -\vec{\nabla}'\boldsymbol{G}(\vec{\boldsymbol{x}},\vec{\boldsymbol{x}}') \tag{13.65}$$

follows

$$\vec{\boldsymbol{q}}(\vec{\boldsymbol{x}}) = -\vec{\nabla}\left(\int_V \vec{\boldsymbol{q}}(\vec{\boldsymbol{x}}')\cdot\vec{\nabla}'\boldsymbol{G}(\vec{\boldsymbol{x}},\vec{\boldsymbol{x}}')dV'\right) - \vec{\nabla}\times\left(\int_V \vec{\boldsymbol{q}}(\vec{\boldsymbol{x}}')\times\vec{\nabla}'\boldsymbol{G}(\vec{\boldsymbol{x}},\vec{\boldsymbol{x}}')dV'\right). \tag{13.66}$$

So it results in the longitudinal part

$$\vec{\boldsymbol{q}}_{\|}(\vec{\boldsymbol{x}}) = -\vec{\nabla}\left(\int_V \vec{\boldsymbol{q}}(\vec{\boldsymbol{x}}')\cdot\vec{\nabla}'\boldsymbol{G}(\vec{\boldsymbol{x}},\vec{\boldsymbol{x}}')dV'\right) \tag{13.67}$$

and the transversal part

$$\vec{\boldsymbol{q}}_{\perp}(\vec{\boldsymbol{x}}) = -\vec{\nabla}\times\left(\int_V \vec{\boldsymbol{q}}(\vec{\boldsymbol{x}}')\times\vec{\nabla}'\boldsymbol{G}(\vec{\boldsymbol{x}},\vec{\boldsymbol{x}}')dV'\right) \tag{13.68}$$

respectively

$$\boldsymbol{\Phi}(\vec{\boldsymbol{x}}) = -\int_V \vec{\boldsymbol{q}}(\vec{\boldsymbol{x}}')\cdot\vec{\nabla}'\boldsymbol{G}(\vec{\boldsymbol{x}},\vec{\boldsymbol{x}}')dV' \tag{13.69}$$

and the vector potential

[5] respectively $\boldsymbol{G}(\vec{\boldsymbol{x}},\vec{\boldsymbol{x}}') = -\frac{1}{4\pi}\frac{1}{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}'|}$

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{x}}) = -\int_V \vec{\boldsymbol{q}}(\vec{\boldsymbol{x}}') \times \vec{\nabla}' \boldsymbol{G}(\vec{\boldsymbol{x}}, \vec{\boldsymbol{x}}') dV'. \tag{13.70}$$

Part IV.

Relativistic fluctuations

14. Introduction

In this part results of the turbulence theory are used finding evolution equations for General Relativity. This possibility decisively follows from the fact that the background of the above fluctuations are specially assigned geometrodynamics. Establishing the Einstein Equations suitable evolution equations are not yet found. Solutions are only known for special cases. To solve these equations further constraints must be introduced. The complexity of this difficulty is marked by Mathias Blau[1] with *"tremendously complicated set of equations, and trying to learn and say something about general properties of solutions to these equations is very challenging"*. Moreover, it is basically cloudy how to calculate turbulent cosmological motions, which can not even be defined in the nonrelativistical case.

The Einstein Equations are

$$\mathbf{R}_{\mu\nu} - \frac{1}{2}\mathbf{g}_{\mu\nu}\mathbf{R} = 8\pi \cdot G_N \mathbf{T}_{\mu\nu}. \tag{14.1}$$

Both sides of these equations vanishing after using the divergence operator for Einstein was an essential criterion deriving his equations. The left side of the equations only obtains geometrical quantities of Space-Time: metric tensor $\mathbf{g}_{\mu\nu}$, Ricci Curvature Tensor $\mathbf{R}_{\mu\nu}$ and the Ricci Scalar $\mathbf{R}$. The right side of the equations obtains besides a constant the energy momentum tensor $\mathbf{T}_{\mu\nu}$, only. This one is described in relevant literature for an ideal fluid by the tensor[37]

$$\mathbf{T}_{\mu\nu} = \boldsymbol{p}\mathbf{g}_{\mu\nu} + (\rho + p/c^2)\mathbf{v}_{\mu} \cdot \mathbf{v}_{\nu} \tag{14.2}$$

assuming that the present fluctuation of Space-Time makes no contribution. This assumption does not present a problem as long as the observed curvatures are only considered in connection with sufficiently weak fluctuations.

The known energy-, momentum- and continuum equations for an ideal fluid with laminar flow arise after operating the divergence on the above tensor. For turbulent fluctuations no Lagrangian is existent. In this case an energy momentum tensor can not be formulated. The questionable existence of this tensor presents a relevant problem for the evaluation of the Einstein Equations as Stephen Hawking noted [22] page 64:

“ *The conditions ... do not tell one how to construct the energy-momentum tensor for a given set of fields, or whether it is unique. In practice one relies heavily on one's intuitive knowledge of what energy and momentum are. However, there is a definite and unique formula for the energy-momentum tensor in the case that the equations of the fields can be derived from a Lagrangian.*“

By the unification of gravitational and electromagnetic fluctuations found in this treatise it is shown that at first an energy momentum tensor is assigned to gravitational waves and secondly an even greater importance belongs to the Einstein-Equations. On the other hand the current formulation of the linearised equation of gravitational waves is questioned.

The above derived turbulence theory is at first pure geometrodynamics of a continuum uniquely related to turbulently moved matter and secondly, avoiding physical assumptions, exact with this in mind. Its physical evaluation using the equation of continuity leads to a smeared matter distribution. The structure of turbulent geometrodynamics is in essence characterised by two communicating vector fields, a vortex field and an accompanying vector curvature field. There is a further geometrodynamics of the continuum, based on fluctuations of deformation vector fields. This geometrodynamics is described by quasilinear, generalized Maxwell Equations.
Both concepts of geometrodynamics in Euclidian space may be used for the installation of evolution equations of General Relativity, as they do not explicitly contain matter but only motion quantities or their derivatives without using the Einstein Equations. The motion quantities are mapped into the observation space. The Einstein Space may be devided into a spatial part with the signature $(+,+,+)$ and a temporal part with a sinature $(-)$. The spatial part corresponds to a 3-dimensional Riemannian space, called Riemannian hypersurface $\boldsymbol{S}$. This may be seen as a deformation of a suitable observation space $\boldsymbol{B}$ or coordinate space. This flat observation space is abstractly to be seen, as an actual human observer lives in an at least slightly curved space. The physics considered from such an observation space allows **comparatively** simple mathematical formulations.

Such considerations lead at first to evolution equations of turbulently moved matter by mapping the motion quatities into the observation space and secondly to gravitational waves from a new perspective interpreting the mapping from the observation space into the hypersurface as deformation. The constant propagation speed according deformation fluctuations in chapter 12 led to the linear Maxwell Equations.
That is why a unification of the Maxwell Field and the gravitational field is only a small step in mind. The Einstein Equations prove to be more than the key

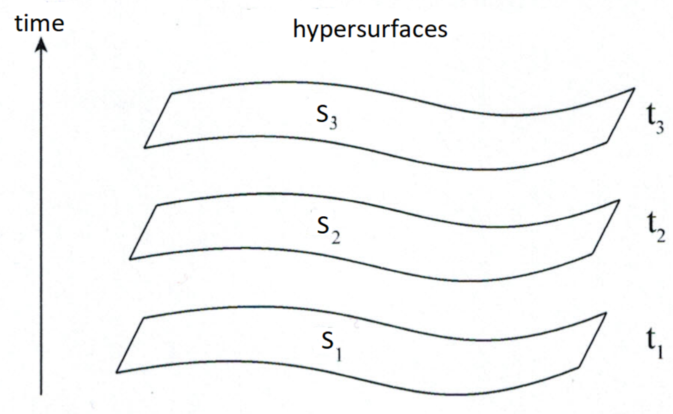


Figure 14.1.: **hypersurfaces** $\mathbf{S_i}$

equations of the macroscopic Space-Time and may be fundamental equations of the micro-world with natural causality. So the possibility of quantizing the gravitational field in the sense of quantum electrodnamics seems to be plausible. This possibility is not used in this treatise.

15. Evolution of matter distributions by cosmological turbulences

The Einstein Equations of General Relativity consist of 10 non linear coupled equations for a 4-dimensional space. Studying systems of astrophysical relevance determined by strong and dynamical gravitational fields analytical solutions appear excluded on one side and known numerical methods prove to be extremely laborious and complicated on the other side.

The equation system (13.17) succeeds in a drastic facilitation considering cosmological turbulent motions.

$$\boxed{\begin{array}{l}
\dfrac{\partial \vec{\mathbf{v}}}{\partial t} + \dfrac{1}{2}\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}^2 - 2\vec{\mathbf{v}} \times \vec{\boldsymbol{\omega}} = \vec{\mathbf{q}} \\
\hline
\dfrac{\partial}{\partial t}\vec{\boldsymbol{\omega}} - \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{a}} = \dfrac{1}{2}\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}} \\
\dfrac{\partial}{\partial t}\vec{\mathbf{b}} + \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{\omega}} = \dfrac{1}{2}\vec{\mathbf{b}}\left[\dfrac{\vec{\boldsymbol{\omega}}}{\omega^2} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}}\right] \\
\dfrac{\partial}{\partial t}\vec{\mathbf{F}} + 2\vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{\omega}}) = 0 \\
\text{with } \vec{\mathbf{v}} = \vec{\boldsymbol{\omega}} \times \dfrac{\vec{\mathbf{b}}}{b^2},\ \vec{\mathbf{a}} = \vec{\mathbf{v}} \times \vec{\boldsymbol{\omega}},\ \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{v}} \perp \vec{\mathbf{v}},\ \dfrac{\partial \vec{\mathbf{v}}}{\partial t} = \vec{\mathbf{E}},\ (2\vec{\omega})^2\vec{\mathbf{F}}^{-1} = \left(\dfrac{\partial \vec{\mathbf{v}}}{\partial t}\right)
\end{array}}$$

This is enabled by the fact that the equation system consists of motion quantities ,only, that means velocities and their derivations (no masses or mass densities, an equation system of pure geometrodynamics). It describes turbulent geometrodynamics in a 3+1-dimensional Euclidian observation space. The motion quantities may be understood as images of mappings from the hypersurface of the Einstein Space into the Euclidian Space (the inverse of the mapping from Euclidian Space into a

