# Division by Zero Calculus in Trigonometric Functions

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April 1, 2019

**Abstract:** In this paper, we will introduce the division by zero calculus in triangles and trigonometric functions as the first stage in order to see the elementary properties.

**Key Words:** Zero, division by zero, division by zero calculus, $0/0 = 1/0 = z/0 = 0$, triangle, trigonometric function, Laurent expansion.

**2010 Mathematics Subject Classification:** 30E20, 30C20, 00A05, 00A09.

## 1 Division by zero calculus

The essence of the division by zero is given by the simple division by zero calculus. For this conclusion, see the papers in the references.

Therefore, we will simply introduce the division by zero calculus. For any Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{-1} C_n(z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n(z-a)^n, \tag{1.1}$$

we **define** the identity

$$f(a) = C_0. \tag{1.2}$$

By considering derivatives in (1.1), we **define** any order derivatives of the function $f$ at the singular point $a$; that is,

$$f^{(n)}(a) = n!C_n.$$

In addition, we will refer to the naturality of this division by zero calculus.

Recall the Cauchy integral formula for an analytic function $f(z)$; for an analytic function $f(z)$ around $z = a$ and for a smooth simple Jordan closed curve $\gamma$ enclosing one time the point $a$, we have

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a} dz.$$

Even when the function $f(z)$ has any singularity at the point $a$, we assume that this formula is valid as the division by zero calculus. We define the value of the function $f(z)$ at the singular point $z = a$ with the Cauchy integral.

**The basic idea of the above may be considered that we can consider the value of a function by some mean value of the function.**

The division by zero calculus opens a new world since Aristotele-Euclid. See, in particular, [2] and also the references for recent related results.

On February 16, 2019 Professor H. Okumura introduced the surprising news in Research Gate:

José Manuel Rodríguez Caballero
Added an answer
In the proof assistant Isabelle/HOL we have $x/0 = 0$ for each number $x$. This is advantageous in order to simplify the proofs. You can download this proof assistant here: **https://isabelle.in.tum.de/**.

J.M.R. Caballero kindly showed surprisingly several examples by the system that

$$\tan \frac{\pi}{2} = 0,$$

$$\log 0 = 0,$$

$$\exp \frac{1}{x}(x = 0) = 1,$$

and others. Furthermore, for the presentation at the annual meeting of the Japanese Mathematical Society at the Tokyo Institute of Technology:

March 17, 2019; 9:45-10:00 in Complex Analysis Session, *Horn torus models for the Riemann sphere from the viewpoint of division by zero* with [2],

he kindly sent the message:

It is nice to know that you will present your result at the Tokyo Institute of Technology. Please remember to mention Isabelle/HOL, which is a software in which x/0 = 0. This software is the result of many years of research and a millions of dollars were invested in it. If x/0 = 0 was false, all these money was for nothing. Right now, there is a team of mathematicians formalizing all the mathematics in Isabelle/HOL, where x/0 = 0 for all x, so this mathematical relation is the future of mathematics. https://www.cl.cam.ac.uk/ lp15/Grants/Alexandria/

Surprisingly enough, he sent his e-mail at 2019.3.30.18:42 as follows:

Nevertheless, you can use that $x/0 = 0$, following the rules from Isabelle/HOL and you will obtain no contradiction. Indeed, you can check this fact just downloading Isabelle/HOL: https://isabelle.in.tum.de/

and copying the following code

```
theory DivByZeroSatoih imports Complex Main
begin
theorem T: ‹x/0 + 2000 = 2000› for x :: complex by simp
end
```

and he referred to horn torus models [2] in the system.

Meanwhile, on ZERO, S. K. Sen and R. P. Agarwal [23] published its long history and many important properties of zero. See also R. Kaplan [3] and E. Sondheimer and A. Rogerson [25] on the very interesting books on zero and infinity. In particular, for the fundamental relation of zero and infinity, we stated the simple and fundamental relation in [22] that

**The point at infinity is represented by zero; and zero is the definite complex number and the point at infinity is considered by the limiting idea**

and that is represented geometrically with the horn torus model [2].

S. K. Sen and R. P. Agarwal [23] referred to the paper [4] in connection with division by zero, however, their understandings on the paper seem to be not suitable (not right) and their ideas on the division by zero seem to be traditional, indeed, they stated as the conclusion of the introduction of the book that:

**"Thou shalt not divide by zero" remains valid eternally.**

However, in [21] we stated simply based on the division by zero calculus that

**We Can Divide the Numbers and Analytic Functions by Zero with a Natural Sense.**

They stated in the book many meanings of zero over mathematics, deeply.

In this paper, we will introduce the division by zero calculus in triangles and trigonometric functions as the first stage in order to see the elementary properties.

