On the Ricci Scalar, Ricci Tensor and the Riemannian Curvature Tensor

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Abstract

The article in the first two sections proves decisively that the Ricci scalar and the norm of the Ricci tensor are constants on the manifold. In the subsequent sections Ricci tensor and Riemannian curvature tensor turn out to be null tensors. The Ricci scalar works out to zero.

Introduction

The Ricci scalar[1] and the norm of the Ricci tensor[2] are not only invariants but they are constants on a given manifold, independent of the space time coordinates. This idea is established in the initial stages of the article. Subsequent calculations show that the Ricci tensor and Riemannian curvature tensor are the null tensors. Consequently the Ricci scalar works out to zero.

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Part I

Ricci Scalar

Covariant derivative of a scalar is equivalent to its partial derivative

We prove first

$$\frac{\partial}{\partial x^{\gamma}}\left(B_{\alpha\beta}A^{\alpha\beta}\right)=A_{\alpha\beta}\nabla_{\gamma}B^{\alpha\beta}+B_{\alpha\beta}\nabla_{\gamma}A_{\alpha\beta} \quad (1)$$

Proof:

We consider the following relations

$$\nabla_{\gamma}A^{\alpha\beta}=A^{\alpha\beta}{}_{;\gamma}=\frac{\partial A^{\alpha\beta}}{\partial x^{\gamma}}+\Gamma_{\gamma s}{}^{\alpha}A^{s\beta}+\Gamma_{\gamma s}{}^{\beta}A^{\alpha s}$$

$$\nabla_{\gamma}B_{\alpha\beta}=B_{\alpha\beta;\gamma}=\frac{\partial B_{\alpha\beta}}{\partial x^{\gamma}}+\Gamma^{s}{}_{\gamma\alpha}B_{s\beta}+\Gamma^{s}{}_{\gamma\beta}B_{\alpha s}$$

[The above relations do not assume $A^{\alpha\beta}$ and $B_{\alpha\beta}$ as symmetric tensors]

We obtain,

$$\frac{\partial}{\partial x^\gamma}\left(B_{\alpha\beta}A^{\alpha\beta}\right) = B_{\alpha\beta}\left(\nabla_\gamma A^{\alpha\beta} - \Gamma_{\gamma s}{}^{\alpha} A^{s\beta} - \Gamma_{\gamma s}{}^{\beta} A^{\alpha s}\right) + A^{\alpha\beta}\left(\nabla_\gamma B_{\alpha\beta} + \Gamma^{s}{}_{\gamma\alpha} B_{s\beta} + \Gamma^{s}{}_{\gamma\beta} B_{\alpha s}\right)$$

$$\frac{\partial}{\partial x^\gamma}\left(B_{\alpha\beta}A^{\alpha\beta}\right) = B_{\alpha\beta}\left(-\Gamma_{\gamma s}{}^{\alpha} A^{s\beta} - \Gamma_{\gamma s}{}^{\beta} A^{\alpha s}\right) + A^{\alpha\beta}\left(\Gamma^{s}{}_{\gamma\alpha} B_{s\beta} + \Gamma^{s}{}_{\gamma\beta} B_{\alpha s}\right) + A^{\alpha\beta}\nabla_\gamma B_{\alpha\beta} + B^{\alpha\beta}\nabla_\gamma A_{\alpha\beta}$$

$$= -\Gamma_{\gamma s}{}^{\alpha} g^{s\beta} B_{\alpha\beta} - \Gamma_{\gamma s}{}^{\beta} g^{\alpha s} B_{\alpha\beta} + \Gamma^{s}{}_{\gamma\alpha} A^{\alpha\beta} T_{s\beta} + \Gamma^{s}{}_{\gamma\beta} A^{\alpha\beta} B_{s\alpha} + A^{\alpha\beta}\nabla_\gamma B_{\alpha\beta} + B^{\alpha\beta}\nabla_\gamma A_{\alpha\beta}$$

$$\frac{\partial}{\partial x^\gamma}\left(B_{\alpha\beta}A^{\alpha\beta}\right) = \left(-\Gamma_{\gamma s}{}^{\alpha} A^{s\beta} B_{\alpha\beta} + \Gamma^{s}{}_{\gamma\alpha} A^{\alpha\beta} B_{s\beta}\right) + \left(\Gamma^{s}{}_{\gamma\beta} A^{\alpha\beta} B_{\alpha s} - \Gamma_{\gamma s}{}^{\beta} A^{\alpha s} B_{\alpha\beta}\right) + A^{\alpha\beta}\nabla_\gamma B_{\alpha\beta} + B^{\alpha\beta}\nabla_\gamma A_{\alpha\beta} \quad (2)$$

[In the above α, s, β are dummy indices]

We work out the two parentheses separately.

With the second term in the first parenthesis to the right we interchange as follows

$$\alpha \leftrightarrow s$$

$$\left(-\Gamma_{\gamma s}{}^{\alpha} A^{s\beta} B_{\alpha\beta} + \Gamma^{s}{}_{\gamma\alpha} A^{\alpha\beta} B_{s\beta}\right) = \left(-\Gamma_{\gamma s}{}^{\alpha} A^{s\beta} T_{\alpha\beta} + \Gamma^{\alpha}{}_{\gamma s} A^{s\beta} B_{\alpha\beta}\right) = 0$$

We do not have to worry about reflections on the left side of (5)because alpha and beta on the left side also disappear on contraction.

Indeed recalling (2) and using the relation:$B_{\alpha\beta}A^{\alpha\beta} = B_{\mu\nu}A^{\mu\nu}$we may rewrite it [equation (2)] in the following form :

$$\frac{\partial}{\partial x^\gamma}\left(B_{\mu\nu}A^{\mu\nu}\right) = B_{\alpha\beta}\left(\nabla_\gamma A^{\alpha\beta} - \Gamma_{\gamma s}{}^{\alpha} A^{s\beta} - \Gamma_{\gamma s}{}^{\beta} A^{\alpha s}\right) + A^{\alpha\beta}\left(\nabla_\gamma B_{\alpha\beta} + \Gamma^{s}{}_{\gamma\alpha} B_{s\beta} + \Gamma^{s}{}_{\gamma\beta} B_{\alpha s}\right) + A^{\alpha\beta}\nabla_\gamma B_{\alpha\beta} + B^{\alpha\beta}\nabla_\gamma A_{\alpha\beta}$$

There is no α, β on the left side of the above.