Riemannian space). The accelleration $\vec{\mathbf{q}}$ in the Riemannian hypersurface is seen as the result of the Ricci Curvature of the enfolding Einstein Space. $\vec{\mathbf{v}}(\vec{\mathbf{x}}, \boldsymbol{t_0})$ and $\frac{\partial \vec{\mathbf{v}}}{\partial t}|_{\boldsymbol{t_0}}$ are used as initial values at time $\boldsymbol{t_0}$. Out of this happen $\vec{\mathbf{b}}(\vec{\mathbf{x}}, \boldsymbol{t_0})$, $\vec{\boldsymbol{\omega}}(\vec{\mathbf{x}}, \boldsymbol{t_0})$ and $\vec{\mathbf{q}}(\vec{\mathbf{x}}, \boldsymbol{t_0})$ as initial fields related to a unique solution of the suitable Cauchy problem.
From the determined initial conditions

$$\begin{aligned} \left(\vec{\mathbf{v}}(\vec{\mathbf{x}}, \boldsymbol{t_0}), \quad \frac{\partial \vec{\mathbf{v}}}{\partial t}|_{\boldsymbol{t_0}}\right) &\longrightarrow \vec{\mathbf{q}}(\vec{\mathbf{x}}, \boldsymbol{t_0}) \\ \vec{\mathbf{v}}(\vec{\mathbf{x}}, \boldsymbol{t_0}) \longrightarrow &\left(\vec{\boldsymbol{\omega}}(\vec{\mathbf{x}}, \boldsymbol{t_0}), \quad \vec{\mathbf{b}}(\vec{\mathbf{x}}, \boldsymbol{t_0})\right) \end{aligned} \tag{15.1}$$

the evolution of the vector fields:

$$\left(\vec{\mathbf{v}}(\vec{\mathbf{x}}, \boldsymbol{t}), \qquad \vec{\boldsymbol{\omega}}(\vec{\mathbf{x}}, \boldsymbol{t}), \qquad \vec{\mathbf{b}}(\vec{\mathbf{x}}, \boldsymbol{t}), \qquad \vec{\mathbf{q}}(\vec{\mathbf{x}}, \boldsymbol{t})\right) \tag{15.2}$$

uniquely result. The equation system (13.17) describes the turbulent matter fluctuations in Space-Time mapped into an observer space. The motion quantities in the observer space are uniquely assigned to the matter motions in the Einstein Space.

With the equation system of turbulent geometrodynamics suitable evolution equations of General Relativity are gained. The smeared density distribution may be obtained using the equation of continuity

$$\frac{\partial}{\partial t}\boldsymbol{\rho} = -\vec{\boldsymbol{\nabla}} \cdot (\boldsymbol{\rho}\vec{\mathbf{v}}). \tag{15.3}$$

The numerical methods solving this equation system are constituted in [40]. Thereby, section 13.2 describes a complete evolution system of General Relativity without using the Einstein Equations and without having to formulate the energy momentum tensor $\mathbf{T}_{\boldsymbol{\mu\nu}}$ for turbulent motions. [1] The precision of the numerical calculations primarily depends on the quality of the used initial fields.

Regardless of matter fluctuations the geometrodynamics of space fluctuations is considered. The 4-dimensional geometry of Space-Time is equivalent to the 3-dimensional geometrodynamics of Space. This may be calculated decoupled from the corresponding matter fluctuations if their sources do not need to be considered and so as long as the motion quantities are not influenced by processes interfering the continuous differentiability. In the next chapter the evolution equations of Space-Time and such the geometrodynamics of the Riemannian hypersurface are formulated leading to a new perspective of gravitational waves.

[1] Without the discussed turbulence theory this does not appear to be feasible, too.

16. Space-Time-fluctuations in General Relativity

$$\boxed{\mathbf{R}_{\mu\nu} = 8\pi \cdot G_N \left(\mathbf{T}_{\mu\nu} - \frac{1}{2} \mathbf{g}_{\mu\nu} \mathbf{T} \right)}$$

16.1. Introduction

Electrodynamics with its Maxwell Equations is the only field theory of classical physics students of physics are generally faced with in the frame of theoretical physics (at least in Germany). The Maxwell Equations above are shown formally beeing a limiting case of classical continuum physics. Because of the constant velocity of light they were the reason for setting up the Einsteinian Special Relativity. The adjustment of the electrodynamic field to Space-Time caused many physicists including Albert Einstein to try an identification of these fields with Space-Time fluctuations. Obviously, electromagnetic fluctuations are properties of Space-Time itself, though a prove is missing.

In chapter 12 continuum fluctuations of general vector fields are discussed. Now we consider deformation vector fields $\vec{\mathbf{d}}(\vec{\mathbf{x}}, t)$ with $\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{d}} \neq \mathbf{0}$. They are sufficiently often continuously differentiable. Defining $\vec{\mathbf{e}}$ and $\vec{\mathbf{b}}$ by

$$\begin{aligned} \vec{\mathbf{e}} &= \partial \vec{\mathbf{d}} / \partial t \neq 0 \\ \vec{\mathbf{b}} &= \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{d}} \neq 0 \end{aligned} \tag{16.1}$$

and interchanging the sequence of the operators $\partial/\partial t$ and $\vec{\boldsymbol{\nabla}} \times$

$$\frac{\partial \vec{\mathbf{b}}}{\partial t} = \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{e}} \tag{16.2}$$

directly follows. So this equation is a necessary consequence of the continuous differentiability of $\vec{\mathbf{d}}(\vec{\mathbf{x}}, t)$. The hereto dual equation is found according to chapter 12

with

$$\begin{aligned}
&\frac{\partial}{\partial t}\vec{\mathbf{b}} - \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{e}} = 0 \\
&\frac{\partial}{\partial t}\left(\frac{b^2}{e^2} \cdot \vec{\mathbf{e}}\right) + \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{b}} = 0 \\
&\vec{\mathbf{e}} \times \frac{\vec{\mathbf{b}}}{b^2} = \text{propagation speed}
\end{aligned} \tag{16.3}$$

Assuming the constant speed of light the Maxwell Equations of vacuum[1]are obtained:

$$\boxed{\begin{aligned}
&\frac{\partial}{\partial t}\vec{\mathbf{b}} - \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{e}} = 0 \\
&\frac{1}{c^2}\frac{\partial}{\partial t}\vec{\mathbf{e}} + \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{b}} = 0 \\
&\vec{\mathbf{e}} \times \frac{\vec{\mathbf{b}}}{b^2} = \vec{\mathbf{c}} = \text{propagation speed of light.}
\end{aligned}} \tag{16.4}$$

16.2. Space-Time of General Relativity and its Riemannian hypersurface

First, the Riemannian hypersurface of Space-Time is considered as deformation of an Euclidian space. For a precise mathematical definition of the Riemannian space [33] is noted.

The Riemannian space is generally defined by a manifold, which consists of a point set, charts or coordinate systems and a symmetrical metric tensor field. Riemannian space and a suitable Euclidian space are one to one linked by the coordinate system. The according mapping is in mathematics not explicitly used as all considerations are abstractly concerned with the connections of the Riemannian space itself not interesting what kind of picture succeeds in the observational coordinate space. The metric tensor arises in the point $P(\vec{\boldsymbol{x}}) \in \boldsymbol{M}$ with $\vec{\boldsymbol{x}} \in \boldsymbol{E}$ (Euclidian space) by scalar products of the tangential vectors $\vec{\boldsymbol{g}}_{\boldsymbol{i}}$.

$$\boldsymbol{g_{ij}}(P(\vec{\boldsymbol{x}})) = \vec{\boldsymbol{g}}_{\boldsymbol{i}}(P(\vec{\boldsymbol{x}})) \cdot \vec{\boldsymbol{g}}_{\boldsymbol{j}}(P(\vec{\boldsymbol{x}})) \tag{16.5}$$

By free choice of the coordinate system $\boldsymbol{g_{ij}}(P(\vec{\boldsymbol{x}}))$ may be determined in one point $(P(\vec{\boldsymbol{x}}))$. But this does not simultaneously hold for the neighborhood of this point.

[1]The Maxwell Equations are usually presented by $\vec{\mathbf{e}} \rightarrow -\vec{\mathbf{e}}$

The isomorphic mapping from Euclidian space into the Riemannian hypersurface is brought to physical life when interpreted as deformation of the Euclidian space, both spaces, Euclidian and Riemannian space, tangentially merging in one point. Here the deformation vector field $\vec{\boldsymbol{d}} = \vec{\boldsymbol{d}}(\vec{\boldsymbol{x}}, t)$ vanishes. These time dependent mappings can be interpreted as gravitational waves. The Riemannian hypersurface arises from

$$\vec{\boldsymbol{y}}(\vec{\boldsymbol{x}}, t) = \vec{\boldsymbol{d}}(\vec{\boldsymbol{x}}, t) + \vec{\boldsymbol{x}} \quad . \tag{16.6}$$

The gradient on the deformed field is described by

$$\vec{\boldsymbol{\nabla}}\vec{\boldsymbol{y}} = \left(\boldsymbol{\partial}_i \boldsymbol{y}_j\right) \tag{16.7}$$

and detailed

$$\left(\boldsymbol{\partial}_i \boldsymbol{y}_j\right) = \begin{pmatrix} \boldsymbol{\partial}_1 \boldsymbol{y}_1 & \boldsymbol{\partial}_1 \boldsymbol{y}_2 & \boldsymbol{\partial}_1 \boldsymbol{y}_3 \\ \boldsymbol{\partial}_2 \boldsymbol{y}_1 & \boldsymbol{\partial}_2 \boldsymbol{y}_2 & \boldsymbol{\partial}_2 \boldsymbol{y}_3 \\ \boldsymbol{\partial}_3 \boldsymbol{y}_1 & \boldsymbol{\partial}_3 \boldsymbol{y}_2 & \boldsymbol{\partial}_3 \boldsymbol{y}_3 \end{pmatrix} \qquad i, j = 1, 2, 3. \tag{16.8}$$

