Calculation of dipole radiation

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The effect of the retarded electromagnetic field of an oscillating dipole on this very dipole is calculated. In this way, we have verified the method that is used to prove the spin radiation of a rotating dipole.
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1. Introduction

As is well known, a linear electric dipole oscillator $\mathbf{d}$ radiates energy flux, i.e. power, mainly in the plane perpendicular to the dipole [1 (67.8)], [2 (9.24)], [3 (2.74)]

$$P = \omega^4 \mathrm{d}^2 /(12\pi\varepsilon_0 c^3), \tag{1.1}$$

while a rotating dipole $\mathbf{d}$ radiates energy flux mainly along the axis of rotation of the dipole [1 § 67, Problem 1]

$$P = \omega^4 \mathrm{d}^2 /(6\pi\varepsilon_0 c^3) \tag{1.2}$$

and emits angular momentum flux, i.e. torque, mainly in the plane of rotation of the dipole [1 (75.7)], [4 Appendix 1 (17)]

$$dL_z / dt = \underset{L}{\tau}_z = \omega^3 \mathrm{d}^2 /(6\pi\varepsilon_0 c^3)\,. \tag{1.3}$$

In [5-7], it is shown that the angular momentum flux (1.3) is an orbital angular momentum flux and is not a radiation. In this regard, we add a quote here: “The angular momentum is contained in that region of the field in which the product EH decreases as r^{-3}” (W. Heitler) [8].

However, in [5-7], it is shown that a spin angular momentum flux

$$dS_z / dt = \underset{S}{\tau}_z = \omega^3 \mathrm{d}^2 /(12\pi\varepsilon_0 c^3) \tag{1.4}$$

is radiated mainly along the axis of rotation of the rotating dipole. This flux is not described now by the modern electrodynamics.

In Figures 1 and 2, angular distributions of the powers and of the angular momentum fluxes are depicted

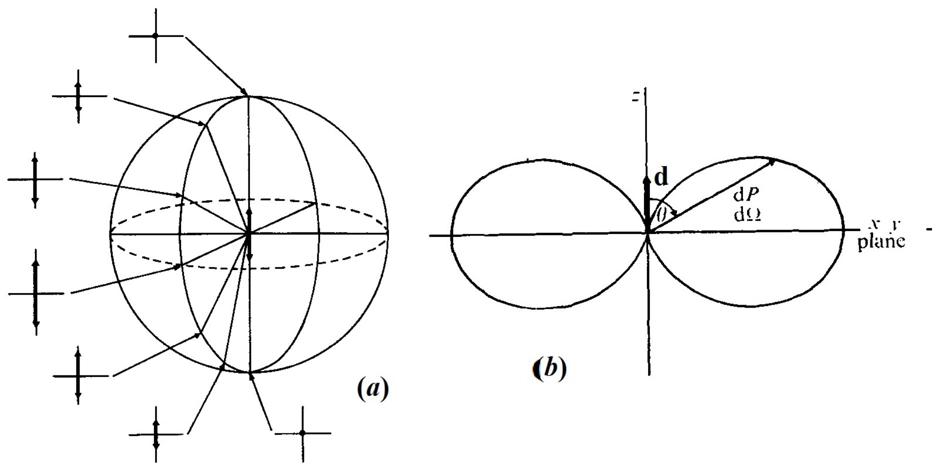

Fig. 1. An electric dipole oscillating parallel to the z-axis.
(a) Polarization of the electric field seen by looking from different direction [9].
(b) Angular distribution of the radiated energy [1 (67.7)], [2 (9.23)], [3 (2.72]

$$dP / d\Omega = \omega^4 \mathrm{d}^2 \sin^2\theta / 32\pi^2\varepsilon_0 c^3\,. \tag{1.5}$$