With the second term in the second parenthesis

$$\beta \leftrightarrow s$$

$$\left(\Gamma^{s}{}_{\gamma\beta} A^{\alpha\beta} B_{\alpha s} - \Gamma_{\gamma s}{}^{\beta} A^{\alpha s} B_{\alpha\beta}\right) = \left(\Gamma^{s}{}_{\gamma\beta} A^{\alpha\beta} B_{\alpha s} - \Gamma_{\gamma\beta}{}^{s} A^{\alpha\beta} B_{\alpha s}\right) = 0$$

$$\frac{\partial}{\partial x^\gamma}\left(B_{\alpha\beta}A^{\alpha\beta}\right) = A^{\alpha\beta}\nabla_\gamma B_{\alpha\beta} + B^{\alpha\beta}\nabla_\gamma A_{\alpha\beta}$$

Field Equations[3][4]

$$R^{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta} = \frac{8\pi G}{c^4} T^{\alpha\beta} \quad (3)$$

$$g_{\alpha\beta}\left(R^{\alpha\beta}-\frac{1}{2}Rg^{\alpha\beta}\right)=\frac{8\pi G}{c^4}g_{\alpha\beta}T^{\alpha\beta}$$

$$\left(Rg_{\alpha\beta}{}^{\alpha\beta}-\frac{1}{2}Rg_{\alpha\beta}g^{\alpha\beta}\right)=\frac{8\pi G}{c^4}g_{\alpha\beta}T^{\alpha\beta}$$

$$R-2R=\frac{8\pi G}{c^4}g_{\alpha\beta}T^{\alpha\beta}$$

$$g_{\alpha\beta}T^{\alpha\beta}=-\frac{c^4}{8\pi G}R\ (4)$$

$$\nabla_i\left(g_{\alpha\beta}T^{\alpha\beta}\right)=-\frac{c^4}{8\pi G}\frac{\partial R}{\partial x^i}$$

$$g_{\alpha\beta}\nabla_i T^{\alpha\beta}+T^{\alpha\beta}\nabla_i g_{\alpha\beta}=-\frac{c^4}{8\pi G}\frac{\partial R}{\partial x^i}$$

$$g_{\alpha\beta}\nabla_i T^{\alpha\beta}==-\frac{c^4}{8\pi G}\nabla_i R=-\frac{c^4}{8\pi G}\frac{\partial R}{\partial x^i}\ (5)$$

$$\nabla_j\left(g_{\alpha\beta}\nabla_i T^{\alpha\beta}\right)=-\frac{c^4}{8\pi G}\frac{\partial^2 R}{\partial x^j\partial x^i}$$

$$g_{\alpha\beta}\nabla_j\nabla_i T^{\alpha\beta}+\left(\nabla_j g_{\alpha\beta}\right)\left(\nabla_i T^{\alpha\beta}\right)=-\frac{c^4}{8\pi G}\nabla_j\nabla_i R$$

$$g_{\alpha\beta}\nabla_j\nabla_i T^{\alpha\beta}=-\frac{c^4}{8\pi G}\nabla_j\nabla_i R$$

By similar technique we have

$$g_{\alpha\beta}\nabla_i\nabla_j T^{\alpha\beta}=-\frac{c^4}{8\pi G}\nabla_i\nabla_j R$$

For torsion free fields[typical of General Relativity]

$$\nabla_j\nabla_i R=\nabla_i\nabla_j R$$

[$\nabla_j\nabla_i = \nabla_i\nabla_j$ true of scalars in a torsion free field only: In General Relativity fields are typically torsion free]

$$g_{\alpha\beta}\nabla_j\nabla_i T^{\alpha\beta}=g_{\alpha\beta}\nabla_i\nabla_j T^{\alpha\beta}\ (6)$$

But we know that the covariant derivative operator does not commute for tensors of rank one or higher:

$$\nabla_j \nabla_i T^{\alpha\beta} \neq \nabla_i \nabla_j T^{\alpha\beta}$$

Therefore (6) is an equation and not an identity that holds for every i, j pair. Thus we have six(4C2) such equations while the number of space time coordinates is four.[$g_{\alpha\beta}$ and $T^{\alpha\beta}$ are functions of space and time]

Disregarding dependence on coordinates, though hypothetically in order to make our arguments stronger, there are ten independent components of the stress energy tensor considering its symmetric nature. If the metric coefficients are considered as known quantities we may express six components of $T^{\alpha\beta}$ in terms of the other four $T^{\alpha\beta}$ components and the metric components. The ten supposedly independent components of the stress energy tensor are not really independent

Moreover $\nabla_i T^{\alpha\beta} = 0$falls in line with $\nabla_\beta T^{\alpha\beta} = 0$.We may consider $\nabla_\beta T^{\alpha\beta} = 0$ as the seventh equation. Moreover $\nabla_i T^{\alpha\beta} = 0$ guarantees the independence of the ten stress energy tensor components: it eventually guarantees the independence of the ten Field Equations of Einstein.

Therefore it seems most plausible that

$$\nabla_i T^{\alpha\beta} = 0; i = 0,1,2,3 \ (7)$$

Therefore from(5)

$$\nabla_i R = \frac{\partial R}{\partial x^i} = 0; i = 0,1,2,3 \ (8)$$

Therefore R is independent of space time coordinates. But implicit dependence of R on space and time coordinates is there[example: Einstein Hilbert Action formulation[5]]

The Ricci Tensor

We consider the Field equations

$$R^{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta} = \frac{8\pi G}{c^4} T^{\alpha\beta}$$

$$\nabla_i \left(R^{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta} \right) = \frac{8\pi G}{c^4} \nabla_i T^{\alpha\beta}$$

Applying (6) we have,

$$\nabla_i \left(R^{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta} \right) = 0$$

$$\nabla_i R^{\alpha\beta} + \frac{1}{2} R \nabla_i g^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} \nabla_i R = 0$$

Applying (7) and the fact $\nabla_i g^{\alpha\beta} = 0$ we obtain

$$\nabla_i R^{\alpha\beta} = 0 \quad (9)$$

$$\nabla_i \left(R_{\alpha\beta} R^{\alpha\beta} \right) = R_{\alpha\beta} \nabla_i R^{\alpha\beta} + R^{\alpha\beta} \nabla_i R_{\alpha\beta}$$

Applying (9) we have

$$\nabla_i \left(R_{\alpha\beta} R^{\alpha\beta} \right) = 0; i = 0,1,2,3$$

$$\frac{\partial \left(R_{\alpha\beta} R^{\alpha\beta} \right)}{\partial x^i} = 0$$

Therefore the norm of the Ricci Tensor is independent of space time coordinates

Dot Product Preserving Transport

In parallel transport[6] the two vectors the transported parallel to themselves.In dot product preserving transport product the dot is preserved but the two vectors individually are not transported parallel to themselves.

