Proof that Mersenne Prime Numbers are Infinite and that Even Perfect Numbers are Infinite

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17 November 2018

Abstract

Mersenne prime is a prime number that is one less than a power of two. That is, it is a prime number of the form $M_n = 2^n - 1$ for some integer n. They are named after Marin Mersenne, a French Minim friar, who studied them in the early 17th century.

The exponents n which give Mersenne primes are 2, 3, 5, 7, 13, 17, 19, 31, ... (sequence A000043) and the resulting Mersenne primes are 3, 7, 31, 127, 8191, 131071, 524287, 2147483647, ...

If n is a composite number then so is $2^n - 1$. More generally, numbers of the form $M_n = 2^n - 1$ without the primality requirement may be called Mersenne numbers. Sometimes, however, Mersenne numbers are defined to have the additional requirement that n be prime. The smallest composite Mersenne number with prime exponent n is $2^{11} - 1 = 2047 = 23 \times 89$.

Mersenne primes M_p are also noteworthy due to their connection to perfect numbers.

A new Mersenne prime was found in December 2017. As of January 2018, 50 Mersenne primes are now known. The largest known prime number $2^{77,232,917} - 1$ is a Mersenne prime. Many fundamental questions about Mersenne primes remain unresolved. It is not even known whether the set of Mersenne primes is finite or infinite. Ever since M_{521} was proven prime in 1952, the largest known prime has always been Mersenne primes, which shows that Mersenne primes become large quickly. Since the prime numbers are infinite, and since all large primes discovered since 1952 have been Mersenne primes, this seems to be evidence indicating the infinitude of Mersenne primes since there has to continually be an infinite number of large primes, even if we don't find them. Additional evidence, is that since prime numbers are infinite, there exist an infinite number of Mersenne numbers of form $2^p – 1$, meaning there exist an infinite number of Mersenne numbers that are candidates for Mersenne primes. However, as with $2^{11} – 1$, we know not all Mersenne numbers of form $2^p – 1$ are primes. All of this evidence makes it reasonable to conjecture that there exist an infinite number of Mersenne primes. First we will provide additional evidence indicating an infinite number of Mersenne primes. Then we will provide the proof.

Additional Evidence Indicating Infinite Mersenne Primes

In reference 6, Dirichlet proves that every Mersenne prime, M_p is a prime number of the form 4*n* + 3 for some integer "*n*".

In reference 7, Dirichlet also proves that there are infinitely many primes of the form 4*n* + 3 where *n* is a nonnegative integer. Therefore, there are an infinite number of prime numbers of form 4*n* + 3, which indicates there is enough room to have an infinite number of Mersenne prime, M_p, of form 4*n* + 3, yet additional evidence.

The strongest evidence comes from the following probability of Mersenne primes. Let $M_p = 2^p - 1$, for all Mersenne primes. For each prime *p*, the probability that M_p is prime is given as follows (from reference 8):

$$\frac{1}{\ln(2^p - 1)} \approx \frac{1}{p(\ln(2))}$$

Thus, the expected total number of Mersenne primes is the sum of the individual probabilities:

$$\frac{1}{\ln(2)} \sum_{p=1}^{p=\infty} \frac{1}{p}$$

Where, the sum runs over all primes. But it is known that

$$\sum_{p=1}^{p=\infty} \frac{1}{p}$$

is divergent, which strongly suggests that the number of Mersenne primes is infinite (reference 8).

Proof of Infinite Mersenne Primes

The divergence of the harmonic series was independently proved by Johann Bernoulli in 1689 in a counter-intuitive manner (reference 1). His proof is worthy of deep study, as it shows the counter-intuitive nature of infinity. We will use Bernoulli's proof and apply it toward proving the Mersenne prime numbers are infinite.

Let the finite set of, p, Mersenne primes be listed in reverse order from the largest to smallest Mersenne primes as follows:

$n_1 = M_1 = 2^{p1} - 1$ = largest Mersenne prime

$n_2 = M_2 = 2^{p2} - 1$ = second largest Mersenne prime

$n_3 = M_3 = 2^{p3} - 1$ = third largest Mersenne prime

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$n_p = M_p = 2^p - 1$ = smallest Mersenne prime number = 3

This reverse ordering of the finite set of Mersenne prime numbers is key to our proof. We assume that the following Mersenne prime reciprocal series have a finite sum, which we call S.

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \cdots + \frac{1}{n_p} > \frac{1}{2n_1} + \frac{1}{3n_2} + \frac{1}{4n_3} + \cdots + \frac{1}{kn_p} = S$$

Where, k is the denominator factor for the smallest Mersenne prime number that exists in our finite set.