## 2 Trigonometric functions

In order to see how elementary of the division by zero, we will see the division by zero in trigonometric functions as the fundamental object. Even the cases of triangles and trigonometric functions, we can derive new concepts and results.

Even the case

$$\tan x = \frac{\sin x}{\cos x},$$

we have the identity, for $x = \pi/2$

$$0 = \frac{1}{0}.$$

Of course, for identities for analytic functions, they are still valid even at isolated singular points with the division by zero calculus. Here, we will see many more direct applications of the division by zero $1/0 = 0/0 = 0$ as in the above with some meanings.

Note that from the inversion of the both sides

$$\cot x = \frac{\cos x}{\sin x},$$

for example, we have, for $x = 0$,

$$0 = \frac{1}{0}.$$

By this general method, we can consider many problems.

We will consider a triangle ABC with $BC = a, CA = b, AB = c$. Let $\theta$ be the angle of the side BC and the bisector line of A. Then, we have the identity

$$\tan\theta = \frac{c+b}{c-b}\tan\frac{A}{2}, \quad b < c.$$

For $c = b$, we have

$$\tan\theta = \frac{2b}{0}\tan\frac{A}{2}.$$

Of course, $\theta = \pi/2$; that is,

$$\tan\frac{\pi}{2} = 0.$$

Here, we used

$$\frac{2b}{0} = 0$$

and we did not consider that by the division by zero calculus

$$\frac{c+b}{c-b} = 1 + \frac{2b}{c-b}$$

and for $c = b$

$$\frac{c+b}{c-b} = 1.$$

In the Napier's formula

$$\frac{a+b}{a-b} = \frac{\tan(A+B)/2}{\tan(A-B)/2},$$

there is no problem for $a = b$ and $A = B$.

Masakazu Nihei derived the result (H. Okumura sent his result at 2018.11.29.10:06):

For the bisector line of A in the above formula, if we consider a general line with a common point with BC, then we have

$$\tan\theta = \frac{2bc\sin A}{(b-c)(b+c)}.$$

Here, for $b = c$, of course, we have $\theta = \pi/2$ and $\tan\frac{\pi}{2} = 0$.

Similarly, in the formula

$$\frac{b-c}{b+a}\frac{1}{\tan\frac{A}{2}}+\frac{b+c}{b-c}\tan\frac{A}{2}=\frac{2}{\sin(B-C)},$$

for $b=c$, $B=C$, and we have

$$0+\frac{2c}{0}\tan\frac{A}{2}=\frac{2}{0},$$

that is right.

We have the formula

$$\frac{a^2+b^2-c^2}{a^2-b^2+c^2}=\frac{\tan B}{\tan C}.$$

If $a^2+b^2-c^2=0$, then by the Pythagorean theorem $C=\pi/2$. Then,

$$0=\frac{\tan B}{\tan\frac{\pi}{2}}=\frac{\tan B}{0}.$$

Meanwhile, for the case $a^2-b^2+c^2=0$, $B=\pi/2$, and we have

$$\frac{a^2+b^2-c^2}{0}=\frac{\tan\frac{\pi}{2}}{\tan C}=0.$$

In the formula

$$\frac{a^2+b^2+c^2}{2abc}=\frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c},$$

for the case $a=0$, with $b=c$ and $B=C=\pi/2$ the identity holds.

Meanwhile, the lengths $f$ and $f'$ of the bisector lines of A and in the out of the triangle ABC are given by

$$f=\frac{2bc\cos\frac{A}{2}}{b+c}$$

and

$$f'=\frac{2bc\sin\frac{A}{2}}{b-c},$$

respectively.

If $b = c$, then we have $f' = 0$, by the division by zero. However, note that, from

$$f' = 2 \sin \frac{A}{2} \left( c + \frac{c^2}{b - c} \right),$$

by the division by zero calculus, for $b = c$, we have

$$f' = 2b \sin \frac{A}{2} = a.$$

The result $f' = 0$ is a popular property, but the result $f' = a$ is also an interesting popular property. See [8].

Let H be the perpendicular leg of A to the side BC and let E and M be the mid points of AH and BC, respectively. Let $\theta$ be the angle of EMB ($b > c$). Then, we have

$$\frac{1}{\tan \theta} = \frac{1}{\tan C} - \frac{1}{\tan B}.$$

If $B = C$, then $\theta = \pi/2$ and $\tan(\pi/2) = 0$.

In the formula

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

if $b$ or $c$ is zero, then, by the division by zero, we have $\cos A = 0$. Therefore, then we should understand as $A = \pi/2$.

This result may be derived from the formulas

$$\sin^2 \frac{A}{2} = \frac{(a - b + c)(a + b - c)}{4bc}$$

and

$$\cos^2 \frac{A}{2} = \frac{(a + b + c)(-a + b + c)}{4bc},$$

by applying the division by zero calculus.