We have due to the preservation of dot product,

$$t^i \nabla_i \left(g_{\alpha\beta} u^\alpha v^\beta \right) = 0 \ (10)$$

Since each vector is not transported parallel to itself we have

$$t^i \nabla_i u^\alpha \neq 0; t^i \nabla_i v^\beta \neq 0 \quad (11)$$

.

We transform to a frame of reference where t^i has only one non zero component.

$t^{k\prime} \nabla_{k\prime} \left(g_{\alpha\beta}{}' u^{\alpha\prime} v^{\beta\prime} \right) = 0$[no summation on k’: prime denotes the new frame of reference and not differentiation]

$$\nabla_k' \left(g_{\alpha\beta}{}' u^{\alpha\prime} v^{\beta\prime} \right) = 0 (12)$$

$$u^{\alpha\prime} v^{\beta\prime} \nabla_{i\prime} \left(g_{\alpha\beta}{}' \right) + g_{\alpha\beta}{}' \nabla_{i\prime} \left(u^{\alpha\prime} v^{\beta\prime} \right) = 0$$

Since $\nabla_i \left(g_{\alpha\beta} \right) = 0$,we have,

$$g_{\alpha\beta}{}'\nabla_i\left(u^{\alpha\prime}v^{\beta\prime}\right)=0 \quad (13)$$

The vectors $u^{\alpha\prime}$and $v^{\beta\prime}$ and consequently their individual components are *arbitrary*. Therefore

$$g'_{\alpha\beta}=0 \Rightarrow g_{\alpha\beta}=0 \quad (14)$$

[the null tensor remains null in all frames of reference]That implies that the Riemann tensor , Ricci tensor and the Ricci scalar are all zero valued objects.

Part II

$$R=-\frac{8\pi G}{c^4}g_{\alpha\beta}T^{\alpha\beta}$$

$$g_{\alpha\beta}T^{\alpha\beta}=-\frac{c^4}{8\pi G}R$$

Trial solution

$$T^{\alpha\beta}=-\frac{c^4}{8\pi G}\frac{R}{4}g^{\alpha\beta}+X^{\alpha\beta};\, g_{\alpha\beta}X^{\alpha\beta}=0 \ (15)$$

$$\nabla_\beta T^{\alpha\beta}=0$$

$$\nabla_\beta\left(-\frac{c^4}{8\pi G}\frac{R}{4}g^{\alpha\beta}+X^{\alpha\beta}\right)=0$$

Since R is constant we have,

$$\nabla_\beta X^{\alpha\beta}=0$$

$$X^{\alpha\beta}=kg^{\alpha\beta}+l\left(R^{\alpha\beta}-\frac{1}{2}Rg^{\alpha\beta}\right)+Y^{\alpha\beta};\nabla_\beta Y^{\alpha\beta}=0 \ (16)$$

[k and l are constants that have different dimensions]

Our trial solution for $T^{\alpha\beta}$ now,

$$T^{\alpha\beta}=-\frac{c^4}{8\pi G}\frac{R}{4}g^{\alpha\beta}+kg^{\alpha\beta}+l\left(R^{\alpha\beta}-\frac{1}{2}Rg^{\alpha\beta}\right)+Y^{\alpha\beta};\nabla_\beta Y^{\alpha\beta}=0 \quad (17)$$

$$g_{\alpha\beta}T^{\alpha\beta}=-\frac{c^4}{8\pi G}\frac{R}{4}g_{\alpha\beta}g^{\alpha\beta}+kg_{\alpha\beta}g^{\alpha\beta}+l\left(g_{\alpha\beta}R^{\alpha\beta}-\frac{1}{2}Rg_{\alpha\beta}g^{\alpha\beta}\right)+g_{\alpha\beta}Y^{\alpha\beta}$$

$$-\frac{c^4}{8\pi G}R=-\frac{c^4}{8\pi G}R+4k+l(R-2R)+g_{\alpha\beta}Y^{\alpha\beta}$$

$$0=4k-lR+g_{\alpha\beta}Y^{\alpha\beta}$$

$$g_{\alpha\beta}Y^{\alpha\beta} = lR - 4k \quad (18)$$

Again since $\nabla_\beta g^{\alpha\beta} = 0$ and $\nabla_\beta Y^{\alpha\beta} = 0$

$$g_{\alpha\beta}\nabla_\beta Y^{\alpha\beta} + Y^{\alpha\beta}\nabla_\beta g^{\alpha\beta} = 0 \Rightarrow \nabla_\beta\left(g_{\alpha\beta}Y^{\alpha\beta}\right) = 0 \Rightarrow \nabla_\beta\left(g_{\mu\nu}Y^{\mu\nu}\right) = 0 \Rightarrow \partial_\beta\left(g_{\mu\nu}Y^{\mu\nu}\right) = 0$$

$g_{\mu\nu}Y^{\mu\nu}$is independent of space time coordinates

From field equations $\frac{c^4}{8\pi G}\left(R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta}\right) = T^{\alpha\beta}$