Defining the spatially tangential vector $\vec{\boldsymbol{t}}_i$ with

$$\vec{\boldsymbol{t}}_i = \boldsymbol{\partial}_i \vec{\boldsymbol{y}} = \left(\boldsymbol{\partial}_i \boldsymbol{y}_1, \boldsymbol{\partial}_i \boldsymbol{y}_2, \boldsymbol{\partial}_i \boldsymbol{y}_3\right), \tag{16.9}$$

one obtains the spatial metric tensor $\boldsymbol{t}_{ij} = \vec{\boldsymbol{t}}_i \cdot \vec{\boldsymbol{t}}_j$ by

$$\left(\boldsymbol{t_{ij}}\right) = \left(\boldsymbol{\partial}_i \boldsymbol{y}_j\right) \cdot \left(\boldsymbol{\partial}_i \boldsymbol{y}_j\right)^{\boldsymbol{T}} \tag{16.10}$$

and

$$\boldsymbol{t_{ij}} = \boldsymbol{\partial}_i \boldsymbol{y}_1 \cdot \boldsymbol{\partial}_j \boldsymbol{y}_1 + \boldsymbol{\partial}_i \boldsymbol{y}_2 \cdot \boldsymbol{\partial}_j \boldsymbol{y}_2 + \boldsymbol{\partial}_i \boldsymbol{y}_3 \cdot \boldsymbol{\partial}_j \boldsymbol{y}_3 \tag{16.11}$$

as part of the metric tensors of Space-Time

$$\left(\mathbf{g}_{\nu\mu}\right) = \left(\mathbf{g}_{\mu\nu}\right) = \begin{pmatrix} \mathbf{g_{00}} & \mathbf{g_{01}} & \mathbf{g_{02}} & \mathbf{g_{03}} \\ \mathbf{g_{10}} & \mathbf{t_{11}} & \mathbf{t_{12}} & \mathbf{t_{13}} \\ \mathbf{g_{20}} & \mathbf{t_{21}} & \mathbf{t_{22}} & \mathbf{t_{23}} \\ \mathbf{g_{30}} & \mathbf{t_{31}} & \mathbf{t_{32}} & \mathbf{t_{33}} \end{pmatrix} \qquad \mu, \nu = 0, 1, 2, 3 \ . \tag{16.12}$$

The metric-tensor elements $\boldsymbol{t}_{ij}$ of the spatial hypersurface are components of the metric-tensor element set $\boldsymbol{g}_{\mu\nu}$ of Space-Time. The corresponding statement does not hold for the Ricci Curvature Tensor. The Ricci Tensor elements $\boldsymbol{r}_{ij}$ of the Riemannian hypersurface as subspace of Space-Time are not part of the Ricci Tensor element set $\boldsymbol{R}_{\mu\nu}$ of the overall space.

$$\left(\mathbf{R}_{\nu\mu}\right) = \left(\mathbf{R}_{\mu\nu}\right) = \begin{pmatrix} \mathbf{R_{00}} & \mathbf{R_{01}} & \mathbf{R_{02}} & \mathbf{R_{03}} \\ \mathbf{R_{10}} & \mathbf{R_{11}} & \mathbf{R_{12}} & \mathbf{R_{13}} \\ \mathbf{R_{20}} & \mathbf{R_{21}} & \mathbf{R_{22}} & \mathbf{R_{23}} \\ \mathbf{R_{30}} & \mathbf{R_{31}} & \mathbf{R_{32}} & \mathbf{R_{33}} \end{pmatrix} \neq \begin{pmatrix} \mathbf{R_{00}} & \mathbf{R_{01}} & \mathbf{R_{02}} & \mathbf{R_{03}} \\ \mathbf{R_{10}} & \mathbf{r_{11}} & \mathbf{r_{12}} & \mathbf{r_{13}} \\ \mathbf{R_{20}} & \mathbf{r_{21}} & \mathbf{r_{22}} & \mathbf{r_{23}} \\ \mathbf{R_{30}} & \mathbf{r_{31}} & \mathbf{r_{32}} & \mathbf{r_{33}} \end{pmatrix} \tag{16.13}$$

$$\text{i.e.} \qquad \boldsymbol{r}_{ij} \neq \boldsymbol{R}_{ij} \qquad i,j = 1,2,3$$

Initially, it is the plan to express the Ricci Tensor of Space Time by the Ricci Tensor of the spatial hypersurface and its time dependent metric tensor

$$\boldsymbol{R}_{ij} = \boldsymbol{R}_{ij}(\boldsymbol{r}_{ij}, \boldsymbol{t}_{ij}) \qquad i,j = 1,2,3. \tag{16.14}$$

Formulating the energy momentum tensor of the right side of the Einstein equations

$$\mathbf{R}_{\mu\nu} - \frac{1}{2}\mathbf{g}_{\mu\nu}\mathbf{R} = 8\pi \cdot G_N \mathbf{T}_{\mu\nu} \qquad \mu,\nu = 0,1,2,3$$

by the related deformation fluctuations using its electromagnetic interpretation the unification of gravitational and electromagnetic field is outlined in the following chapter.

Originating from the Einstein equations

$$\mathbf{R}_{\mu\nu} - \frac{1}{2}\mathbf{g}_{\mu\nu}\mathbf{R} = 8\pi \cdot G_N \mathbf{T}_{\mu\nu} \tag{16.15}$$

one obtains by contraction

$$\mathbf{trace}\left(\mathbf{R}_{\mu\nu} - \frac{1}{2}\mathbf{g}_{\mu\nu}\mathbf{R}\right) = \mathbf{g}^{\mu\mu}\left(\mathbf{R}_{\mu\mu} - \frac{1}{2}\mathbf{g}_{\mu\mu}\mathbf{R}\right) = -\mathbf{R} = 8\pi \cdot G_N \mathbf{T}^{\mu}_{\mu} = 8\pi \cdot G_N \mathbf{T} \tag{16.16}$$

an alternative form of the Einstein Equations

$$\mathbf{R}_{\mu\nu} = 8\pi \cdot G_N \left(\mathbf{T}_{\mu\nu} - \frac{1}{2}\mathbf{g}_{\mu\nu}\mathbf{T}\right). \tag{16.17}$$

16.3. The Ricci Tensor in the origin of a local inertial-system

The Riemannian curvature tensor $\mathbf{R}^{\mu}_{.\nu\alpha\beta}$ is described in any coordinate system by the Christoffel symbols

$$\boldsymbol{\Gamma}^{\boldsymbol{\mu}}_{\boldsymbol{\nu\alpha}} = \left\{ \begin{array}{c} \mu \\ \nu\ \alpha \end{array} \right\} = \frac{1}{2}\mathbf{g}^{\mu\lambda}\Big[\partial_\nu \mathbf{g}_{\alpha\lambda} + \partial_\alpha \mathbf{g}_{\lambda\nu} - \partial_\lambda \mathbf{g}_{\nu\alpha}\Big] \tag{16.18}$$

$$\mathbf{R}^{\mu}_{.\nu\alpha\beta} = \frac{\partial \boldsymbol{\Gamma}^{\mu}_{\nu\beta}}{\partial \mathbf{x}^{\alpha}} - \frac{\partial \boldsymbol{\Gamma}^{\mu}_{\nu\alpha}}{\partial \mathbf{x}^{\beta}} + \boldsymbol{\Gamma}^{\mu}_{\rho\alpha}\boldsymbol{\Gamma}^{\rho}_{\nu\beta} - \boldsymbol{\Gamma}^{\mu}_{\rho\beta}\boldsymbol{\Gamma}^{\rho}_{\nu\alpha}. \tag{16.19}$$

In the origin $\vec{\mathbf{x}_0}$ of a local inertial system [1] the partial derivatives with respect to coordinates of the metric tensor $\mathbf{g}_{\lambda\nu}$ vanish such that

$$\boldsymbol{\Gamma}^{\boldsymbol{\mu}}_{\boldsymbol{\nu\alpha}}(\vec{\mathbf{x}_0}) = \mathbf{0} \tag{16.20}$$

and

$$\mathbf{R}^{\mu}_{.\nu\alpha\beta}(\vec{\mathbf{x}_0}) = \frac{\partial \boldsymbol{\Gamma}^{\mu}_{\nu\beta}}{\partial \mathbf{x}^{\alpha}} - \frac{\partial \boldsymbol{\Gamma}^{\mu}_{\nu\alpha}}{\partial \mathbf{x}^{\beta}}. \tag{16.21}$$

In the origin of the coordinate system the metric tensor itself equals the Minkowski tensor.

$$\mathbf{g}_{\mu\nu}(\vec{\mathbf{x}_0}) = \eta_{\mu\nu} = \begin{pmatrix} -\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix} \tag{16.22}$$

Written out one obtains

$$\mathbf{R}^{\mu}_{.\nu\alpha\beta}(\vec{\mathbf{x}_0}) = \frac{1}{2}\eta^{\mu\lambda}\frac{\partial}{\partial x^{\alpha}}\Big[\partial_\nu \mathbf{g}_{\beta\lambda} + \partial_\beta \mathbf{g}_{\lambda\nu} - \partial_\lambda \mathbf{g}_{\nu\beta}\Big] - \frac{1}{2}\eta^{\mu\lambda}\frac{\partial}{\partial x^{\beta}}\Big[\partial_\nu \mathbf{g}_{\alpha\lambda} + \partial_\alpha \mathbf{g}_{\lambda\nu} - \partial_\lambda \mathbf{g}_{\nu\alpha}\Big] \tag{16.23}$$

$\Longrightarrow$

$$\mathbf{R}^{\mu}_{.\nu\alpha\beta}(\vec{\mathbf{x}_0}) = \frac{1}{2}\eta^{\mu\lambda}\Big[\partial_\alpha\partial_\nu \mathbf{g}_{\beta\lambda} + \partial_\alpha\partial_\beta \mathbf{g}_{\lambda\nu} - \partial_\alpha\partial_\lambda \mathbf{g}_{\nu\beta}\Big] - \frac{1}{2}\eta^{\mu\lambda}\Big[\partial_\beta\partial_\nu \mathbf{g}_{\alpha\lambda} + \partial_\beta\partial_\alpha \mathbf{g}_{\lambda\nu} - \partial_\beta\partial_\lambda \mathbf{g}_{\nu\alpha}\Big] \tag{16.24}$$

$\Longrightarrow$

$$\mathbf{R}^{\mu}_{.\nu\alpha\beta}(\vec{\mathbf{x}_0}) = \frac{1}{2}\eta^{\mu\lambda}\Big[\partial_\alpha\partial_\nu\mathbf{g}_{\beta\lambda} + \partial_\beta\partial_\lambda\mathbf{g}_{\nu\alpha} - \partial_\alpha\partial_\lambda\mathbf{g}_{\nu\beta} - \partial_\beta\partial_\nu\mathbf{g}_{\alpha\lambda}\Big] \tag{16.25}$$

$\Longrightarrow$

$$\mathbf{R}_{\mu\nu\alpha\beta}(\vec{\mathbf{x}_0}) = \frac{1}{2}\Big[\partial_\alpha\partial_\nu\mathbf{g}_{\beta\lambda} + \partial_\beta\partial_\lambda\mathbf{g}_{\nu\alpha} - \partial_\alpha\partial_\lambda\mathbf{g}_{\nu\beta} - \partial_\beta\partial_\nu\mathbf{g}_{\alpha\lambda}\Big]. \tag{16.26}$$

After contraction there is the associated Ricci Tensor

$$\mathbf{R}_{\mu\nu}(\vec{\mathbf{x}_0}) = \frac{1}{2}\Big[\partial_\mu\partial_\alpha\mathbf{g}^{\alpha}_{\nu} + \partial_\nu\partial^\alpha\mathbf{g}_{\mu\alpha} - \partial_\alpha\partial^\alpha\mathbf{g}_{\mu\nu} - \partial_\nu\partial_\mu\mathbf{g}^{\alpha}_{\alpha}\Big] \tag{16.27}$$

and as $\partial_\alpha\partial^\alpha = \Box$ means the D'Alembert-Operator $\Longrightarrow$

$$\boxed{\mathbf{R}_{\mu\nu}(\vec{\mathbf{x}_0}) = \frac{1}{2}\Big[\partial_\mu\partial_\alpha\mathbf{g}^{\alpha}_{\nu} + \partial_\nu\partial_\alpha\mathbf{g}^{\alpha}_{\mu} - \Box\mathbf{g}_{\mu\nu} - \partial_\nu\partial_\mu\mathbf{g}\Big]}\,. \tag{16.28}$$

This result may be obtained by linearization of the Riemannian curvature tensor, too. Choosing point $(\vec{\mathbf{x}_0})$ as the origin of a local inertial system, linearization is not necessary.