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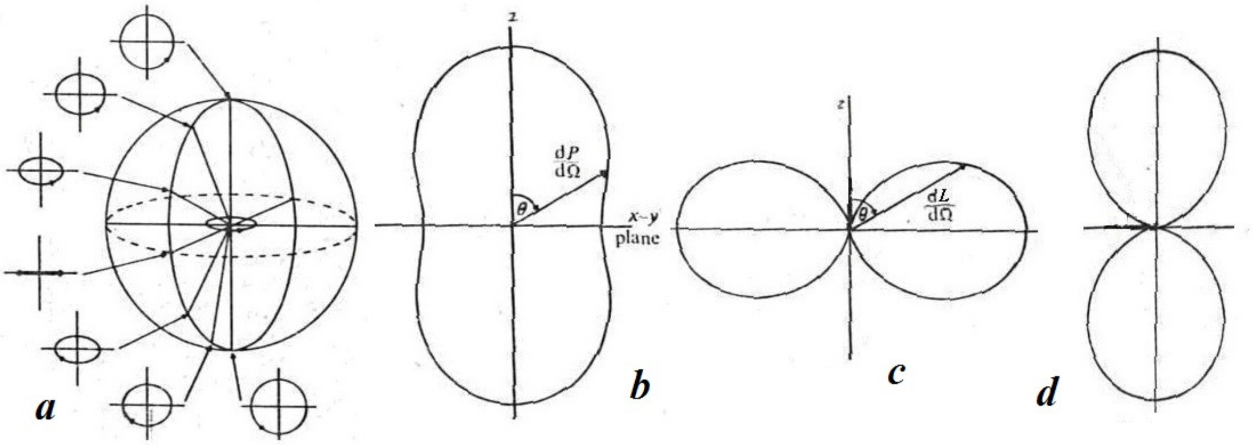

Fig. 2. An electric dipole rotating in the *x-y* plane

(a) Polarization of the electric field seen by looking from different direction [9].

(b) Angular distribution of the radiated energy [1, § 67, Problem 1], [2, Table 9.1], [10]

$$dP / d\Omega = \omega^4 \mathrm{d}^2 (\cos^2 \theta + 1) / 32\pi^2 \varepsilon_0 c^3 . \tag{1.6}$$

(c) Angular distribution of the orbital angular momentum flux [5-7,10]

$$dL_z / dtd\Omega = \omega^3 \mathrm{d}^2 \sin^2 \theta / 16\pi^2 \varepsilon_0 c^3 . \tag{1.7}$$

(d) Angular distribution of the spin angular momentum flux [5-7]

$$dS_z / dtd\Omega = \omega^3 \mathrm{d}^2 \cos^2 \theta / 16\pi^2 \varepsilon_0 c^3 . \tag{1.8}$$

It is important that the total flux of the angular momentum, orbital plus spin, (1.3) and (1.4),

$$dL_z / dt + dS_z / dt == \omega^3 \mathrm{d}^2 /(4\pi\varepsilon_0 c^3), \tag{1.9}$$

was found in the work [7] by counting the effect of the retarded electromagnetic field of the rotating dipole on the dipole itself. In this article, we demonstrate the validity of this method of counting by the use it when calculating the radiation of energy by an oscillating dipole (1.1).

2. The use of the Jefimenko's generalizations

We will obtain the value (1.1) as a result of the action of the electromagnetic field on the dipole itself, according to the formula for the volume power density

$$P_\wedge = -(\mathbf{j} \cdot \mathbf{E}) ; \tag{2.1}$$

here $\mathbf{j}$ and $\mathbf{E}$ are the dipole current density and the electric field strength in the dipole, respectively, and the subscript $\wedge$ of $P_\wedge$ denotes "density", $dP = P_\wedge d^3x$.

In this paper, the electric field near the dipole is calculated by the known formula taking into account the retardation [2 (6.55)]:

$$\mathbf{E}(\mathbf{x},t) = \frac{1}{4\pi\varepsilon_0} \int d^3x' \left\{ \frac{\hat{\mathbf{r}}}{r^2} [\rho(\mathbf{x}',t')]_{\text{ret}} + \frac{\hat{\mathbf{r}}}{cr} [\partial_{t'} \rho(\mathbf{x}',t')]_{\text{ret}} - \frac{1}{c^2 r} [\partial_{t'} \mathbf{j}(\mathbf{x}',t')]_{\text{ret}} \right\}, \tag{2.2}$$

and we consider an "elementary vibrator" as a dipole. It means that the current of the dipole is the same at all points, and the charges $\breve{q}$ are only at the ends (see Figure 3).