Using the above with (50) we obtain,

$$\frac{c^4}{8\pi G}\left(R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta}\right) = -\frac{c^4}{8\pi G}\frac{R}{4}g^{\alpha\beta} + kg^{\alpha\beta} + l\left(R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta}\right) + Y^{\alpha\beta}$$

$$\frac{c^4}{8\pi G}\left(R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta}\right) = -\frac{c^4}{8\pi G}\frac{R}{4}g^{\alpha\beta} + \frac{1}{4}\left(lR - g_{\alpha\beta}Y^{\alpha\beta}\right)g^{\alpha\beta} + lR^{\alpha\beta} - \frac{1}{2}lRg^{\alpha\beta} + Y^{\alpha\beta}$$

$$\frac{c^4}{8\pi G}\left(R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta}\right) = -\frac{c^4}{8\pi G}\frac{R}{4}g^{\alpha\beta} + \frac{lR}{4}g^{\alpha\beta} - \frac{1}{4}g_{\alpha\beta}g^{\alpha\beta}Y^{\alpha\beta} + lR^{\alpha\beta} - \frac{1}{2}lRg^{\alpha\beta} + Y^{\alpha\beta}$$

$$\frac{c^4}{8\pi G}\left(R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta}\right) = -\frac{c^4}{8\pi G}\frac{R}{4}g^{\alpha\beta} + \frac{lR}{4}g^{\alpha\beta} + lR^{\alpha\beta} - \frac{1}{2}lRg^{\alpha\beta}$$

$$\frac{c^4}{8\pi G}\left(R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta}\right) = -\frac{c^4}{8\pi G}\frac{R}{4}g^{\alpha\beta} - \frac{lR}{4}g^{\alpha\beta} + lR^{\alpha\beta}$$

$$\frac{c^4}{8\pi G}R^{\alpha\beta} = \frac{c^4}{8\pi G}\frac{R}{4}g^{\alpha\beta} - \frac{lR}{4}g^{\alpha\beta} + lR^{\alpha\beta}$$

$$\left(\frac{c^4}{8\pi G} - l\right)R^{\alpha\beta} = \frac{R}{4}g^{\alpha\beta}\left(\frac{c^4}{8\pi G} - l\right)$$

$$R^{\alpha\beta} = \frac{R}{4}g^{\alpha\beta} \quad (19)$$

Equation (52) holds unless $\frac{c^4}{8\pi G} - l = 0$. Therefore an alternative technique of deriving (52) has been provided.

Recalling (50)

$$T^{\alpha\beta} = -\frac{c^4}{8\pi G}\frac{R}{4}g^{\alpha\beta} + kg^{\alpha\beta} + l\left(R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta}\right) + Y^{\alpha\beta}$$

$$T^{\alpha\beta} = -\frac{c^4}{8\pi G}\frac{R}{4}g^{\alpha\beta} + kg^{\alpha\beta} + l\left(\frac{R}{4}g^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta}\right) + Y^{\alpha\beta}$$

$$T^{\alpha\beta} = -\frac{c^4}{8\pi G}\frac{R}{4}g^{\alpha\beta} + kg^{\alpha\beta} - l\frac{R}{4}g^{\alpha\beta} + Y^{\alpha\beta}$$

$$T^{\alpha\beta} = -\frac{c^4}{8\pi G}\frac{R}{4}g^{\alpha\beta} + \left(\frac{4k - lR}{4}\right)g^{\alpha\beta} + Y^{\alpha\beta}$$

$$T^{\alpha\beta} = -\frac{c^4}{8\pi G}\frac{R}{4}g^{\alpha\beta} - \left(\frac{g_{\alpha\beta}g^{\alpha\beta}Y^{\alpha\beta}}{4}\right) + Y^{\alpha\beta}$$

$$T^{\alpha\beta} = -\frac{c^4}{8\pi G}\frac{R}{4}g^{\alpha\beta} \ (20)$$

In this section, so far, we have not used the fact that R is a constant

Alternative Technique for deriving equation (19):

We have,

$$g_{\alpha\beta}R^{\alpha\beta} = R$$

$$R^{\alpha\beta} = \frac{R}{4}g^{\alpha\beta} + \xi^{\alpha\beta}; g_{\alpha\beta}\xi^{\alpha\beta} = 0 \ (21)$$

Using the above expression for $R^{\alpha\beta}$ and the fact that R is constant in the field equations we have

$$\nabla_\beta \xi^{\alpha\beta} = 0$$

$$\xi^{\alpha\beta} = kg^{\mu\nu} + l\left(R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta}\right) + Y^{\alpha\beta}; \nabla_\beta Y^{\alpha\beta} = 0 \ (22)$$

Therefore,

$$R^{\alpha\beta} = \frac{R}{4}g^{\alpha\beta} + kg^{\alpha\beta} + l\left(R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta}\right) + Y^{\alpha\beta}; \nabla_\beta Y^{\alpha\beta} = 0 \ (23)$$

$$g_{\alpha\beta}R^{\alpha\beta} = \frac{R}{4}g_{\alpha\beta}g^{\alpha\beta} + kg_{\alpha\beta}g^{\alpha\beta} + l\left(g_{\alpha\beta}R^{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}g^{\alpha\beta}\right) + g_{\alpha\beta}Y^{\alpha\beta};$$

$$g_{\alpha\beta}R^{\alpha\beta} = \frac{R}{4}g_{\alpha\beta}g^{\alpha\beta} + kg_{\alpha\beta}g^{\alpha\beta} + l(R - 2R) + g_{\alpha\beta}Y^{\alpha\beta}$$

$$R = R + 4k - lR + g_{\alpha\beta}Y^{\alpha\beta}$$

$$g_{\alpha\beta}Y^{\alpha\beta} = IR - 4k \ (24)$$

Equation(23) implies

$$R^{\alpha\beta}(1 - l) = g^{\alpha\beta}\left(\frac{R}{4} + k - \frac{1}{2}Rl\right) + Y^{\alpha\beta}$$

$$\Rightarrow R^{\alpha\beta} = \frac{1}{1-l} g^{\alpha\beta} \left(\frac{R}{4} + k - \frac{1}{2} Rl \right) + \frac{1}{1-l} Y^{\alpha\beta}$$

$$\Rightarrow R^{\alpha\beta} = \frac{1}{1-l} g^{\alpha\beta} \left(\frac{R + 4k - 2Rl}{4} \right) + \frac{1}{1-l} Y^{\alpha\beta}$$