16.4. The Ricci Tensor of the Einstein Space in dependence of temporal fluctuations of its Riemannian hypersurface

The following relations correspond to [26] Landau Lifschitz volume 2 page.308-309. A time orthogonal coordinate system is always possible. In contrary to [26], we do not equate the velocity of light with 1.

$$\text{Def:} \qquad \varkappa_{\mathbf{ij}} = \frac{\partial\mathbf{g}_{\mathbf{ij}}}{\partial(c\boldsymbol{t})} \tag{16.29}$$

$\mathbf{r}_{\mathbf{ij}}$ means the Ricci Tensor of the Riemannian hypersurface.
$\Longrightarrow$

$$
\begin{aligned}
\mathbf{R_{00}} =& -\frac{1}{2}\frac{\partial \varkappa_i^i}{\partial(c\boldsymbol{t})} - \frac{1}{4}\varkappa_i^j\varkappa_j^i \\
\mathbf{R_{0i}} =& \frac{1}{2}\left(\varkappa^j_{\mathrm{i};j} - \varkappa^j_{\mathrm{j};i}\right) \\
\mathbf{R_{ij}} =& \mathbf{r_{ij}} + \frac{1}{2}\frac{\partial \varkappa_{ij}}{\partial(c\boldsymbol{t})} + \frac{1}{4}\left(\varkappa_{\mathrm{i}j}\varkappa^k_{\mathrm{k}} - 2\varkappa_i^k\varkappa_{jk}\right)
\end{aligned}
\tag{16.30}
$$

i, j, k pass through $1, 2, 3$. ";" means partial derivation, here.

Thus the geometry of Space-Time may be opened up from geometrodynamics of space. Gravitational waves existing the energy momentum tensor $\mathbf{T}_{\mu\nu} \neq \mathbf{0}$ is given in the considered Space-Time area even if there is no matter. [2]

[2] in contrary to Penrose [34] page 467 "The energy-momentum tensor in empty space is zero."

17. Unification of Maxwell Field and gravitational field

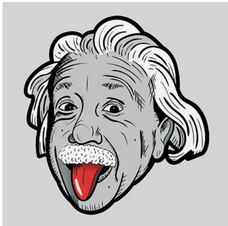


Figure 17.1.: Maybe, Einstein would have had fun at this theory

17.1. Gravitational waves corresponding to electromagnetic Fluctuations

The deformation fluctuations of space and its as electromagnetic fluctuations noticed phenomena are subsequently faced to each other in a limited volume area as fourier developments . The considerations are performed based on treatments of natural vibrations of the electomagnetic field in vacuum in accordance to [26]. The usual electric field $\vec{\mathbf{E}}$ is replaced by $-\vec{\mathbf{E}}$, without loss of generality. An explicit dependency of the viewed overall volume in the canonical variables and such in the resulting energy density and the electromagnetic fields is avoided by modified normalisation of the canonical variables, in contrast to [26].
In pure field theories energy densities and accellerations should occur as **primary** quantities not energies and forces. The energy in one point $(\vec{\mathbf{x}}, t)$ is always zero but not the energy density. Analogically, the same is true for the relation of accelleration and force.

deformation fluctuations	**electromagnetic fluctuations**

From

$\vec{\mathbf{d}} =$ deformation vectorfield | $\vec{\mathbf{A}} =$ vector potential

$\frac{\partial}{\partial t}\vec{\mathbf{b}} - \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{e}} = 0$ | $\frac{\partial}{\partial t}\vec{\mathbf{B}} - \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}} = 0$

$\frac{1}{c^2}\frac{\partial}{\partial t}\vec{\mathbf{e}} + \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{b}} = 0$ | $\frac{1}{c^2}\frac{\partial}{\partial t}\vec{\mathbf{E}} + \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}} = 0$

and

$\vec{\mathbf{e}} = \partial\vec{\mathbf{d}}/\partial t \neq 0$ | $\vec{\mathbf{E}} = \partial\vec{\mathbf{A}}/\partial t \neq 0$

$\vec{\mathbf{b}} = \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{d}} \neq 0$ | $\vec{\mathbf{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}} \neq 0$

one obtains

$\frac{1}{\boldsymbol{c^2}}\frac{\partial^2\vec{\mathbf{d}}}{\partial t^2} = \boldsymbol{\Delta}\vec{\mathbf{d}}$ | $\frac{1}{\boldsymbol{c^2}}\frac{\partial^2\vec{\mathbf{A}}}{\partial t^2} = \boldsymbol{\Delta}\vec{\mathbf{A}}$

Deformation field and according vector potential field are formally described by

$\vec{\mathbf{d}} = \sum_{\vec{\mathbf{k}}} \vec{\mathbf{d}}_{\vec{\mathbf{k}}} = \sum_{\vec{\mathbf{k}}} \vec{\mathbf{a}}_{\vec{\mathbf{k}}} \boldsymbol{e}^{i\vec{\mathbf{k}}\vec{\mathbf{r}}} + \vec{\mathbf{a}^*}_{\vec{\mathbf{k}}} \boldsymbol{e}^{-i\vec{\mathbf{k}}\vec{\mathbf{r}}}$ | $\vec{\mathbf{A}} = \sum_{\vec{\mathbf{k}}} \vec{\mathbf{A}}_{\vec{\mathbf{k}}} = \sum_{\vec{\mathbf{k}}} \vec{\mathfrak{A}}_{\vec{\mathbf{k}}} \boldsymbol{e}^{i\vec{\mathbf{k}}\vec{\mathbf{r}}} + \vec{\mathfrak{A}}^*_{\vec{\mathbf{k}}} \boldsymbol{e}^{-i\vec{\mathbf{k}}\vec{\mathbf{r}}}$

and it follows

$\ddot{\vec{\mathbf{d}}}_{\vec{\mathbf{k}}} + c^2k^2\vec{\mathbf{d}}_{\vec{\mathbf{k}}} = \mathbf{0}$ | $\ddot{\vec{\mathbf{A}}}_{\vec{\mathbf{k}}} + c^2k^2\vec{\mathbf{A}}_{\vec{\mathbf{k}}} = \mathbf{0}$

with

$\vec{\boldsymbol{e}} = \dot{\vec{\boldsymbol{d}}} = \sum_{\vec{\mathbf{k}}} \dot{\vec{\mathbf{d}}}_{\vec{\mathbf{k}}} = \sum_{\vec{\mathbf{k}}}\left(\dot{\vec{\mathbf{a}}}_{\vec{\mathbf{k}}} \boldsymbol{e}^{i\vec{\mathbf{k}}\vec{\mathbf{r}}} + \dot{\vec{\mathbf{a}}}^*_{\vec{\mathbf{k}}} \boldsymbol{e}^{-i\vec{\mathbf{k}}\vec{\mathbf{r}}}\right)$ | $\vec{\boldsymbol{E}} = \dot{\vec{\boldsymbol{A}}} = \sum_{\vec{\mathbf{k}}} \dot{\vec{\mathbf{A}}}_{\vec{\mathbf{k}}} = \sum_{\vec{\mathbf{k}}}\left(\dot{\vec{\mathfrak{A}}}_{\vec{\mathbf{k}}} \boldsymbol{e}^{i\vec{\mathbf{k}}\vec{\mathbf{r}}} + \dot{\vec{\mathfrak{A}}}^*_{\vec{\mathbf{k}}} \boldsymbol{e}^{-i\vec{\mathbf{k}}\vec{\mathbf{r}}}\right)$

and

$\vec{\boldsymbol{b}} = -\boldsymbol{i}\sum_{\vec{\mathbf{k}}} \vec{\mathbf{k}} \times \left(\vec{\mathbf{a}}_{\vec{\mathbf{k}}} \boldsymbol{e}^{i\vec{\mathbf{k}}\vec{\mathbf{r}}} + \vec{\mathbf{a}^*}_{\vec{\mathbf{k}}} \boldsymbol{e}^{-i\vec{\mathbf{k}}\vec{\mathbf{r}}}\right)$ | $\vec{\boldsymbol{B}} = -\boldsymbol{i}\sum_{\vec{\mathbf{k}}} \vec{\mathbf{k}} \times \left(\vec{\mathfrak{A}}_{\vec{\mathbf{k}}} \boldsymbol{e}^{i\vec{\mathbf{k}}\vec{\mathbf{r}}} + \vec{\mathfrak{A}}^*_{\vec{\mathbf{k}}} \boldsymbol{e}^{-i\vec{\mathbf{k}}\vec{\mathbf{r}}}\right)$

$\mathbf{k_1} = \frac{2\pi\cdot n_x}{L_x},\ \mathbf{k_2} = \frac{2\pi\cdot n_y}{L_y},\ \mathbf{k_3} = \frac{2\pi\cdot n_z}{L_z};$

$\vec{\mathbf{k}} = (\mathbf{k_1}, \mathbf{k_2}, \mathbf{k_3})$

$\mathbf{a}_{\mathbf{k}_i} \sim \boldsymbol{e}^{-i\omega_{\mathbf{k}_i} t}, \qquad \omega_{\mathbf{k}_i} = ck_i$ | $\mathfrak{A}_{\vec{\mathbf{k}}_i} \sim \boldsymbol{e}^{-i\omega_{\mathbf{k}_i} t}, \qquad \omega_{\mathbf{k}_i} = ck_i$

The wave vectors are calculated in a sufficiently great volume $\boldsymbol{V} = \boldsymbol{L_x} \cdot \boldsymbol{L_y} \cdot \boldsymbol{L_z}$.

$\mathcal{E} = \frac{1}{8\pi}\int_{V_0}(\boldsymbol{E^2}/\boldsymbol{c^2} + \boldsymbol{B^2})dV$ means the energy of the field in volume V_0.