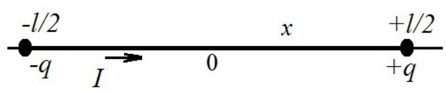

Figure 3. Elementary vibrator

$$\breve{\mathrm{d}} = \breve{q} l = ql \exp(-i\omega t) , \tag{2.3}$$

Expression (2.3) is a complex dipole moment (the symbol *breve* denotes complex quantities). The current of the dipole is obtained by differentiation:

$$\breve{I} = \partial_t \breve{\mathrm{d}}/l = -i\omega q \exp(-i\omega t) . \tag{2.4}$$

The first term of expression (2.2),

$$E_1^x = \frac{1}{4\pi\varepsilon_0} \int d^3x' \left\{ \frac{1}{r^2} [\rho(\mathbf{x}', t')]_{\mathrm{ret}} \right\},$$

is simply the retarded Coulomb field at the point x. Therefore, replacing

$$d^3x'\rho \to d\breve{q}' , \quad t \to t - (l/2 \pm x)/c , \quad r \to l/2 \pm x$$

and taking into account the direction of the electric field, we obtain the electric field strength from the both charges at the point x:

$$E_1^x = \frac{q}{4\pi\varepsilon_0} \left[\frac{-\exp[-i\omega t + i\omega(l/2 + x)/c]}{(l/2 + x)^2} - \frac{\exp[-i\omega t + i\omega(l/2 - x)/c]}{(l/2 - x)^2} \right]. \tag{2.5}$$

The corresponding contribution of this term to the power generated by the dipole is given by formula (2.1) (we replace $jd^3x \to Idx$ from (2.4), and the bar, instead of the breve, means complex conjugation)

$$\begin{aligned} P_1 &= \frac{-1}{2} \int_{-l/2}^{l/2} dx \Re\{\bar{I} E_1^x\} \\ &= -\frac{\omega q^2}{8\pi\varepsilon_0} \int_{-l/2}^{l/2} dx \Re\left\{ i \exp(i\omega t) \left[\frac{-\exp[-i\omega t + i\omega(l/2 + x)/c]}{(l/2 + x)^2} - \frac{\exp[-i\omega t + i\omega(l/2 - x)/c]}{(l/2 - x)^2} \right] \right\} \\ &= -\frac{\omega q^2}{8\pi\varepsilon_0} \int_{-l/2}^{l/2} dx \left[\frac{\sin[\omega(l/2 + x)/c]}{(l/2 + x)^2} + \frac{\sin[\omega(l/2 - x)/c]}{(l/2 - x)^2} \right]. \end{aligned} \tag{2.6}$$

Taking into account the small size of the dipole, we consider only two terms of the expansion of the sine in a series

$$P_1 = -\frac{\omega q^2}{8\pi\varepsilon_0} \int_{-l/2}^{l/2} dx \left[\frac{\omega}{c(l/2 + x)} - \frac{\omega^3 (l/2 + x)}{6c^3} + \frac{\omega}{c(l/2 - x)} - \frac{\omega^3 (l/2 - x)}{6c^3} \right]. \tag{2.7}$$

Similarly to formula (2.5), we find the electric field provided by the second term of formula (2.2)

$$E_2^x = \frac{i\omega q}{4\pi\varepsilon_0 c} \left[\frac{\exp[-i\omega t + i\omega(l/2 + x)/c]}{(l/2 + x)} + \frac{\exp[-i\omega t + i\omega(l/2 - x)/c]}{(l/2 - x)} \right]. \tag{2.8}$$

In contrast to formula (2.5), this formula contains *i*.