Using (24) on the above relation

$$R^{\alpha\beta} = \frac{1}{1-l} g^{\alpha\beta} \left(\frac{R + Rl - g_{\alpha\beta} Y^{\alpha\beta} - 2Rl}{4} \right) + \frac{1}{1-l} Y^{\alpha\beta}$$

$$\Rightarrow R^{\alpha\beta} = \frac{1}{1-l} g^{\alpha\beta} \left(\frac{R - Rl - g_{\alpha\beta} Y^{\alpha\beta}}{4} \right) + \frac{1}{1-l} Y^{\alpha\beta}$$

$$\Rightarrow R^{\alpha\beta} = \frac{1}{1-l} g^{\alpha\beta} \left(\frac{R - Rl}{4} \right) - \frac{g^{\alpha\beta} g_{\alpha\beta} Y^{\alpha\beta}}{4(1-l)} + \frac{1}{1-l} Y^{\alpha\beta}$$

$$\Rightarrow R^{\alpha\beta} = \frac{1}{1-l} g^{\alpha\beta} \left(\frac{R - Rl}{4} \right) - \frac{Y^{\alpha\beta}}{(1-l)} + \frac{1}{1-l} Y^{\alpha\beta}$$

Equation (19) gets repeated

$$\Rightarrow R^{\alpha\beta} = \frac{R}{4} g^{\alpha\beta}$$

Considering R as constant,

The Enigma

From the field equations we have

$$T^{\alpha\beta} = \frac{c^4}{8\pi G} \left(R^{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta} \right)$$

But we have obtained

$$T^{\alpha\beta} = -\frac{c^4}{8\pi G} \frac{R}{4} g^{\alpha\beta}$$

What is the inner story? The relation $R = g_{\alpha\beta} R^{\alpha\beta}$ is independent of the field equations. It is used with the field equations to deduce $g_{\alpha\beta} T^{\alpha\beta} = -\frac{c^4}{8\pi G} R$. Again this relation is used to derive

$$T^{\alpha\beta} = -\frac{c^4}{8\pi G} \frac{R}{4} g^{\alpha\beta}$$

which is obviously more specific than the stand alone content of the field equations.

Again $R^{\alpha\beta} = \frac{R}{4} g^{\alpha\beta}$is derived from the parent relation $R = g_{\alpha\beta} R^{\alpha\beta}$. Nevertheless in the intermediate steps. we have used the field equations to derive $\nabla_\beta \xi^{\alpha\beta} = 0$.Thus $R = g_{\alpha\beta} R^{\alpha\beta}$is the outcome of both the field equations and the formula $R^{\alpha\beta} = \frac{R}{4} g^{\alpha\beta}$.It is richer in content that if it were derived from the field equations alone.

The Line Element and the Symmetric Nature of the Metric Coefficients

So long as we are on the same manifold, the line element is preserved. This is not true for distinct manifolds

Example: A room with a flat floor and a hemispherical roof is considered. A small arc is drawn on the roof and its projection is taken on the floor. With this transformation

$$ds'^2 \neq ds^2$$

Only if

$$ds'^2 = ds^2$$

then $g_{\mu\nu}$behaves as a rank two tensor. Indeed

$$ds'^2 = ds^2$$

$$\Rightarrow \bar{g}_{\mu\nu} d\bar{x}^\mu d\bar{x}^\nu = g_{\alpha\beta} dx^\alpha dx^\beta$$

$$= g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \bar{x}^\mu} d\bar{x}^\mu \frac{\partial x^\beta}{\partial \bar{x}^\nu} d\bar{x}^\nu$$

$$= g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} d\bar{x}^\mu d\bar{x}^\nu$$

$$\Rightarrow \bar{g}_{\mu\nu} = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu}$$

We revisit the idea[7] that the metric tensor is a symmetric tensor. Indeed

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

By interchanging the dummy indices μ and ν we have,

$$ds^2 = g_{\nu\mu} dx^\nu dx^\mu$$

$$\Rightarrow ds^2 = \frac{1}{2}(g_{\mu\nu} + g_{\nu\mu})dx^\mu dx^\nu$$

$g_{\mu\nu} + g_{\nu\mu}$ is asymmetric quantity. The relation

$$g_{\mu\nu}dx^\mu dx^\nu = \frac{1}{2}(g_{\mu\nu} + g_{\nu\mu})dx^\mu dx^\nu$$

is true for arbitrary $dx^\mu$and dx^ν.Therefore

$$g_{\mu\nu} = \frac{1}{2}(g_{\mu\nu} + g_{\nu\mu})$$

$$\Rightarrow g_{\mu\nu} = g_{\nu\mu}$$

For an *arbitrary* non singular transformation a symmetric tensor has to be produced. That is impossible unless the metric tensor is the null tensor. This corroborates our inference from the "Dot Product Preserving Transport.

Part III

[Alternative Treatment: Brute Force Calculations]

From the Field equations

$$\left(R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R\right)\left(R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}\right) = \left(\frac{8\pi G}{c^4}\right)^2 T^{\alpha\beta}T_{\alpha\beta}$$

$$R_{\alpha\beta}R^{\alpha\beta} - 2\frac{1}{2}Rg_{\alpha\beta}R^{\alpha\beta} + \frac{1}{4}R^2 g^{\alpha\beta}g_{\alpha\beta} = \left(\frac{8\pi G}{c^4}\right)^2 T^{\alpha\beta}T_{\alpha\beta}$$

$$R_{\alpha\beta}R^{\alpha\beta} - R^2 + R^2 = \left(\frac{8\pi G}{c^4}\right)^2 T^{\alpha\beta}T_{\alpha\beta}$$

$$R_{\alpha\beta}R^{\alpha\beta} = \left(\frac{8\pi G}{c^4}\right)^2 T^{\alpha\beta}T_{\alpha\beta} \quad (25)$$