The energy density of the field is $\mathfrak{E} = \frac{1}{8\pi}\sum_{\vec{\mathbf{k}}}(\boldsymbol{E}^{\mathbf{2}}_{\vec{\mathbf{k}}}/\boldsymbol{c^2} + \boldsymbol{B}^{\mathbf{2}}_{\vec{\mathbf{k}}})$

deformation fluctuations | **electromagnetic fluctuations**

Now, the following vectorial quantities (canonical variables) are defined:

$$\vec{\mathbf{q}}_{\vec{\mathbf{k}}} = \sqrt{\tfrac{1}{4\pi c^2}}(\vec{\mathbf{a}}_{\vec{\mathbf{k}}} + \vec{\mathbf{a}}^*_{\vec{\mathbf{k}}}) \qquad \vec{\mathbf{Q}}_{\vec{\mathbf{k}}} = \sqrt{\tfrac{1}{4\pi c^2}}(\vec{\mathfrak{A}}_{\vec{\mathbf{k}}} + \vec{\mathfrak{A}}^*_{\vec{\mathbf{k}}})$$

$$\vec{\mathbf{p}}_{\vec{\mathbf{k}}} = -i\omega_{\vec{\mathbf{k}}}\sqrt{\tfrac{1}{4\pi c^2}}(\vec{\mathbf{a}}_{\vec{\mathbf{k}}} - \vec{\mathbf{a}}^*_{\vec{\mathbf{k}}}) = \dot{\vec{\mathbf{q}}}_{\vec{\mathbf{k}}} \qquad \vec{\mathbf{P}}_{\vec{\mathbf{k}}} = -i\omega_{\vec{\mathbf{k}}}\sqrt{\tfrac{1}{4\pi c^2}}(\vec{\mathfrak{A}}_{\vec{\mathbf{k}}} - \vec{\mathfrak{A}}^*_{\vec{\mathbf{k}}}) = \dot{\vec{\mathbf{Q}}}_{\vec{\mathbf{k}}}$$

$$\vec{\mathbf{q}}_{\mathbf{k}_i} \sim \boldsymbol{cos}(\omega_{\mathbf{k}_i}t), \quad \vec{\mathbf{p}}_{\mathbf{k}_i} \sim \boldsymbol{sin}(\omega_{\mathbf{k}_i}t) \qquad \vec{\mathbf{Q}}_{\mathbf{k}_i} \sim \boldsymbol{cos}(\omega_{\mathbf{k}_i}t), \quad \vec{\boldsymbol{P}}_{\mathbf{k}_i} \sim \boldsymbol{sin}(\omega_{\boldsymbol{k}_i}t)$$

Obviously, they are real and resolved according to complex quantities they give

$$\vec{\mathbf{a}}_{\mathbf{k_j}} = \tfrac{i}{k_j}\sqrt{\pi}(\vec{\mathbf{p}}_{\mathbf{k_j}} - i\omega_{\vec{\mathbf{k}}_j}\vec{\mathbf{q}}_{\mathbf{k_j}}) \qquad \vec{\mathfrak{A}}_{\mathbf{k_j}} = \tfrac{i}{k_j}\sqrt{\pi}(\vec{\mathbf{P}}_{\mathbf{k_j}} - i\omega_{\mathbf{k_j}}\vec{\mathbf{Q}}_{\mathbf{k_j}})$$
$$\vec{\mathbf{a}}^*_{\mathbf{k_j}} = -\tfrac{i}{k_j}\sqrt{\pi}(\vec{\mathbf{p}}_{\mathbf{k_j}} + i\omega_{\mathbf{k_j}}\vec{\mathbf{q}}_{\mathbf{k_j}}) \qquad \vec{\mathfrak{A}}^*_{\mathbf{k_j}} = -\tfrac{i}{k_j}\sqrt{\pi}(\vec{\mathbf{P}}_{\mathbf{k_j}} + i\omega_{\mathbf{k_j}}\vec{\mathbf{Q}}_{\mathbf{k_j}}).$$

Thus one obtains as expansion by characteristic vibrations (in concise presentation):

$$\vec{\mathbf{d}} = \sqrt{4\pi}\textstyle\sum_{\vec{\mathbf{k}}}\frac{1}{k}\left(ck\vec{\mathbf{q}}_{\vec{\mathbf{k}}}cos(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}) - \vec{\mathbf{p}}_{\vec{\mathbf{k}}}sin(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}})\right) \qquad \vec{\mathbf{A}} = \sqrt{4\pi}\textstyle\sum_{\vec{\mathbf{k}}}\frac{1}{k}\left(ck\vec{\mathbf{Q}}_{\vec{\mathbf{k}}}cos(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}) - \vec{\mathbf{P}}_{\vec{\mathbf{k}}}sin(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}})\right)$$
$$\vec{\mathbf{e}} = \sqrt{4\pi}\textstyle\sum_{\vec{\mathbf{k}}}\boldsymbol{c}\left(ck\vec{\mathbf{q}}_{\vec{\mathbf{k}}}sin(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}) + \vec{\mathbf{p}}_{\vec{\mathbf{k}}}cos(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}})\right) \qquad \vec{\mathbf{E}} = \sqrt{4\pi}\textstyle\sum_{\vec{\mathbf{k}}}\boldsymbol{c}\left(ck\vec{\mathbf{Q}}_{\vec{\mathbf{k}}}sin(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}) + \vec{\mathbf{P}}_{\vec{\mathbf{k}}}cos(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}})\right)$$
$$\vec{\mathbf{b}} = -\sqrt{4\pi}\textstyle\sum_{\vec{\mathbf{k}}}\frac{1}{k}\vec{\mathbf{k}}\times[ck\vec{\mathbf{q}}_{\vec{\mathbf{k}}}sin(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}) + \vec{\mathbf{p}}_{\vec{\mathbf{k}}}cos(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}})] \qquad \vec{\mathbf{B}} = -\sqrt{4\pi}\textstyle\sum_{\vec{\mathbf{k}}}\frac{1}{k}\vec{\mathbf{k}}\times[ck\vec{\mathbf{Q}}_{\vec{\mathbf{k}}}sin(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}) + \vec{\mathbf{P}}_{\vec{\mathbf{k}}}cos(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}})]$$

respectively noted for the single modes:

$$\vec{\mathbf{d}}_{k_j} = \sqrt{4\pi}\tfrac{1}{k_j}\left(ck_j\vec{\mathbf{q}}_{\mathbf{k_j}}cos(\vec{\mathbf{k_j}}\cdot\vec{\mathbf{r}}) - \vec{\mathbf{p}}_{\mathbf{k_j}}sin(\vec{\mathbf{k_j}}\cdot\vec{\mathbf{r}})\right) \qquad \vec{\mathbf{A}}_{k_j} = \sqrt{4\pi}\tfrac{1}{k_j}\left(ck_j\vec{\mathbf{Q}}_{\mathbf{k_j}}cos(\vec{\mathbf{k_j}}\cdot\vec{\mathbf{r}}) - \vec{\mathbf{P}}_{\mathbf{k_j}}sin(\vec{\mathbf{k_j}}\cdot\vec{\mathbf{r}})\right)$$
$$\vec{\mathbf{e}}_{k_j} = \sqrt{4\pi}\boldsymbol{c}\left(ck_j\vec{\mathbf{q}}_{\mathbf{k_j}}sin(\vec{\mathbf{k_j}}\cdot\vec{\mathbf{r}}) + \vec{\mathbf{p}}_{\mathbf{k_j}}cos(\vec{\mathbf{k_j}}\cdot\vec{\mathbf{r}})\right) \qquad \vec{\mathbf{E}}_{k_j} = \sqrt{4\pi}\boldsymbol{c}\left(ck_j\vec{\mathbf{Q}}_{\mathbf{k_j}}sin(\vec{\mathbf{k_j}}\cdot\vec{\mathbf{r}}) + \vec{\mathbf{P}}_{\mathbf{k_j}}cos(\vec{\mathbf{k_j}}\cdot\vec{\mathbf{r}})\right)$$
$$\vec{\mathbf{b}}_{k_j} = -\sqrt{4\pi}\tfrac{1}{k_j}\vec{\mathbf{k_j}}\times[ck_j\vec{\mathbf{q}}_{\mathbf{k_j}}sin(\vec{\mathbf{k_j}}\cdot\vec{\mathbf{r}}) + \vec{\mathbf{p}}_{\mathbf{k_j}}cos(\vec{\mathbf{k_j}}\cdot\vec{\mathbf{r}})] \qquad \vec{\mathbf{B}}_{k_j} = -\sqrt{4\pi}\tfrac{1}{k_j}\vec{\mathbf{k_j}}\times[ck_j\vec{\mathbf{Q}}_{\mathbf{k_j}}sin(\vec{\mathbf{k_j}}\cdot\vec{\mathbf{r}}) + \vec{\mathbf{P}}_{\mathbf{k_j}}cos(\vec{\mathbf{k_j}}\cdot\vec{\mathbf{r}})]$$

with $\mathfrak{E} = \sum_{\vec{\mathbf{k}}}\mathfrak{E}_{\vec{\mathbf{k}}} = \frac{1}{2}\sum_{\vec{\mathbf{k}}}(\boldsymbol{E}^2_{\vec{\mathbf{k}}}/\boldsymbol{c^2} + \boldsymbol{B}^2_{\vec{\mathbf{k}}})$ and $\mathcal{E} = \sum_{\vec{\mathbf{k}}}\mathcal{E}_{\vec{\mathbf{k}}} = \frac{1}{2}\sum_{\vec{\mathbf{k}}}\int_{V_0}(\boldsymbol{E}^2_{\vec{\mathbf{k}}}/\boldsymbol{c^2} + \boldsymbol{B}^2_{\vec{\mathbf{k}}})dV.$
respectively $\mathfrak{E}_{\vec{\mathbf{k_j}}} = \frac{1}{2}(\boldsymbol{E}^2_{\vec{\mathbf{k_j}}}/\boldsymbol{c^2} + \boldsymbol{B}^2_{\vec{\mathbf{k_j}}})$ and $\mathcal{E}_{\vec{\mathbf{k_j}}} = \frac{1}{2}\int_{V_0}(\boldsymbol{E}^2_{\vec{\mathbf{k_j}}}/\boldsymbol{c^2} + \boldsymbol{B}^2_{\vec{\mathbf{k_j}}})dV.$