We now proceed to derive a contradiction in the following manner. First we rewrite each term occurring in S thus:

$$\frac{1}{3n_2} = \frac{2}{6n_2} = \frac{1}{6n_2} + \frac{1}{6n_2}\ ,\ \frac{1}{4n_3} = \frac{3}{12n_3} = \frac{1}{12n_3} + \frac{1}{12n_3} + \frac{1}{12n_3}\ , \ldots,$$

Next we write the resulting fractions in an array as shown below:

$$\begin{array}{ccccccc}
\frac{1}{2n_1} & \frac{1}{6n_2} & \frac{1}{12n_3} & \frac{1}{20n_4} & \frac{1}{30n_5} & \frac{1}{42n_6} & \frac{1}{56n_7} \cdots \\
 & \frac{1}{6n_2} & \frac{1}{12n_3} & \frac{1}{20n_4} & \frac{1}{30n_5} & \frac{1}{42n_6} & \frac{1}{56n_7} \cdots \\
 & & \frac{1}{12n_3} & \frac{1}{20n_4} & \frac{1}{30n_5} & \frac{1}{42n_6} & \frac{1}{56n_7} \cdots \\
 & & & \frac{1}{20n_4} & \frac{1}{30n_5} & \frac{1}{42n_6} & \frac{1}{56n_7} \cdots \\
 & & & & \frac{1}{30n_5} & \frac{1}{42n_6} & \frac{1}{56n_7} \cdots \\
 & & & & & \frac{1}{42n_6} & \frac{1}{56n_7} \cdots \\
 & & & & & & \frac{1}{56n_7} \cdots
\end{array}$$

Note that the column sums are just the fractions of the Mersenne primes; thus S is the sum of all the fractions occurring in the array. As Bernoulli did, we now sums the rows using the telescoping technique. Next we assign symbols to the row sums as shown below,

$$A = \frac{1}{2n_1} + \frac{1}{6n_2} + \frac{1}{12n_3} + \frac{1}{20n_4} + \frac{1}{30n_5} + \frac{1}{42n_6} + \frac{1}{56n_7} + \ldots ,$$

$$B = \frac{1}{6n_2} + \frac{1}{12n_3} + \frac{1}{20n_4} + \frac{1}{30n_5} + \frac{1}{42n_6} + \frac{1}{56n_7} + \ldots ,$$

$$C = \frac{1}{12n_3} + \frac{1}{20n_4} + \frac{1}{30n_5} + \frac{1}{42n_6} + \frac{1}{56n_7} + \dots ,$$

$$D = \frac{1}{20n_4} + \frac{1}{30n_5} + \frac{1}{42n_6} + \frac{1}{56n_7} + \dots ,$$

We now rearrange as follows:

$$A = (\frac{1}{n_1} - \frac{1}{2n_1}) + (\frac{1}{2n_2} - \frac{1}{3n_2}) + (\frac{1}{3n_3} - \frac{1}{4n_3}) + (\frac{1}{4n_4} - \frac{1}{5n_4}) + \dots$$

Since, $n_1 > n_2 > n_3 > n_4$

$$A = \frac{1}{n_1} + (\frac{1}{2n_2} - \frac{1}{2n_1}) + (\frac{1}{3n_3} - \frac{1}{3n_2}) + (\frac{1}{4n_4} - \frac{1}{4n_3}) + (\frac{1}{5n_5} - \frac{1}{5n_4}) + \dots$$

Since, $\left(\frac{1}{2n_2} - \frac{1}{2n_1}\right) > 0$, $\left(\frac{1}{3n_3} - \frac{1}{3n_2}\right) > 0$, $(\frac{1}{4n_4} - \frac{1}{4n_3}) > 0$, $(\frac{1}{5n_5} - \frac{1}{5n_4}) > 0$

Then, $A > \frac{1}{n_1}$

$$B = (\frac{1}{2n_2} - \frac{1}{3n_2}) + (\frac{1}{3n_3} - \frac{1}{4n_3}) + (\frac{1}{4n_4} - \frac{1}{5n_4}) + (\frac{1}{5n_5} - \frac{1}{6n_5})\dots$$

Since, $n_1 > n_2 > n_3 > n_4$, the same rearranging that we did with *A* can be done with *B*.

Then, $B > \frac{1}{2n_2}$

$$C = \left(\frac{1}{3n_3} - \frac{1}{4n_3}\right) + \left(\frac{1}{4n_4} - \frac{1}{5n_4}\right) + \left(\frac{1}{5n_5} - \frac{1}{6n_5}\right) + \left(\frac{1}{6n_5} - \frac{1}{7n_5}\right) \ldots$$

Since, $n_1 > n_2 > n_3 > n_4$, the same rearranging that we did with *A* can be done with *C*.