This result is also valid in the Mollweide's equation

$$\sin \frac{B - C}{2} = \frac{(b - c) \cos \frac{A}{2}}{a},$$

for $a = 0$ as

$$0 = \frac{(b - c) \cos \frac{A}{2}}{0}.$$

We have the identities, for the radius $R$ of the circumscribed circle of the triangle ABC,

$$S = \frac{ar_A}{2} = \frac{1}{2}bc\sin A$$
$$= \frac{1}{2}a^2\frac{\sin B\sin C}{\sin A}$$
$$= \frac{abc}{4R} = 2R^2\sin A\sin B\sin C = rs, \quad s = \frac{1}{2}(a+b+c).$$

If A is the point at infinity, then, $S = s = r_A = b = c = 0$ and the above identities all valid.

For the identity

$$\tan\frac{A}{2} = \frac{r}{s-a},$$

if the point A is the point at infinity, $A = 0, s-a = 0$ and the identity holds as $0 = r/0$. Meanwhile, if $A = \pi$, then the identity holds as $\tan(\pi/2) = 0 = 0/s$.

In the identities

$$\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cdot\cot\frac{B}{2}\cdot\cot\frac{C}{2}$$

and

$$\cot A + \cot B + \cot C = \cot A\cdot\cot B\cdot\cot C + \csc A\cdot\csc B\cdot\csc C,$$

we see that they are valid for $A = \pi$ and $B = C = 0$.

For the identity

$$\cot(z_1 + z_2) = \frac{\cot z_1\cot z_2 - 1}{\cot z_1 + \cot z_2},$$

for $z_1 = z_2 = \pi/4$, the identity holds.

For a triangle, we have the identity

$$\cot A + \cot(B + C) = 0.$$

For the case $A = \pi/2$, the identity is valid.

For the identity

$$\tan A + \tan B + \tan C = \tan A\cdot\tan B\cdot\tan C,$$

for $A = \pi/2$, the identity is valid.

In the sine theorem:

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R,$$

for $A=\pi$, $B=C=0$ and then we have

$$\frac{a}{0}=\frac{b}{0}=\frac{c}{0}=0.$$

In the formula

$$\frac{\cos A\cos B}{ab}+\frac{\cos B\cos C}{bc}+\frac{\cos C\cos A}{ca}=\frac{\sin^2 A}{a^2}$$

for $a=0$, $A=0$, $b=c$, $B=C=\pi/2$, the identity is valid.

In the formula

$$R=\frac{abc}{4S},$$

for $S=0$, we have

$$R=0$$

(H. Okumura: 2017.9.5.7:40).

In the formula

$$\cos A+\cos B=\frac{2(a+b)}{c}\sin^2\frac{C}{2},$$

for $c=0$, we have $b=c$ and $A=B=\pi/2$ and the identity is valid.

In a triangle ABC, let $H$ be the orthocenter and $J$ be the common point of the three perpendicular bisectors. Then, we have

$$AH=d_a=\frac{a}{\tan A}$$

and

$$\text{the distance of } J \text{ to the line } BC=h_a=\frac{a}{2\tan A}.$$

For $A=\pi/2$, we have that $d_a=h_2=0$ (V. V. Puha: 2018.7.12.18:10).

For the tangential function, note the following identities.

In the formula

$$\tan\frac{\theta}{2}=\frac{\sin\theta}{1+\cos\theta}=\pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}},$$

for $\theta = \pi$, we have that 0=0/0.

For the inversions and from $x = 0$, we have

$$\frac{1}{0} = \frac{2}{0} = \pm\sqrt{\frac{2}{0}} = 0.$$

In the formula

$$\tan z_1 \pm \tan z_2 = \frac{\sin(z_1 + z_2)}{\sin z_1 \sin z_2},$$

for $z_1 = \pi/2, z_2 = 0$, we have that 0=1/0.

In the formula

$$\tan\frac{x}{2} = \frac{1 \pm \sqrt{1 - \sin^2 x}}{\sin x},$$

for $x = \pi$, we have

$$0 = \frac{0}{0}.$$

In the elementary identity

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta},$$

for the case $\alpha = \beta = \pi/4$, we have

$$\tan\frac{\pi}{2} = \frac{1 + 1}{1 - 1 \cdot 1} = \frac{2}{0} = 0.$$

Further note that it is valid for $\alpha + \beta = 0$ and $\alpha + \beta = \pi/2$ (H. Okumura: 2018.8.7.8:03). Furthermore, the formula

$$\tan(\alpha + \beta) = \frac{\frac{1}{\tan\alpha} + \frac{1}{\tan\beta}}{\frac{1}{\tan\alpha\tan\beta} - 1},$$

is valid for $\alpha = \pi/2$ or $\beta = \pi/2$ and for $\alpha + \beta = \pi/2$ (H. Okumura: 2018.7.11.21:05).