They may formally considered as running waves moving discrete quantities of harmonic oscillators with the Hamilton Functions

$$\mathbf{H} = \sum_{\vec{\mathbf{k}}}\mathbf{H}_{\vec{\mathbf{k}}} = \sum_{\vec{\mathbf{k}}}\frac{1}{2}(\mathbf{p}^2_{\vec{\mathbf{k}}} + \omega^2_{\vec{\mathbf{k}}}\mathbf{q}^2_{\vec{\mathbf{k}}}), \qquad \mathcal{H} = \sum_{\vec{\mathbf{k}}}\mathcal{H}_{\vec{\mathbf{k}}} = \sum_{\vec{\mathbf{k}}}\frac{1}{2}(\mathbf{P}^2_{\vec{\mathbf{k}}} + \omega^2_{\vec{\mathbf{k}}}\mathbf{Q}^2_{\vec{\mathbf{k}}}) \tag{17.1}$$

and the oscillator equations

$$\ddot{\vec{\mathbf{q}}}_{\vec{\mathbf{k}}} + \omega^{\mathbf{2}}_{\vec{\mathbf{k}}}\vec{\mathbf{q}}_{\vec{\mathbf{k}}} = \mathbf{0}, \qquad \ddot{\vec{\mathbf{Q}}}_{\vec{\mathbf{k}}} + \omega^{\mathbf{2}}_{\vec{\mathbf{k}}}\vec{\mathbf{Q}}_{\vec{\mathbf{k}}} = \mathbf{0} \tag{17.2}$$

.

$$\mathbf{H} = \sum_{\vec{\mathbf{k}}} \mathbf{H}_{\vec{\mathbf{k}}} \qquad \mathbf{H}_{\vec{\mathbf{k}}} = \frac{1}{2}(\mathbf{p}_{\vec{\mathbf{k}}}^2 + \omega_{\vec{\mathbf{k}}}^2 \mathbf{q}_{\vec{\mathbf{k}}}^2), \qquad \mathcal{H} = \sum_{\vec{\mathbf{k}}} \mathcal{H}_{\vec{\mathbf{k}}} \qquad \mathcal{H}_{\vec{\mathbf{k}}} = \frac{1}{2}(\mathbf{P}_{\vec{\mathbf{k}}}^2 + \omega_{\vec{\mathbf{k}}}^2 \mathbf{Q}_{\vec{\mathbf{k}}}^2) \tag{17.3}$$

17.2. The energy-momentum-tensor of the electromagnetic field

The energy momentum density tensor for the electromagnetic field (generally called Energy momentum tensor) in covariant components [38] is written with the choosen signature $(-1, 1, 1, 1)$

$$\mathbf{T}_{\mu\nu} = \frac{1}{4\pi}\left(\mathbf{F}_{\mu}^{\alpha}\mathbf{F}_{\alpha\nu} - \frac{1}{4}\mathbf{g}_{\mu\nu}\mathbf{F}_{\alpha\beta}\mathbf{F}^{\alpha\beta}\right) \tag{17.4}$$

It is symmetric: $\mathbf{T}_{\mu\nu} = \mathbf{T}_{\nu\mu}$.
One obtains the Faraday-tensor of the electromagnetic field from

$$\mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu \qquad \mu, \nu = 0, 1, 2, 3 \tag{17.5}$$

and detailed (they are chosen respectively the form of the above Maxwell Equations)

$$\begin{aligned}
\mathbf{F}_{0i} &= \partial_0 \mathbf{A}_i - \partial_i \mathbf{A}_0 = \mathbf{E}_i / c, \qquad i = 1, 2, 3\\
\mathbf{F}_{i0} &= \partial_i \mathbf{A}_0 - \partial_0 \mathbf{A}_i = -\mathbf{E}_i / c, \qquad i = 1, 2, 3\\
\mathbf{F}_{12} &= \partial_1 \mathbf{A}_2 - \partial_2 \mathbf{A}_1 = \mathbf{B}_3\\
\mathbf{F}_{13} &= \partial_1 \mathbf{A}_3 - \partial_3 \mathbf{A}_1 = -\mathbf{B}_2\\
\mathbf{F}_{23} &= \partial_2 \mathbf{A}_3 - \partial_3 \mathbf{A}_2 = \mathbf{B}_1
\end{aligned} \tag{17.6}$$

$$\Longrightarrow \mathbf{F}_{\mu\nu} = -\mathbf{F}_{\nu\mu}$$

$$\partial_\rho \mathbf{F}_{\mu\nu} + \partial_\mu \mathbf{F}_{\nu\rho} + \partial_\nu \mathbf{F}_{\rho\mu} = 0$$

and in greater detail

$$\begin{aligned}\partial_1\mathbf{F_{23}}+\partial_3\mathbf{F_{12}}+\partial_2\mathbf{F_{31}}&=\mathbf{0}\\ \partial_2\mathbf{F_{30}}+\partial_0\mathbf{F_{23}}+\partial_3\mathbf{F_{02}}&=\mathbf{0}\\ \partial_3\mathbf{F_{01}}+\partial_1\mathbf{F_{30}}+\partial_0\mathbf{F_{13}}&=\mathbf{0}^{\cdot}\\ \partial_0\mathbf{F_{12}}+\partial_2\mathbf{F_{01}}+\partial_1\mathbf{F_{20}}&=\mathbf{0}\end{aligned}$$

The indices correspond to $0\to ct$, $1\to x$, $2\to y$, $3\to z$ complying with the following electrodynamic equations of vacuum[1]

$$\boldsymbol{div\vec{B}=0} \qquad \text{and} \qquad \frac{\partial}{\partial t}\vec{\mathbf{B}}-\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{E}}=0.$$

The expressions of the covariant and contravariant Faraday-tensors considering the minkowski tensor

$$\boldsymbol{\eta_{\mu\nu}}=\begin{pmatrix}\mathbf{-1}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}\end{pmatrix} \tag{17.7}$$

lead to

$$\mathbf{F}_{\mu\nu}=\begin{pmatrix}\mathbf{0}&\mathbf{E_1}/\boldsymbol{c}&\mathbf{E_2}/\boldsymbol{c}&\mathbf{E_3}/\boldsymbol{c}\\ -\mathbf{E_1}/\boldsymbol{c}&\mathbf{0}&\mathbf{B_3}&-\mathbf{B_2}\\ -\mathbf{E_2}/\boldsymbol{c}&-\mathbf{B_3}&\mathbf{0}&\mathbf{B_1}\\ -\mathbf{E_3}/\boldsymbol{c}&\mathbf{B_2}&-\mathbf{B_1}&\mathbf{0}\end{pmatrix}\qquad \mathbf{F}^{\mu\nu}=\begin{pmatrix}\mathbf{0}&-\mathbf{E_1}/\boldsymbol{c}&-\mathbf{E_2}/\boldsymbol{c}&-\mathbf{E_3}/\boldsymbol{c}\\ \mathbf{E_1}/\boldsymbol{c}&\mathbf{0}&\mathbf{B_3}&-\mathbf{B_2}\\ \mathbf{E_2}/\boldsymbol{c}&-\mathbf{B_3}&\mathbf{0}&\mathbf{B_1}\\ \mathbf{E_3}/\boldsymbol{c}&\mathbf{B_2}&-\mathbf{B_1}&\mathbf{0}\end{pmatrix} \tag{17.8}$$

$$\mathbf{F}^{\mu}_{\nu}=\begin{pmatrix}\mathbf{0}&-\mathbf{E_1}/\boldsymbol{c}&-\mathbf{E_2}/\boldsymbol{c}&-\mathbf{E_3}/\boldsymbol{c}\\ -\mathbf{E_1}/\boldsymbol{c}&\mathbf{0}&\mathbf{B_3}&-\mathbf{B_2}\\ -\mathbf{E_2}/\boldsymbol{c}&-\mathbf{B_3}&\mathbf{0}&\mathbf{B_1}\\ -\mathbf{E_3}/\boldsymbol{c}&\mathbf{B_2}&-\mathbf{B_1}&\mathbf{0}\end{pmatrix} \quad . \tag{17.9}$$

Thus the covariant components of the electromagnetic energy momentum tensor are written

[1] the polarity reversal $\vec{\boldsymbol{E}}\longrightarrow-\vec{\boldsymbol{E}}$ recognised

$$\mathbf{T}_{\mu\nu} = \frac{1}{4\pi}\begin{pmatrix} \boldsymbol{Q} & (\frac{\vec{\boldsymbol{E}}}{\boldsymbol{c}} \times \vec{\boldsymbol{B}})_1 & (\frac{\vec{\boldsymbol{E}}}{\boldsymbol{c}} \times \vec{\boldsymbol{B}})_2 & (\frac{\vec{\boldsymbol{E}}}{\boldsymbol{c}} \times \vec{\boldsymbol{B}})_3 \\ (\frac{\vec{\boldsymbol{E}}}{\boldsymbol{c}} \times \vec{\boldsymbol{B}})_1 & -[\frac{\boldsymbol{E_1^2}}{\boldsymbol{c^2}} + \boldsymbol{B_1^2} - \boldsymbol{Q}] & -\frac{\mathbf{E_1E_2}}{\mathbf{c^2}} - \mathbf{B_1B_2} & -\frac{\mathbf{E_1E_3}}{\mathbf{c^2}} - \mathbf{B_1B_3} \\ (\frac{\vec{\boldsymbol{E}}}{\boldsymbol{c}} \times \vec{\boldsymbol{B}})_2 & -\frac{\mathbf{E_1E_2}}{\mathbf{c^2}} - \mathbf{B_1B_2} & -[\frac{\boldsymbol{E_2^2}}{\boldsymbol{c^2}} + \boldsymbol{B_2^2} - \boldsymbol{Q}] & -\frac{\mathbf{E_2E_3}}{\mathbf{c^2}} - \mathbf{B_2B_3} \\ (\frac{\vec{\boldsymbol{E}}}{\boldsymbol{c}} \times \vec{\boldsymbol{B}})_3 & -\frac{\mathbf{E_1E_3}}{\mathbf{c^2}} - \mathbf{B_1B_3} & -\frac{\mathbf{E_2E_3}}{\mathbf{c^2}} - \mathbf{B_2B_3} & -[\frac{\boldsymbol{E_3^2}}{\boldsymbol{c^2}} + \boldsymbol{B_3^2} - \boldsymbol{Q}] \end{pmatrix} \tag{17.10}$$

$$\text{with} \qquad \boldsymbol{Q} = \frac{\mathbf{1}}{\mathbf{2}}(\frac{\boldsymbol{E^2}}{\boldsymbol{c^2}} + \boldsymbol{B^2})$$

The trace of the electromagnetic energy momentum tensors vanishes

$$\mathbf{T} = \mathbf{0} \tag{17.11}$$

and the Einstein Equations simplify to

$$\mathbf{R}_{\boldsymbol{ij}} = 8\pi \cdot G_N \mathbf{T}_{\boldsymbol{ij}}. \tag{17.12}$$

For further considerations the following eigenwave is choosen:

$$\mathbf{E_2} = \mathbf{E_3} = \mathbf{B_1} = \mathbf{B_3} = \mathbf{0}, \qquad \mathbf{E_1} \neq \mathbf{0}, \qquad \mathbf{B_2} \neq \mathbf{0}. \tag{17.13}$$

