The Dirac Equation and the Mass of the Fermions

Anamitra Palit

palit.anamitra@gmail.com

Cell: +91 9163892336

Abstract

It is a fact from the standard Model that fermions should be massless. The article brings out this fact in a mathematical manner along with the uncanny attributes like the left handed Dirac spinor moving along the z direction should be independent of the z coordinate.

Introduction

It is a prediction of the standard model that all fermions[1] should be massless unless there is some mechanism to provide them with mass. The article derives such a fact in a strictly mathematical sense. Uncanny attributes like the left handed Dirac spinor moving along the z direction should be independent of the z coordinate have been revealed.

We consider the Dirac equation[2] for a right handed spinor moving in the z-direction[Dirac Pauli representation[3]]

$$i\hbar\frac{\partial\psi_R}{\partial t}I = c\alpha_z\hat{p}_z\psi_R + m_0c^2\beta\psi_R \quad (1.1)$$

$$i\hbar\frac{\partial\psi_R}{\partial t}I + i\hbar c\alpha_z\frac{\partial\psi_R}{\partial z} = m_0c^2\beta\psi_R (1.2)$$

$$\psi_R = N\begin{pmatrix}\chi\\ \frac{\sigma_z p_z}{E+m}\chi\end{pmatrix}e^{-ip.x}; \chi = \begin{pmatrix}1\\0\end{pmatrix}$$

We left multiply both sides of (1.2) by $\Sigma_x = \begin{pmatrix}\sigma_x & 0\\ 0 & \sigma_x\end{pmatrix}; \sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$

$$\Sigma_x\left(i\hbar\frac{\partial\psi_R}{\partial t} + i\hbar c\alpha_z\frac{\partial\psi_R}{\partial z}\right) = m_0c^2\Sigma_x\beta\psi_R \quad (2)$$

Now

$$\Sigma_x\alpha_z = \begin{pmatrix}\sigma_x & 0\\ 0 & \sigma_x\end{pmatrix}\begin{pmatrix}0 & \sigma_z\\ \sigma_z & 0\end{pmatrix} = \begin{pmatrix}0 & \sigma_x\sigma_z\\ \sigma_x\sigma_z & 0\end{pmatrix}$$

where

$$\alpha_z = \begin{pmatrix}0 & \sigma_z\\ \sigma_z & 0\end{pmatrix}; \Sigma_x = \begin{pmatrix}\sigma_x & 0\\ 0 & \sigma_x\end{pmatrix}$$

$$\alpha_z\Sigma_x = \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix}\begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} = \begin{pmatrix} 0 & \sigma_z\sigma_x \\ \sigma_z\sigma_x & 0 \end{pmatrix}$$

Since $\sigma_z\sigma_x = -\sigma_x\sigma_z$

$$\Sigma_x\alpha_z = -\alpha_z\Sigma_x \ \ (3)$$

Again

$$\Sigma_x\beta = \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \begin{pmatrix} \sigma_x & 0 \\ 0 & -\sigma_x \end{pmatrix}; \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$$\beta\Sigma_x = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} = \begin{pmatrix} \sigma_x & 0 \\ 0 & -\sigma_x \end{pmatrix}$$

$$\Sigma_x\beta = \beta\Sigma_x \ \ (4)$$

By (3) and (4),(2) reduces to

$$\left(i\hbar\frac{\partial\Sigma_x\psi_R}{\partial t} + i\hbar c\Sigma_x\alpha_z\frac{\partial\psi_R}{\partial z}\right) = m_0c^2\Sigma_x\beta\psi_R$$

$$\Rightarrow \left(i\hbar\frac{\partial\Sigma_x\psi_R}{\partial x} - i\hbar c\alpha_z\frac{\partial(\Sigma_x\psi_R)}{\partial z}\right) = m_0c^2\beta\Sigma_x\psi_R \ (5)$$

[by (3)]

Now,

$$\Sigma_x\psi_R = \psi_L \ (6)$$

$$\psi_R = N\begin{pmatrix} \chi \\ \frac{\sigma_z p_z}{E+m}\chi \end{pmatrix} e^{-ip.x}$$

Indeed

$$\Sigma_x\psi_R = \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} N\begin{pmatrix} \chi \\ \frac{\sigma_z p_z}{E+m}\chi \end{pmatrix} e^{-ip.x} = N\begin{pmatrix} \sigma_x\chi \\ \frac{\sigma_z p_z}{E+m}\sigma_x\chi \end{pmatrix} e^{-ip.x}$$

$$\sigma_x\chi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \chi'$$

$$\Sigma_x\psi_R = N\begin{pmatrix} \chi' \\ \frac{\sigma_z p_z}{E+m}\chi' \end{pmatrix} e^{-ip.x} = \psi_L$$

Therefore by (6),(5)reduces to

$$\left(i\hbar\frac{\partial\psi_L}{\partial t}-i\hbar c\alpha_z\frac{\partial\psi_L}{\partial z}\right)=m_0c^2\beta\psi_L(7)$$

We consider a left handed Dirac spinor moving in the z-direction

$$i\hbar\frac{\partial\psi_L}{\partial t}+i\hbar c\alpha_z\frac{\partial\psi_L}{\partial z}=m_0c^2\beta\psi_L\ (8)$$

From(6) and (7) by subtraction we have ,

$$i\hbar c\alpha_z\frac{\partial\psi_L}{\partial z}=0$$

The left handed spinor moving in the z direction is independent of the z – coordinate.

Therefore from(1.1)

$$i\hbar\frac{\partial\psi_L}{\partial t}\mathrm{I}=m_0c^2\beta\psi_L(10)$$

$$i\hbar\frac{\partial\psi_L}{\partial t}\mathrm{I}_{4\times 4}=m_0c^2\begin{pmatrix}I_{2\times 2} & 0\\ 0 & -I_{2\times 2}\end{pmatrix}\psi_L$$

$$i\hbar\frac{\partial\psi_L}{\partial t}=+m_0c^2\ (11.1)$$

$$i\hbar\frac{\partial\psi_L}{\partial t}=-m_0c^2(11.2)$$

Either $m_0=0$or $\psi_L$independent of time. The equation $\frac{\partial\psi_L}{\partial t}=0$ implies energy $\left(E\equiv i\hbar\frac{\partial}{\partial t}\Rightarrow i\hbar\frac{\partial\psi}{\partial t}=\varepsilon\psi\right)$is zero which is impossible for massive particles.

Conclusion

Therefore mass has to be zero unless there is some interaction. Interaction will change the free solutions of the Dirac equation to solutions incorporating interaction. Relations (3) ,(4) and (6) will no longer hold. Mass will not be zero. For massive particles this interaction has always got to stay[even in the absence of known interactions]. Hence it is some primordial interaction provided by a primordial field: the Higg's field. Without such a field the Dirac equation cannot support massive particles/.

References

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