From relation

$$R_{\alpha\beta}R^{\alpha\beta} = \text{Constant} \quad (26)$$

$$\Rightarrow T^{\alpha\beta}T_{\alpha\beta} = \text{constant} \quad (27)$$

From (27)we have,

$$T^{\alpha\beta}\nabla_i T_{\alpha\beta} + T_{\alpha\beta}\nabla_i T^{\alpha\beta} = 0$$

$$T_{\alpha\beta}\nabla_i T^{\alpha\beta} = 0; i = 0,1,2,3 \ 28)$$

Indeed

$$T^{\alpha\beta}\nabla_i T_{\alpha\beta} = g^{\alpha\mu}g^{\beta\nu}T_{\mu\nu}\nabla_i\left(g_{\alpha k}g_{\beta l}T^{lk}\right)$$

$$= g^{\alpha\mu}g^{\beta\nu}T_{\mu\nu}\left[g_{\alpha k}g_{\beta l}\nabla_i\left(T^{lk}\right) + T^{lk}\nabla_i\left(g_{\alpha k}g_{\beta l}\right)\right]$$

$$= g^{\alpha\mu}g^{\beta\nu}g_{\alpha k}g_{\beta l}T_{\mu\nu}\nabla_i\left(T^{lk}\right)$$

$$= \delta_k{}^{\mu}\delta_l{}^{\nu}\mathrm{T}_{\mu\nu}\nabla_i\left(T^{lk}\right)$$

$$= \mathrm{T}_{\mu\nu}\nabla_i T^{\mu\nu} = \mathrm{T}_{\alpha\beta}\nabla_i T^{\alpha\beta}$$

Therefore

$$T_{\alpha\beta}\nabla_i T^{\alpha\beta} = 0 \ (29)$$

From (8)

$$R_{\alpha\beta}R^{\alpha\beta} = constant$$

We have,

$$R_{\alpha\beta}\nabla_i R^{\alpha\beta} + R^{\alpha\beta}\nabla_i R_{\alpha\beta} = 0 \ (30)$$

Now

$$R^{\alpha\beta}\nabla_i R_{\alpha\beta} = g^{\alpha\mu}g^{\beta\nu}R_{\mu\nu}\nabla_i\left(g_{\alpha k}g_{\beta l}R^{lk}\right)$$

$$= g^{\alpha\mu}g^{\beta\nu}R_{\mu\nu}\left[g_{\alpha k}g_{\beta l}\nabla_i\left(R^{lk}\right) + R^{lk}\nabla_i\left(g_{\alpha k}g_{\beta l}\right)\right]$$

$$= g^{\alpha\mu}g^{\beta\nu}g_{\alpha k}g_{\beta l}R_{\mu\nu}\nabla_i\left(R^{lk}\right)$$

$$= \delta_k{}^{\mu}\delta_l{}^{\nu}\mathrm{R}_{\mu\nu}\nabla_i\left(R^{lk}\right)$$

$$= \mathrm{R}_{\mu\nu}\nabla_i R^{\mu\nu} = \mathrm{R}_{\alpha\beta}\nabla_i R^{\alpha\beta}$$

Therefore

$$R_{\alpha\beta}\nabla_i R^{\alpha\beta} = 0 \qquad (31)$$

From (16)

$$T_{\alpha\beta}g^{\alpha\beta} = C$$

$$\Rightarrow g_{\mu\nu}\frac{\partial T^{\mu\nu}}{\partial x^i} + \frac{\partial g_{\mu\nu}}{\partial x^i}T^{\mu\nu} = 0 \quad (32)$$

From(30)

$$T_{\alpha\beta}\nabla_i T^{\alpha\beta} = 0$$

$$\Rightarrow T_{\alpha\beta}\frac{\partial T^{\alpha\beta}}{\partial x^i} + T_{\alpha\beta}g^{\alpha s}\frac{1}{2}\left(\frac{\partial g_{si}}{\partial x^j} + \frac{\partial g_{sj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^s}\right)T^{j\beta} + T_{\alpha\beta}g^{\beta s}\frac{1}{2}\left(\frac{\partial g_{si}}{\partial x^j} + \frac{\partial g_{sj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^s}\right)T^{\alpha j} = 0$$

$$T_{\alpha\beta}\frac{\partial T^{\alpha\beta}}{\partial x^i} + T_{\alpha\beta}T^{j\beta}g^{\alpha s}\frac{1}{2}\left(\frac{\partial g_{si}}{\partial x^j} + \frac{\partial g_{sj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^s}\right) + T_{\alpha\beta}T^{\alpha j}g^{\beta s}\frac{1}{2}\left(\frac{\partial g_{si}}{\partial x^j} + \frac{\partial g_{sj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^s}\right) = 0$$

$$T_{\alpha\beta}\frac{\partial T^{\alpha\beta}}{\partial x^i} + T^s{}_\beta T^{j\beta}\frac{1}{2}\left(\frac{\partial g_{si}}{\partial x^j} + \frac{\partial g_{sj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^s}\right) + T^s{}_\beta T^{j\beta}\frac{1}{2}\left(\frac{\partial g_{si}}{\partial x^j} + \frac{\partial g_{sj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^s}\right) = 0$$

$$T_{\alpha\beta}\frac{\partial T^{\alpha\beta}}{\partial x^i} + T^s{}_\beta T^{j\beta}\left(\frac{\partial g_{si}}{\partial x^j} + \frac{\partial g_{sj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^s}\right) = 0 \ (33)$$

Now considering the fact that dummy indices can always be interchanged without affecting the value of an expression we have,

$$T^j{}_\beta T^{s\beta}\left(\frac{\partial g_{ji}}{\partial x^s} + \frac{\partial g_{js}}{\partial x^i} - \frac{\partial g_{is}}{\partial x^j}\right) = T^s{}_\beta T^{j\beta}\left(\frac{\partial g_{si}}{\partial x^j} + \frac{\partial g_{js}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^s}\right) \ (34)$$

We have

$$0 = 2T_{\alpha\beta}\nabla_i T^{\alpha\beta} = 2T_{\alpha\beta}\frac{\partial T^{\alpha\beta}}{\partial x^i} + 2T^s{}_\beta T^{j\beta}\left(\frac{\partial g_{si}}{\partial x^j} + \frac{\partial g_{sj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^s}\right)$$