$\Longrightarrow$

$$\mathbf{T_{00}} = \frac{1}{\mathbf{8}\boldsymbol{\pi}}\left(\frac{\mathbf{E_1^2}}{\mathbf{c^2}} + \mathbf{B_2^2}\right), \qquad \mathbf{T_{01}} = \mathbf{T_{02}} = \mathbf{0}, \qquad \mathbf{T_{03}} = \frac{1}{\mathbf{4}\boldsymbol{\pi}}(\frac{\vec{\mathbf{E}}_\mathbf{1}}{\boldsymbol{c}} \times \vec{\mathbf{B}}_\mathbf{2}) \tag{17.14}$$

$$\mathbf{T_{ik}} = \mathbf{0} \qquad \text{für } \boldsymbol{i} \neq \boldsymbol{k} \qquad \boldsymbol{i}, \boldsymbol{k} = \mathbf{1}, \mathbf{2}, \mathbf{3} \tag{17.15}$$

$$\mathbf{T_{11}} = \frac{-1}{\mathbf{8}\boldsymbol{\pi}}\left(\frac{\mathbf{E_1^2}}{\mathbf{c^2}} - \mathbf{B_2^2}\right), \qquad \mathbf{T_{22}} = \frac{1}{\mathbf{8}\boldsymbol{\pi}}\left(\frac{\mathbf{E_1^2}}{\mathbf{c^2}} - \mathbf{B_2^2}\right) \tag{17.16}$$

$$\mathbf{T_{33}} = \frac{1}{\mathbf{8}\boldsymbol{\pi}}\left(\frac{\mathbf{E_1^2}}{\mathbf{c^2}} + \mathbf{B_2^2}\right) \tag{17.17}$$

17.3. The quantitative relation of electromagnetic and gravitational waves

The quantitative connection is achieved via the Einstein Equations

$$\mathbf{R}_{\boldsymbol{\mu\nu}} = 8\pi \cdot G_N \mathbf{T}_{\boldsymbol{\mu\nu}}.$$

The description of a natural oscillation takes place using deformation interpretation by

$$\boxed{\begin{aligned} \vec{\boldsymbol{d}}_{k_i} =& \sqrt{4\pi}\frac{1}{k_i}\left(ck_i\vec{\mathbf{q}}_{\boldsymbol{k_i}}cos(\vec{\boldsymbol{k_i}}\cdot\vec{\boldsymbol{r}}) - \vec{\boldsymbol{p}}_{\boldsymbol{k_i}}sin(\vec{\boldsymbol{k_i}}\cdot\vec{\boldsymbol{r}})\right) \\ \vec{\boldsymbol{e}}_{k_i} =& \sqrt{4\pi}\boldsymbol{c}\left(ck_i\vec{\boldsymbol{q}}_{\vec{\boldsymbol{k_i}}}sin(\vec{\boldsymbol{k_i}}\cdot\vec{\boldsymbol{r}}) + \vec{\boldsymbol{p}}_{\vec{\boldsymbol{k_i}}}cos(\vec{\boldsymbol{k_i}}\cdot\vec{\boldsymbol{r}})\right) \\ \vec{\boldsymbol{b}}_{k_i} =& -\sqrt{4\pi}\frac{1}{k_i}\vec{\boldsymbol{k_i}}\times\left[ck_i\vec{\boldsymbol{q}}_{\vec{\boldsymbol{k_i}}}sin(\vec{\boldsymbol{k_i}}\cdot\vec{\boldsymbol{r}}) + \vec{\boldsymbol{p}}_{\vec{\boldsymbol{k_i}}}cos(\vec{\boldsymbol{k_i}}\cdot\vec{\boldsymbol{r}})\right], \end{aligned}} \tag{17.18}$$

and using the electromagnetic field interpretation by

$$\boxed{\begin{aligned} \vec{\boldsymbol{A}}_{k_i} =& \sqrt{4\pi}\frac{1}{k}\left(ck_i\vec{\boldsymbol{Q}}_{\vec{\boldsymbol{k_i}}}cos(\vec{\boldsymbol{k_i}}\cdot\vec{\ }) - \vec{\boldsymbol{P}}_{\vec{\boldsymbol{k_i}}}sin(\vec{\boldsymbol{k_i}}\cdot\vec{\boldsymbol{r}})\right) \\ \vec{\boldsymbol{E}}_{k_i} =& \sqrt{4\pi}\boldsymbol{c}\left(ck_i\vec{\boldsymbol{Q}}_{\vec{\boldsymbol{k_i}}}sin(\vec{\boldsymbol{k_i}}\cdot\vec{\boldsymbol{r}}) + \vec{\boldsymbol{P}}_{\vec{\boldsymbol{k_i}}}cos(\vec{\boldsymbol{k_i}}\cdot\vec{\boldsymbol{r}})\right) \\ \mathrm{B}_{k_i} =& -\sqrt{4\pi}\frac{1}{k}\vec{\boldsymbol{k_i}}\times\left[ck_i\vec{\boldsymbol{Q}}_{\vec{\boldsymbol{k_i}}}sin(\vec{\boldsymbol{k_i}}\cdot\vec{\boldsymbol{r}}) + \vec{\boldsymbol{P}}_{\vec{\boldsymbol{k_i}}}cos(\vec{\boldsymbol{k_i}}\cdot\vec{\boldsymbol{r}})\right] \end{aligned}} \tag{17.19}$$

with their corresponding energy density and energy in a volume surrounding the coordinate origin ($\vec{\mathbf{x}}_0$).

$$\boxed{\begin{aligned} \mathfrak{E}_{k_i} =& \frac{1}{2}\left(\frac{\boldsymbol{E}^2_{k_i}}{\boldsymbol{c^2}} + \boldsymbol{B}^2_{k_i}\right) && \text{Energiedichte} \\ \mathcal{E}_{k_i} =& \frac{1}{2}\int_{V_0}\left(\frac{\boldsymbol{E}^2_{k_i}}{\boldsymbol{c^2}} + \boldsymbol{B}^2_{k_i}\right)dV && \text{Energie} \end{aligned}} \tag{17.20}$$

.

The metric tensor of an elementary wave with $\vec{\mathbf{q}}_{\vec{\mathbf{k}}} \parallel \vec{\mathbf{e}}_x$, $\vec{\mathbf{p}}_{\vec{\mathbf{k}}} \parallel \vec{\mathbf{e}}_y$ and $\vec{\mathbf{k}} \parallel \vec{\mathbf{e}}_z$, $\vec{\mathbf{k}} \times \vec{\mathbf{q}}_{\vec{\mathbf{k}}} \parallel \vec{\mathbf{e}}_y$ is given by the tangential vectors:

$$\vec{\boldsymbol{t}}_i = \boldsymbol{\partial}_i\vec{\boldsymbol{y}} = (\boldsymbol{\partial}_i\boldsymbol{y}_1, \boldsymbol{\partial}_i\boldsymbol{y}_2, \boldsymbol{\partial}_i\boldsymbol{y}_3), \qquad \vec{\mathbf{y}} = \vec{\mathbf{d}} + \vec{\mathbf{x}}$$

$\Longrightarrow$

$$\vec{\boldsymbol{t}}_z = \boldsymbol{\partial_z}\vec{\boldsymbol{y}} = (\boldsymbol{\partial}_z\boldsymbol{d}_x, \mathbf{0}, \mathbf{1})\,.$$

With $\vec{\mathbf{k}} \cdot \vec{\mathbf{r}} = \boldsymbol{k} \cdot \boldsymbol{z} = \boldsymbol{\omega}_k/c \cdot \boldsymbol{z}$ one obtains

$$\vec{\boldsymbol{t}}_z = \left(-\sqrt{4\pi}\omega_k\vec{\mathbf{q}}_{\vec{\mathbf{k}}}sin(\boldsymbol{\omega}_k/c \cdot \boldsymbol{z}), -\sqrt{4\pi}\vec{\mathbf{p}}_{\vec{\mathbf{k}}}cos(\boldsymbol{\omega}_k/c \cdot \boldsymbol{z}), 1\right). \tag{17.21}$$

As searched spatial metric tensor element remains

$$\boldsymbol{t}_{zz} = 4\pi\left(\omega_k^2\mathbf{q}_{\vec{\mathbf{k}}}^2 sin^2(\boldsymbol{\omega}_k/c \cdot \boldsymbol{z}) + \mathbf{p}_{\vec{\mathbf{k}}}^2 cos^2(\boldsymbol{\omega}_k/c \cdot \boldsymbol{z})\right) + \mathbf{1} \tag{17.22}$$

with

$$\mathbf{q_k} = \mathbf{u_k}\boldsymbol{cos}(\omega_\mathbf{k}t), \qquad \mathbf{p_k} = \mathbf{v_k}\boldsymbol{sin}(\omega_\mathbf{k}t). \tag{17.23}$$

The purpose is the evaluation of the equation

$$\mathbf{R}_{\boldsymbol{zz}} = 8\pi \cdot G_N\mathbf{T}_{\boldsymbol{zz}}. \tag{17.24}$$

It is appropriate to note, that

$$\mathbf{T}_{zz} = \frac{1}{\boldsymbol{8\pi}}\left(\frac{\mathbf{E_x^2}}{c^2} + \mathbf{B_y^2}\right) = \frac{\mathfrak{E}_\mathbf{k}}{4\pi}. \tag{17.25}$$

Starting from the Riemannian curvature tensor

$$\mathbf{R}^\sigma_{.\nu\alpha\beta} = \partial_\alpha\boldsymbol{\Gamma}^\sigma_{\nu\beta} - \partial_\beta\boldsymbol{\Gamma}^\sigma_{\nu\alpha} + \boldsymbol{\Gamma}^\sigma_{\rho\alpha}\boldsymbol{\Gamma}^\rho_{\nu\beta} - \boldsymbol{\Gamma}^\sigma_{\rho\beta}\boldsymbol{\Gamma}^\rho_{\nu\alpha}. \tag{17.26}$$

with

$$\boldsymbol{\Gamma}^{\boldsymbol{\mu}}_{\boldsymbol{\nu\alpha}} = \frac{1}{2}\mathbf{g}^{\mu\lambda}\Big[\partial_\nu\mathbf{g}_{\alpha\lambda} + \partial_\alpha\mathbf{g}_{\lambda\nu} - \partial_\lambda\mathbf{g}_{\nu\alpha}\Big] \tag{17.27}$$

leads by contraction to the Ricci tensor

$$\mathbf{R}_{\mu\nu} = \mathbf{R}^\sigma_{.\mu\nu\sigma} = \partial_\nu\boldsymbol{\Gamma}^\sigma_{\mu\sigma} - \partial_\sigma\boldsymbol{\Gamma}^\sigma_{\mu\nu} + \boldsymbol{\Gamma}^\sigma_{\rho\nu}\boldsymbol{\Gamma}^\rho_{\mu\sigma} - \boldsymbol{\Gamma}^\sigma_{\rho\sigma}\boldsymbol{\Gamma}^\rho_{\mu\nu}. \tag{17.28}$$

