Proof of Landau's Fourth Problem

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26 November 2018

Abstract

At the 1912 International Congress of Mathematicians, Edmund Landau listed four basic problems about prime numbers. These problems were characterised in his speech as "unattackable at the present state of mathematics" and are now known as Landau's problems. They are as follows:

1. Goldbach's conjecture: Can every even integer greater than 2 be written as the sum of two primes?
2. Twin prime conjecture: Are there infinitely many primes p such that $p + 2$ is prime?
3. Legendre's conjecture: Does there always exist at least one prime between consecutive perfect squares?
4. Are there infinitely many primes p such that $p - 1$ is a perfect square? In other words: Are there infinitely many primes of the form $n^2 + 1$?

We will solve Landau's fourth problem by proving there are infinitely many primes of the form $n^2 + 1$.

Proof of Infinite Primes of Form $n^2 + 1$

The divergence of the harmonic series was independently proved by Johann Bernoulli in 1689 in a counter-intuitive manner (reference 1). His proof is worthy of deep study, as it shows the counter-intuitive nature of infinity. We will use Bernoulli's proof and apply it toward proving the prime numbers are infinite.

Let the finite set of, p, primes of form $n^2 + 1$ be listed in reverse order from the largest to smallest primes of form $n^2 + 1$ as follows:

$$n_1 = 2^{p1} - 1 = \text{largest prime of form } n^2 + 1$$

$n_2 = 2^{p2} - 1$ = second largest prime of form $n^2 + 1$

$n_3 = 2^{p3} - 1$ = third largest prime of form $n^2 + 1$

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$n_p = 2^p - 1$ = smallest prime of form $n^2 + 1$

This reverse ordering of the finite set of primes of form $n^2 + 1$ is key to our proof. We assume that the following primes of form $n^2 + 1$ reciprocal series have a finite sum, which we call *S*.

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \cdots + \frac{1}{n_p} > \frac{1}{2n_1} + \frac{1}{3n_2} + \frac{1}{4n_3} + \cdots + \frac{1}{kn_p} = S$$

Where, *k* is the denominator factor for the smallest prime of form $n^2 + 1$ that exists in our finite set.

We now proceed to derive a contradiction in the following manner. First we rewrite each term occurring in *S* thus:

$$\frac{1}{3n_2} = \frac{2}{6n_2} = \frac{1}{6n_2} + \frac{1}{6n_2}, \quad \frac{1}{4n_3} = \frac{3}{12n_3} = \frac{1}{12n_3} + \frac{1}{12n_3} + \frac{1}{12n_3}, \ldots,$$

Next we write the resulting fractions in an array as shown below:

$$\begin{array}{ccccccc} \frac{1}{2n_1} & \frac{1}{6n_2} & \frac{1}{12n_3} & \frac{1}{20n_4} & \frac{1}{30n_5} & \frac{1}{42n_6} & \frac{1}{56n_7} \cdots \\ & \frac{1}{6n_2} & \frac{1}{12n_3} & \frac{1}{20n_4} & \frac{1}{30n_5} & \frac{1}{42n_6} & \frac{1}{56n_7} \cdots \\ & & \frac{1}{12n_3} & \frac{1}{20n_4} & \frac{1}{30n_5} & \frac{1}{42n_6} & \frac{1}{56n_7} \cdots \\ & & & \frac{1}{20n_4} & \frac{1}{30n_5} & \frac{1}{42n_6} & \frac{1}{56n_7} \cdots \end{array}$$

$$\frac{1}{30n_5} \quad \frac{1}{42n_6} \quad \frac{1}{56n_7} \cdots$$

$$\frac{1}{42n_6} \quad \frac{1}{56n_7} \cdots$$

$$\frac{1}{56n_7} \cdots$$

Note that the column sums are just the fractions of the primes of form $n^2 + 1$; thus S is the sum of all the fractions occurring in the array. As Bernoulli did, we now sums the rows using the telescoping technique. Next we assign symbols to the row sums as shown below,

$$A = \frac{1}{2n_1} + \frac{1}{6n_2} + \frac{1}{12n_3} + \frac{1}{20n_4} + \frac{1}{30n_5} + \frac{1}{42n_6} + \frac{1}{56n_7} + \dots,$$

$$B = \frac{1}{6n_2} + \frac{1}{12n_3} + \frac{1}{20n_4} + \frac{1}{30n_5} + \frac{1}{42n_6} + \frac{1}{56n_7} + \dots,$$

$$C = \frac{1}{12n_3} + \frac{1}{20n_4} + \frac{1}{30n_5} + \frac{1}{42n_6} + \frac{1}{56n_7} + \dots,$$

$$D = \frac{1}{20n_4} + \frac{1}{30n_5} + \frac{1}{42n_6} + \frac{1}{56n_7} + \dots,$$

We now rearrange as follows:

$$A = (\frac{1}{n_1} - \frac{1}{2n_1}) + (\frac{1}{2n_2} - \frac{1}{3n_2}) + (\frac{1}{3n_3} - \frac{1}{4n_3}) + (\frac{1}{4n_4} - \frac{1}{5n_4}) + \dots$$