Applying (34) on the above,

$$2T_{\alpha\beta}\nabla_i T^{\alpha\beta} = 2T_{\alpha\beta}\frac{\partial T^{\alpha\beta}}{\partial x^i} + T^s{}_\beta T^{j\beta}\left(\frac{\partial g_{si}}{\partial x^j} + \frac{\partial g_{js}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^s}\right) + T^j{}_\beta T^{s\beta}\left(\frac{\partial g_{ji}}{\partial x^s} + \frac{\partial g_{js}}{\partial x^i} - \frac{\partial g_{is}}{\partial x^j}\right) \ (35)$$

$$2T_{\alpha\beta}\nabla_i T^{\alpha\beta} = 2T_{\alpha\beta}\frac{\partial T^{\alpha\beta}}{\partial x^i} + \left(T^j{}_\beta T^{s\beta} - T^s{}_\beta T^{j\beta}\right)\frac{\partial g_{ji}}{\partial x^s} + \left(T^j{}_\beta T^{s\beta} + T^s{}_\beta T^{j\beta}\right)\frac{\partial g_{js}}{\partial x^i} + \left(T^s{}_\beta T^{j\beta} - T^j{}_\beta T^{s\beta}\right)\frac{\partial g_{is}}{\partial x^j} \quad (36)$$

Now,

$$T^j{}_\beta T^{s\beta} = g_{\beta k}T^{jk}T^{s\beta}$$

$$T^s{}_\beta T^{j\beta} = g_{\beta k}T^{sk}T^{j\beta} = g_{k\beta}T^{s\beta}T^{jk} = T^j{}_\beta T^{s\beta}$$

Therefore

$$T_{\beta}{}^{j}T^{s\beta} - T_{\beta}{}^{s}T^{j\beta} = 0 \quad (37)$$

An alternative technique for deriving (37) would be as follows. By the interchange of dummy indices j w and s we may assert that

$$\left(T^{j}{}_{\beta}T^{s\beta} - T^{s}{}_{\beta}T^{j\beta}\right)\frac{\partial g_{ji}}{\partial x^{s}} = \left(T^{j}{}_{\beta}T^{s\beta} + T^{s}{}_{\beta}T^{j\beta}\right)\frac{\partial g_{js}}{\partial x^{i}}$$

$$\left(T^{j}{}_{\beta}T^{s\beta} - T^{s}{}_{\beta}T^{j\beta}\right)\left(\frac{\partial g_{ji}}{\partial x^{s}} + \frac{\partial g_{js}}{\partial x^{i}}\right) = 0$$

$\left(T^{j}{}_{\beta}T^{s\beta} - T^{s}{}_{\beta}T^{j\beta}\right)$is a tensor while $\left(\frac{\partial g_{ji}}{\partial x^{s}} + \frac{\partial g_{js}}{\partial x^{i}}\right)$ is not a tensor and their product is not a tensor. The product may not be zero in all frames of reference unless we have (37): $T^{j}{}_{\beta}T^{s\beta} - T^{s}{}_{\beta}T^{j\beta}$. By quotient law[8], $\left(\frac{\partial g_{ji}}{\partial x^{s}} + \frac{\partial g_{js}}{\partial x^{i}}\right)$ should be a tensor unless $\left(T^{j}{}_{\beta}T^{s\beta} - T^{s}{}_{\beta}T^{j\beta}\right) = 0$The only solution is $T_{\alpha\beta} - kg_{\alpha\beta} = 0$..If $\left(\frac{\partial g_{ji}}{\partial x^{s}} + \frac{\partial g_{js}}{\partial x^{i}}\right) = 0$ in all reference frames it becomes the null tensor. We are considering it to be not so from its transformation.

The alternative technique has been discussed since it will be used in the final stages to bring out important results.

From (36) and (37) we have

$$2T_{\alpha\beta}\nabla_{i}T^{\alpha\beta} = 2T_{\alpha\beta}\frac{\partial T^{\alpha\beta}}{\partial x^{i}} + \left(T_{\beta}{}^{j}T^{s\beta} + T_{\beta}{}^{s}T^{j\beta}\right)\frac{\partial g_{js}}{\partial x^{i}} = 2T_{\alpha\beta}\frac{\partial T^{\alpha\beta}}{\partial x^{i}} + 2T_{\beta}{}^{j}T^{s\beta}\frac{\partial g_{js}}{\partial x^{i}}$$

$$2T_{\alpha\beta}\nabla_{i}T^{\alpha\beta} = 2T_{\alpha\beta}\frac{\partial T^{\alpha\beta}}{\partial x^{i}} + 2T_{\beta}{}^{j}T^{s\beta}\frac{\partial g_{js}}{\partial x^{i}} \quad (38)$$

Using $T_{\alpha\beta}\nabla_{i}T^{\alpha\beta} = 0$ and (38)we have

$$T_{\alpha\beta}\frac{\partial T^{\alpha\beta}}{\partial x^{i}} + T_{\beta}{}^{j}T^{s\beta}\frac{\partial g_{js}}{\partial x^{i}} = 0$$

$$\Rightarrow T_{\alpha\beta}\frac{\partial T^{\alpha\beta}}{\partial x^{i}} + T_{k}{}^{l}T^{mk}\frac{\partial g_{lm}}{\partial x^{i}} = 0 \quad (39)$$

From (32)

$$g_{\alpha\beta}\frac{\partial T^{\alpha\beta}}{\partial x^{i}} + \frac{\partial g_{lm}}{\partial x^{i}}T^{lm} = 0$$

$$g_{\alpha\beta}\frac{\partial T^{\alpha\beta}}{\partial x^i}+\frac{\partial g_{lm}}{\partial x^i}g_k{}^l T^{mk}=0 \;\; (40)$$

From(38) and(39)

$$\frac{\partial T^{\alpha\beta}}{\partial x^i}\left(T_{\alpha\beta}-kg_{\alpha\beta}\right)+\frac{\partial g_{lm}}{\partial x^i}T^{mk}\left(T_k{}^l-kg_k{}^l\right)=0$$

$$\frac{\partial T^{\alpha\beta}}{\partial x^i}\left(T_{\alpha\beta}-kg_{\alpha\beta}\right)+\frac{\partial g_{lm}}{\partial x^i}T^{mk}g^{lp}\left(T_{kp}-kg_{kp}\right)=0$$

$$\frac{\partial T^{\alpha\beta}}{\partial x^i}\left(T_{\alpha\beta}-kg_{\alpha\beta}\right)+\frac{\partial g_{lm}}{\partial x^i}T^{m\alpha}g^{l\beta}\left(T_{\alpha\beta}-kg_{\alpha\beta}\right)=0$$

$$\left(T_{\alpha\beta}-kg_{\alpha\beta}\right)\left(\frac{\partial T^{\alpha\beta}}{\partial x^i}+\frac{\partial g_{lm}}{\partial x^i}T^{m\alpha}g^{l\beta}\right)=0 \;\; (41)$$