The metric tensor after the deformation by the above elementary wave is used in the time orthogonal coordinate system.

$$\eta_{\mu\nu} = \begin{pmatrix} -\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix} \tag{17.29}$$

$$\mathbf{g}_{\mu\nu}(\vec{\mathbf{x}_0}) = \begin{pmatrix} \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{t_{zz}} \end{pmatrix} \qquad \mathbf{g}^{\mu\nu}(\vec{\mathbf{x}_0}) = \begin{pmatrix} \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1/t_{zz}} \end{pmatrix} \tag{17.30}$$

$$\begin{gathered} \mathbf{g}_{\mu\nu} \approx \eta_{\mu\nu} + \mathbf{h}_{\mu\nu}, \qquad \mathbf{g}^{\mu\nu} \approx \eta^{\mu\nu} - \mathbf{h}^{\mu\nu} \\ |\mathbf{h}_{\mu\nu}|,|\mathbf{h}^{\mu\nu}| \ll \mathbf{1} \end{gathered} \tag{17.31}$$

The Ricci tensor is typically written in a linear and non-linear proportion with respect to the Christoffel symbols stripped down.

$$\mathbf{R}^{(1)}_{\mu\nu}(\vec{\mathbf{x}_0}) = \partial_\nu \mathbf{\Gamma}^\sigma_{\mu\sigma} - \partial_\sigma \mathbf{\Gamma}^\sigma_{\mu\nu}, \qquad \mathbf{R}^{(2)}_{\mu\nu}(\vec{\mathbf{x}_0}) = \mathbf{\Gamma}^\sigma_{\rho\nu}\mathbf{\Gamma}^\rho_{\mu\sigma} - \mathbf{\Gamma}^\sigma_{\rho\sigma}\mathbf{\Gamma}^\rho_{\mu\nu} \tag{17.32}$$

Detailed examination of the Christoffel symbols

$$\boldsymbol{\Gamma^\sigma_{\mu\sigma}} = \frac{1}{2}\sum_\sigma\sum_\rho g^{\sigma\rho}\partial_\sigma g_{\mu\rho} + \frac{1}{2}\sum_\sigma\sum_\rho g^{\sigma\rho}\partial_\mu g_{\mu\rho} - \frac{1}{2}\sum_\sigma\sum_\rho g^{\sigma\rho}\partial_\rho g_{\mu\rho} \tag{17.33}$$

$$\begin{aligned} \frac{1}{2}\sum_\sigma\sum_\rho g^{\sigma\rho}\partial_\sigma g_{z\rho} &= \frac{1}{2}\underbrace{g^{00}\partial_0 g_{z0}}_{=0} + \frac{1}{2}g^{zz}\partial_z g_{zz} \\ \frac{1}{2}\sum_\sigma\sum_\rho g^{\sigma\rho}\partial_z g_{\rho\sigma} &= \frac{1}{2}\underbrace{g^{00}\partial_z g_{00}}_{=0} + \frac{1}{2}g^{zz}\partial_z g_{zz} \\ \frac{1}{2}\sum_\sigma\sum_\rho g^{\sigma\rho}\partial_\rho g_{\sigma z} &= \frac{1}{2}\underbrace{g^{00}\partial_0 g_{00}}_{=0} + \frac{1}{2}g^{zz}\partial_z g_{zz} \end{aligned} \tag{17.34}$$

$$\partial_z \mathbf{\Gamma}^\sigma_{\mathbf{z}\sigma} = \frac{1}{2}\partial_z \mathbf{g}^{zz}\partial_z \mathbf{g}_{zz} \tag{17.35}$$

$$\begin{aligned} \partial_\sigma \mathbf{\Gamma}^\sigma_{\mathbf{zz}} =& \frac{1}{2}\partial_0 \mathbf{g}^{00}[\partial_z \underbrace{\mathbf{g}_{z0}}_{=0} + \partial_z \underbrace{\mathbf{g}_{0z}}_{=0} - \partial_0 \mathbf{g}_{zz}] + \frac{1}{2}\partial_z \mathbf{g}^{zz}[\partial_z \mathbf{g}_{zz} + \partial_z \mathbf{g}_{zz} - \partial_z \mathbf{g}_{zz}] \\ =& + \frac{1}{2}\partial_0^2 \mathbf{g}_{zz} + \frac{1}{2}\partial_z \mathbf{g}^{zz}\partial_z \mathbf{g}_{zz} \end{aligned} \tag{17.36}$$

lead for the linear part to

$$\mathbf{R}^{(1)}_{zz}(\vec{\mathbf{x}_0}) = \partial_z \mathbf{\Gamma}^{\sigma}_{z\sigma} - \partial_\sigma \mathbf{\Gamma}^{\sigma}_{zz} = -\frac{1}{2}\partial_0^2 \mathbf{g}_{zz}. \tag{17.37}$$

The nonlinear part is determined for the considered elementary wave by

$$\mathbf{R}^{(2)}_{zz}(\vec{\mathbf{x}_0}) = \mathbf{\Gamma}^{\sigma}_{\rho z}\mathbf{\Gamma}^{\rho}_{z\sigma} - \mathbf{\Gamma}^{\sigma}_{\rho\sigma}\mathbf{\Gamma}^{\rho}_{zz} \tag{17.38}$$

with

$$\mathbf{\Gamma}^{\boldsymbol{\sigma}}_{\boldsymbol{\rho}\mathbf{z}} = \frac{1}{2}\mathbf{g}^{\sigma\sigma}\Big[\partial_\rho \mathbf{g}_{z\sigma} + \partial_z \mathbf{g}_{\sigma\rho} - \partial_\sigma \mathbf{g}_{\rho z}\Big] \tag{17.39}$$

$$\mathbf{\Gamma}^{\boldsymbol{\rho}}_{\mathbf{z}\boldsymbol{\sigma}} = \frac{1}{2}\mathbf{g}^{\rho\rho}\Big[\partial_z \mathbf{g}_{\sigma\rho} + \partial_z \mathbf{g}_{\rho z} - \partial_\rho \mathbf{g}_{z\sigma}\Big] \tag{17.40}$$

Considering the asumed elementary wave the single partial differentiaions $\partial_0,\ \partial_z$ of the metric tensor vanish in the space-time point $(0,0,0,0)$.

Thus one gets

$$\mathbf{R}_{zz}(\vec{\mathbf{x}_0}) = \mathbf{R}^{(1)}_{zz}(\vec{\mathbf{x}_0}) = -\frac{1}{2}\partial_0^2 \mathbf{g}_{zz}(\vec{\mathbf{x}_0}). \tag{17.41}$$

Now using

$$\mathbf{R}_{\boldsymbol{zz}} = 8\pi \cdot G_N \mathbf{T}_{\boldsymbol{zz}}.$$

and concerning

$$\partial_0 = \frac{1}{ic}\partial_t$$

the amplitude of the elementary gravitational wave (electromagnetic wave) gives the quantitative deformation of space by an electrodynamic elementary wave. Such the importance of the Einstein-Equations for microphysics is proved.

$$\boxed{\mathbf{d_k} = \frac{\mathbf{2}}{\omega_k^2}\sqrt{\pi\gamma\mathfrak{E}_\mathbf{k}}}. \tag{17.42}$$

with the constant of gravitation $\gamma = 6.67 \cdot 10^{-11} m^3 kg^{-1} s^{-2}$ and $\mathfrak{E}_\mathbf{k}$ = as energy density. In these considerations the light velocity c does not occur explicitly.

Setting $\mathfrak{E}_\mathbf{k} = 1 Wsec/m^3$ and using $\omega_k^2 = (\mathbf{2}\pi \cdot \nu)^\mathbf{2}$ with $\nu = 50$ this results in $\mathbf{d_k} = 2.933 \cdot 10^{-10} m$. In comparison, the measured atomic radius of H^1 is given by $\approx 2.5 \cdot 10^{-11} m$. **Obviously, that effect has to be considered in practice.**

As Spin 1 is assigned to photons the same has to be assumed for

the graviton. (A photon of giant wavelength from an other perspective, if it is existent.)
The Einstein Equations maybe achieve much more than describing cosmological processes!

18. Summary

The whole relativistic part is determined by the preceding solution of the problem of turbulence.
The constituted equations of evolution on one side for cosmological turbulent matter distribution and on the other side for Space-Time (Geometry of Einstein Space) do not use the Einstein Equations. They are based on different geometrodynamics, the geometrodynamics of turbulent matter transport and the geometrodynamics of deformation for the development of Space-Time (alternativly seen as development of the Riemannian hypersurface in dependence of time). Both equation systems are used independently from each other and depend on initial conditions especially designed for them. Both are limited to continuously differentiable processes. Emerging or collapsing of stars is not considered. In this case the equation systems have to be supplemented by appropriate source formulations.
The Einstein Equations couple Einstein Space-Time with energy momentum density. Thus a relation is established between physically totally different processes, which are treated decoupled in the mentioned evolution equation systems. The Einstein Equations come into play, when Space-Time fluctuations are related to the energy momentum density of an electromagnetic field. In literature, even in treatises of Penrose, the matter free Space-Time has no energy momentum density unless an electromagnetic field has to be considered. Gravitational waves, identified in this treatise by electromagnetic waves of very great wave lenghts, are always to be assumed. So the energy momentum tensor always gets together with one part due to the gravitational wave and another part due to a matter energy momentum tensor.

$$\mathbf{R}_{\boldsymbol{\mu\nu}} - \tfrac{1}{2}\mathbf{g}_{\boldsymbol{\mu\nu}}\mathbf{R} = 8\pi \cdot G_N \mathbf{T}_{\boldsymbol{\mu\nu}}$$

$$\Uparrow \qquad\qquad\qquad \Uparrow$$

Space-Time **Energy-Momentum**

$$\Updownarrow \qquad\qquad\qquad |$$

Geometrodynamics of Deformation
$c = const$

Geometrodynamics of Turbulence

Surely, respecting both parts simultaneously is not meaningful in approximations where Newtonian calculations are sufficient, but near black holes or exploding stars the energy momentum density of Space Time is not insignificant, probably.

Until now, electromagnetism is not directly understood. It is described with detours via mechanical effects though for physicists it has manifested in immediate clearness after more than a century of successful handling. With the described unification electromagnetism is directly led back to the most fundamental terms of physics, space and time. The usually discussed gauge transformations are chosen by the observation space respectivly the coordinate space. The vector potential achieves an absolute significance.

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