Then, $C > \frac{1}{3n_3}$

$$D = \left(\frac{1}{4n_4} - \frac{1}{5n_4}\right) + \left(\frac{1}{5n_5} - \frac{1}{6n_5}\right) + \left(\frac{1}{6n_5} - \frac{1}{7n_5}\right) + \left(\frac{1}{7n_6} - \frac{1}{8n_6}\right) \ldots$$

Since, $n_1 > n_2 > n_3 > n_4$, the same rearranging that we did with *A* can be done with *D*.

Then, $D > \frac{1}{4n_4}$

and so on. Thus the sum *S,* which we had written in the form $A + B + C + D + \ldots$, turns out to be greater than

$$S > \frac{1}{n_1} + \frac{1}{2n_2} + \frac{1}{3n_3} + \frac{1}{4n_4} + \cdots$$

At the start we had defined *S* to be the following finite series,

$$S = \frac{1}{2n_1} + \frac{1}{3n_2} + \frac{1}{4n_3} + \cdots + \frac{1}{kn_\mathrm{p}}$$

And we defined that, $\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \frac{1}{n_4} + \cdots > S = \frac{1}{2n_1} + \frac{1}{3n_2} + \frac{1}{4n_3} + \cdots$

However, we just proved that $S > \frac{1}{n_1} + \frac{1}{2n_2} + \frac{1}{3n_3} + \frac{1}{4n_4} + \cdots > S = \frac{1}{2n_1} + \frac{1}{3n_2} + \frac{1}{4n_3} + \cdots + \frac{1}{kn_{\mathrm{p}}}$

However, this is a contradiction, since in the finite realm S can't be equal to and greater than $\frac{1}{2n_1} + \frac{1}{3n_2} + \frac{1}{4n_3} + \cdots + \frac{1}{kn_{\mathrm{p}}}$ at the same time. Therefore, S must be infinite.

Now we can rewrite the S, the Mersenne prime series as,

$$S > \frac{1}{n_1} + \frac{1}{2n_2} + \frac{1}{3n_3} + \frac{1}{4n_4} + \cdots > \frac{1}{2n_1} + \frac{1}{3n_2} + \frac{1}{4n_3} + \cdots + \frac{1}{kn_{\mathrm{p}}} = S$$

This implies that $S > S$

However, no finite number can satisfy such an equation. Therefore, we have a contradiction and must conclude that $S = \infty$**.** Remember our definition of S from the above series:

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \cdots + \frac{1}{n_{\mathrm{p}}} > S = \infty$$

Therefore, $\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \cdots + \frac{1}{n_{\mathrm{p}}} > \infty$

Therefore, we have proven that the reciprocal Mersenne prime series diverges to infinity. Obviously, this cannot possibly happen if there are only finitely many Mersenne prime reciprocals, therefore the Mersenne prime reciprocals are infinite in number. Since the Mersenne prime reciprocals are infinite in number, the Mersenne prime numbers must be infinite as well.

This proof shows the shows the counter-intuitive nature of infinity, and why it has taken so long to prove the Mersenne primes are infinite, as it is not obvious that the reciprocal Mersenne prime series would diverge. The opposite is true, that it seems that the Mersenne prime series would converge. For example, numerically it has been shown that the sum of reciprocals of Mersenne primes converges to $0.516454178940788565 33\cdots$, however we have just proven that it diverges to infinity extremely slow. Numerically the Mersenne primes grow so fast that the reciprocals for the large primes are rounded off to zero numerically. However, our proof has shown that the infinitesimally small reciprocal Mersenne primes add to infinity and should not have been rounded to zero. Numerically no computer could do these calculations to prevent rounding, additionally to date computers have only found the first 50 Mersenne primes since they grow so rapidly.

The author expresses many thanks to the work of Johann Bernoulli in 1689, without his work this proof would not have been possible. It was solely through the study of Johann Bernoulli's work that the author was inspired to see this divergent proof. The author would also like to express many thanks to Shailesh Shirali's work in which he documented Johann Bernoulli's work in the most fascinating and interesting way.

Proof of Infinite Even Perfect Numbers

Our previous proof that Mersenne primes are infinite, provides a direct proof to the existence of an infinite number of even perfect numbers. First, Euclid proved that $2^{p-1}(2^p - 1)$ is an even perfect number whenever $2^p - 1$ is prime.

Prime numbers of the form $2^p - 1$ are known as Mersenne primes. Therefore, since that every Mersenne prime generates an even perfect number of the form $2^{p-1}(2^p - 1)$, and since we have proven that an infinite number of Mersenne primes, then it follows that an infinite number of even perfect numbers exits.

Again, the author expresses many thanks to the work of Shailesh Shirali for documenting Johann Bernoulli's work, as this proof is dependent on Johann Bernoulli's work on the divergence of the harmonic series.

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