Since, $n_1 > n_2 > n_3 > n_4$

$$A = \frac{1}{n_1} + (\frac{1}{2n_2} - \frac{1}{2n_1}) + (\frac{1}{3n_3} - \frac{1}{3n_2}) + (\frac{1}{4n_4} - \frac{1}{4n_3}) + (\frac{1}{5n_5} - \frac{1}{5n_4}) + \dots$$

Since, $\left(\frac{1}{2n_2} - \frac{1}{2n_1}\right) > 0$, $\left(\frac{1}{3n_3} - \frac{1}{3n_2}\right) > 0$, $(\frac{1}{4n_4} - \frac{1}{4n_3}) > 0$, $(\frac{1}{5n_5} - \frac{1}{5n_4}) > 0$

Then, $A > \frac{1}{n_1}$

$$B = (\frac{1}{2n_2} - \frac{1}{3n_2}) + (\frac{1}{3n_3} - \frac{1}{4n_3}) + (\frac{1}{4n_4} - \frac{1}{5n_4}) + (\frac{1}{5n_5} - \frac{1}{6n_5})\ldots$$

Since, $n_1 > n_2 > n_3 > n_4$, the same rearranging that we did with *A* can be done with *B*.

Then, $B > \frac{1}{2n_2}$

$$C = (\frac{1}{3n_3} - \frac{1}{4n_3}) + (\frac{1}{4n_4} - \frac{1}{5n_4}) + \left(\frac{1}{5n_5} - \frac{1}{6n_5}\right) + (\frac{1}{6n_5} - \frac{1}{7n_5}) \ldots$$

Since, $n_1 > n_2 > n_3 > n_4$, the same rearranging that we did with *A* can be done with *C*.

Then, $C > \frac{1}{3n_3}$

$$D = (\frac{1}{4n_4} - \frac{1}{5n_4}) + \left(\frac{1}{5n_5} - \frac{1}{6n_5}\right) + \left(\frac{1}{6n_5} - \frac{1}{7n_5}\right) + \left(\frac{1}{7n_6} - \frac{1}{8n_6}\right) \ldots$$

Since, $n_1 > n_2 > n_3 > n_4$, the same rearranging that we did with *A* can be done with *D*.

Then, $D > \frac{1}{4n_4}$

and so on. Thus the sum *S,* which we had written in the form $A + B + C + D + \dots$, turns out to be greater than

$$S > \frac{1}{n_1} + \frac{1}{2n_2} + \frac{1}{3n_3} + \frac{1}{4n_4} + \cdots$$

At the start we had defined *S* to be the following finite series,

$$S = \frac{1}{2n_1} + \frac{1}{3n_2} + \frac{1}{4n_3} + \cdots + \frac{1}{kn_{\mathrm{p}}}$$

And we defined that, $\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \frac{1}{n_4} + \cdots > S = \frac{1}{2n_1} + \frac{1}{3n_2} + \frac{1}{4n_3} + \cdots$

However, we just proved that $S > \frac{1}{n_1} + \frac{1}{2n_2} + \frac{1}{3n_3} + \frac{1}{4n_4} + \cdots > S = \frac{1}{2n_1} + \frac{1}{3n_2} + \frac{1}{4n_3} + \cdots + \frac{1}{kn_{\mathrm{p}}}$

However, this is a contradiction, since in the finite realm *S* can't be equal to and greater than $\frac{1}{2n_1} + \frac{1}{3n_2} + \frac{1}{4n_3} + \cdots + \frac{1}{kn_{\mathrm{p}}}$ at the same time. Therefore, *S* must be infinite.

Now we can rewrite the *S*, the primes series of form $n^2 + 1$ as,

$$S > \frac{1}{n_1} + \frac{1}{2n_2} + \frac{1}{3n_3} + \frac{1}{4n_4} + \cdots > \frac{1}{2n_1} + \frac{1}{3n_2} + \frac{1}{4n_3} + \cdots + \frac{1}{kn_{\mathrm{p}}} = S$$

This implies that $S > S$

However, no finite number can satisfy such an equation. Therefore, we have a contradiction and must conclude that $S = \infty$**.** Remember our definition of S from the above series:

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \cdots + \frac{1}{n_\mathrm{p}} > S = \infty$$

$$\text{Therefore, } \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \cdots + \frac{1}{n_\mathrm{p}} > \infty$$

Therefore, we have proven that the reciprocal prime series of form $n^2 + 1$ diverges to infinity. Obviously, this cannot possibly happen if there are only finitely many of form $n^2 + 1$ prime reciprocals of form $n^2 + 1$, therefore the prime reciprocals of form $n^2 + 1$ are infinite in number. Since the prime reciprocals of form $n^2 + 1$ are infinite in number, the prime numbers of form $n^2 + 1$ must be infinite as well.

The author expresses many thanks to the work of Johann Bernoulli in 1689, without his work this proof would not have been possible. It was solely through the study of Johann Bernoulli's work that the author was inspired to see this divergent proof. The author would also like to express many thanks to Shailesh Shirali's work in which he documented Johann Bernoulli's work in the most fascinating and interesting way.

1. On the Infinitude of the Prime Numbers, Shailesh A Shirali, Kishi Valley School (Krishnamurti Foundation of India), Kishi Valley, Anclhra Pradesh, India