.Now the second factor on the left of (36) may be written as:

$$\frac{\partial T^{\alpha\beta}}{\partial x^i}+\frac{\partial g_{lm}}{\partial x^i}T^{m\alpha}g^{l\beta}=\frac{\partial T^{\alpha\beta}}{\partial x^i}+\frac{\partial (T^{m\alpha}g_{lm})}{\partial x^i}g^{l\beta}-g_{lm}g^{l\beta}\frac{\partial T^{m\alpha}}{\partial x^i}$$

$$=\frac{\partial T^{\alpha\beta}}{\partial x^i}+\frac{\partial (T^{m\alpha}g_{lm})}{\partial x^i}g^{l\beta}-\delta_m{}^\beta\frac{\partial T^{m\alpha}}{\partial x^i}$$

$$=\frac{\partial T^{\alpha\beta}}{\partial x^i}+\frac{\partial (T^{m\alpha}g_{lm})}{\partial x^i}g^{l\beta}-\frac{\partial T^{\alpha\beta}}{\partial x^i}$$

$$=\frac{\partial T_l{}^a}{\partial x^i}g^{l\beta}$$

Therefore

$$\frac{\partial T^{\alpha\beta}}{\partial x^i}+\frac{\partial g_{lm}}{\partial x^i}T^{m\alpha}g^{l\beta}=\frac{\partial T_l{}^a}{\partial x^i}g^{l\beta}$$

. The above is not a tensor [derivative of a tenor is not a tensor in the curved space time context]

$$\left(T_{\alpha\beta}-kg_{\alpha\beta}\right)\frac{\partial T_l{}^a}{\partial x^i}g^{l\beta}=0$$

$\left(T_{\alpha\beta}-kg_{\alpha\beta}\right)$ is a rank two covariant tensor.

$$\left(T_{\alpha\beta}-kg_{\alpha\beta}\right)\frac{\partial T_l{}^a}{\partial x^i}g^{l\beta}=0 \; (42)$$

$$\left(T_\alpha{}^l-k\delta_\alpha{}^l\right)\frac{\partial T_l{}^a}{\partial x^i}=0 \;\; (43)$$

$\left(T_{\alpha}{}^{l}-k\delta_{\alpha}{}^{l}\right)$is a rank two mixed tensor.By quotient law $\frac{\partial T_{l}{}^{\alpha}}{\partial x^{i}}$ should be a tensor unless$T_{\alpha}{}^{l}-k\delta_{\alpha}{}^{l}=0$.

Therefore the left side of (17) is not a tensor. It may not transform to zero in all frames of reference unless

$$T_{\alpha}{}^{l}-k\delta_{\alpha}{}^{l}=0$$

$$g_{l\beta}\left(T_{\alpha}{}^{l}-kg_{\alpha}{}^{l}\right)=0$$

$$T_{\alpha\beta}=kg_{\alpha\beta}\ (44)$$

It is important to take note of the fact that equation (43) is derived from tensor equations. Hence it preserves form in all reference frames though it is not expected to do so considering the fact that it is not a tensor equation. By division rule $\frac{\partial T_{l}{}^{\alpha}}{\partial x^{i}}$ should be a tensor unless $T_{\alpha\beta}-kg_{\alpha\beta}=0$The only solution is $T_{\alpha\beta}-kg_{\alpha\beta}=0$..

Similarly by using (31`) and (32) we may prove

$$R_{\alpha\beta}=k'g_{\alpha\beta}\ \ (45)$$

but we have seen in part I that $g_{\alpha\beta}=0$. The same result was derived n the manuscript [Dot Product Preserving Transport]

We recall(43)

$$\left(T_{\alpha}{}^{l}-k\delta_{\alpha}{}^{l}\right)\frac{\partial T_{l}{}^{\alpha}}{\partial x^{i}}=0$$

If $l=\alpha$

$$(T_{\alpha}{}^{\alpha}-k)\frac{\partial T_{\alpha}{}^{\alpha}}{\partial x^{i}}=0$$

$$T_{\alpha}{}^{\alpha}\frac{\partial T_{\alpha}{}^{\alpha}}{\partial x^{i}}-k\frac{\partial T_{\alpha}{}^{\alpha}}{\partial x^{i}}=0$$

$$T_{\alpha}{}^{\alpha}dT_{\alpha}{}^{\alpha}-kdT_{\alpha}{}^{\alpha}=0\Rightarrow\frac{1}{2}dT_{\alpha}{}^{\alpha 2}-kdT_{\alpha}{}^{\alpha}=0$$

$$\Rightarrow\frac{1}{2}T_{\alpha}{}^{\alpha 2}-kT_{\alpha}{}^{\alpha}-c=0\Rightarrow T_{\alpha}{}^{\alpha 2}-2kT_{\alpha}{}^{\alpha}-C=0$$

$$T_{\alpha}{}^{\alpha}=\frac{2k\pm\sqrt{4k^{2}+4C}}{2}\ \ (46)$$

The field is constant .But we have seen$\left(T_{\alpha}{}^{l}-k\delta_{\alpha}{}^{l}\right)=0$

Therefore

$$T_{\alpha}{}^{\alpha} = \frac{2k \pm \sqrt{4k^2 + 4C}}{2} = k \Rightarrow k^2 + C = 0 \quad (47)$$

The point is the right hand side of equation (43) is not expected to have zero on its right hand side in all frames of reference since it is not a tensor equation. Nevertheless the form of (43)is preserved since

$T_{\alpha\beta} - kg_{\alpha\beta} = 0$ which is a tensor equation.

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Conclusion

We have unexpected constants on the manifold like the Ricci Scalar and the norm of the Ricci Tensor. They are independent of the space time coordinates. They are not only invariants but they are also constants. The article renders the fact that the Ricci tensor and the Riemannian curvature tensor are the null tensors. The metric tensor also happens to be a null tensor. There is a requirement for restructuring the subject.

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