Formula (2.1) gives the contribution of the second term, E_2^x, to the power generated by the dipole

$$\begin{aligned} P_2 &= \frac{\omega^2 q^2}{8\pi\varepsilon_0 c} \int_{-l/2}^{l/2} dx \Re\left\{ \exp(i\omega t) \left[\frac{\exp[-i\omega t + i\omega(l/2 + x)/c]}{(l/2 + x)} + \frac{\exp[-i\omega t + i\omega(l/2 - x)/c]}{(l/2 - x)} \right] \right\} \\ &= \frac{\omega^2 q^2}{8\pi\varepsilon_0 c} \int_{-l/2}^{l/2} dx \left[\frac{\cos[\omega(l/2 + x)/c]}{(l/2 + x)} + \frac{\cos[\omega(l/2 - x)/c]}{(l/2 - x)} \right]. \end{aligned} \tag{2.9}$$

Restricting ourselves to the two terms of the cosine expansion in a series, we have

$$P_2 = \frac{\omega^2 q^2}{8\pi\varepsilon_0 c} \int_{-l/2}^{l/2} dx \left[\frac{1}{(l/2 + x)} - \frac{\omega^2 (l/2 + x)}{2c^2} + \frac{1}{(l/2 - x)} - \frac{\omega^2 (l/2 - x)}{2c^2} \right]. \tag{2.10}$$

Surprisingly, the integrals diverging at the ends of the dipole are shortened upon the addition $P_1 + P_2$, and the remaining terms are constants. As $l/6 - l/2 = -l/3$, this part of the power is

$$P_1 + P_2 = -\frac{\omega^2 \mathrm{d}^2}{24\pi\varepsilon_0 c^3} . \tag{2.11}$$

The third term of formula (2.2),

$$E_3^x = \frac{1}{4\pi\varepsilon_0} \int dx' \left\{ -\frac{1}{c^2 r} [\partial_{t'} I(x', t')]_{\mathrm{ret}} \right\},$$

uses the derivative of the current

$$\partial_t \breve{I} = -\omega^2 q \exp(-i\omega t)\,. \qquad (2.12)$$

To calculate the strength at the point x, we divided the region of integration into two parts by the point x

$$E_3^x = -\frac{\omega^2 q}{4\pi\varepsilon_0 c^2}\left\{\int_{-l/2}^{x} dx'\frac{-\exp[-i\omega t + i\omega(x-x')/c}{x-x'} + \int_{x}^{l/2} dx'\frac{-\exp[-i\omega t + i\omega(x'-x)/c}{x'-x}\right\}$$

Using formula (2.1), $dP = -(\mathbf{j}\cdot\mathbf{E})d^3x = -IE^x dx$, and current (5), we obtain the power corresponding to the third term of formula (2.1):

$$P_3 = -\frac{\omega^3 q^2}{8\pi\varepsilon_0 c^2}\int_{-l/2}^{l/2} dx \Re\left\{i\omega\exp(i\omega t)\left[\int_{-l/2}^{x} dx'\frac{\exp[-i\omega t + i\omega(x-x')/c]}{x-x'} + \int_{x}^{l/2} dx'\frac{\exp[-i\omega t + i\omega(x'-x)/c]}{x'-x}\right]\right\}$$

$$= -\frac{\omega^3 q^2}{8\pi\varepsilon_0 c^2}\int_{-l/2}^{l/2} dx\left[\int_{-l/2}^{x} dx'\frac{-\sin[\omega(x-x')/c]}{x-x'} + \int_{x}^{l/2} dx'\frac{-\sin[\omega(x'-x)/c]}{x'-x}\right]. \qquad (2.13)$$

Restricting ourselves to one term in the expansion of the sine in a series, we easily obtain

$$P_3 = \frac{\omega^4 \mathrm{d}^2}{8\pi\varepsilon_0 c^3}\,. \qquad (2.14)$$

Thus, the power radiated by a dipole is equal to the value (1.1)

$$P = P_1 + P_2 + P_3 = \omega^4 \mathrm{d}^2/(12\pi\varepsilon_0 c^3)\,. \qquad (2.15)$$

3. Conclusion

The presented calculation confirms the correctness of the method, which previously proved the existence of spin radiation by a rotating dipole [7], found using the spin tensor [5,6]

I am eternally grateful to Professor Robert Romer for the courageous publication of my question: "Does a plane wave really not carry spin?" (was submitted on 07 October, 1999) [11].

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