In the identity

$$\sqrt{\frac{1 - \sin\alpha}{1 + \sin\alpha}} = \frac{1}{\cos\alpha} - \tan\alpha,$$

for $\alpha = \pi/2$, we have

$$0 = \frac{1}{0} - 0.$$

For the double angle formula

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha},$$

for $\alpha = \pi/2$, we have that

$$0 = \frac{2\cdot 0}{1-0}.$$

In the identity

$$\tan 3\alpha = \frac{2\tan\alpha - \tan^3\alpha}{1-3\tan^2\alpha},$$

for $\alpha = \pi/6$, we have

$$\tan\frac{\pi}{2} = 0,$$

that is right.

In the identities

$$\frac{1+\cos x}{\sin x} = \frac{\sin x}{1-\cos x}$$

and

$$\frac{1-\cos x}{\sin x} = \tan\frac{x}{2}$$

for $x = 0$, we have the identity

$$\frac{0}{0} = 0.$$

In the identities

$$\tan(x+iy) = \frac{\sin 2x + i\sinh 2y}{\cos 2x + \cosh 2y}$$

and

$$\cot(x+iy) = \frac{\sin 2x - i\sinh 2y}{\cosh 2x - \cos 2x}$$

for $x = \pi/2, y = 0$, they are valid.

In the identity

$$\arctan\sqrt{\frac{b}{a}} = \arcsin\sqrt{\frac{b}{a+b}},$$

for $a = 0$, the identity is valid.

We can find similar interesting identities in the following identities:

$$\frac{\sin 3x + \sin x}{\cos 3x + \cos x} = \tan 2x,$$
$$\frac{\sin 3x + \sin x}{\cos 3x - \cos x} = -\cot x$$
and
$$\frac{\sin 3x - \sin x}{\cos 3x + \cos x} = \tan x$$
(H. Okumura: 2019.1.31 and the first one is obtained by M. Nihei).

We consider a triangle ABP with $A(-a, 0)$, $B(a, 0)$, $P(x, y)$ with $a > 0$. Then we have
$$\angle QPA = \tan^{-1}\frac{x+a}{y}$$
and
$$\angle QPB = \tan^{-1}\frac{y-a}{y}.$$
For $y = 0$, we obtain
$$\angle QPA = \tan^{-1}\frac{x+a}{0} = \tan^{-1} 0 = \frac{\pi}{2}$$
and
$$\angle QPB = \tan^{-1}\frac{x-a}{0} = \tan^{-1} 0 = \frac{\pi}{2}.$$
In addition, M. Nihei remarked the following identities through H. Okumura (2018.7.8.12:12; 2019.1.31):
$$\frac{\sin 2x}{1 + \cos x} = \frac{1 - \cos x}{\sin 2x} = \tan x,$$
$$\frac{1 + \sin 2x + \cos 2x}{1 + \sin 2x - \cos 2x} = \cot x,$$
$$\frac{\sin 2x}{1 - \cos 2x} = \frac{1 + \cos 2x}{\sin 2x} = \cot x,$$
and in a triangle
$$\cot A + \cot B + \cot C = \frac{a^2 + b^2 + c^2}{4S}.$$
Consider the case $x = \pi/2$ in the above two formulas and in the triangle, consider the case $A = \pi/2$.

## 3 Triangles

For three points $a, b, c$ on a circle with its center at the origin on the complex $z$-plane with radius $R$, we have

$$|a + b + c| = \frac{|ab + bc + ca|}{R}.$$

If $R = 0$, then $a, b, c = 0$ and we have $0 = 0/0$.

For a circle with radius $R$ and for an inscribed triangle with side lengths $a, b, c$, and further for the inscribed circle with radius $r$ for the triangle, the area $S$ of the triangle is given by

$$S = \frac{r}{2}(a + b + c) = \frac{abc}{4R}.$$

If $R = 0$, then we have

$$S = 0 = \frac{0}{0}$$

(H. Michiwaki: 2017.7.28.). We have the identity

$$r = \frac{2S}{a + b + c}.$$

If $a + b + c = 0$, then we have

$$0 = \frac{0}{0}.$$

For the distance $d$ of the centers of the inscribed circle and circumscribed circle, we have the Euler formula

$$r = \frac{1}{2}R - \frac{d^2}{2R}.$$

If $R = 0$, then we have $d = 0$ and

$$0 = 0 - \frac{0}{0}.$$

Let $r$ be the radius of the inscribed circle of the triangle ABC, and $r_A, r_B, r_C$ be the distances from A, B, C to the lines BC, CA, AB, respectively. Then we have

$$\frac{1}{r} = \frac{1}{r_A} + \frac{1}{r_B} + \frac{1}{r_C}.$$

When A is the point at infinity, then, $r_A = 0$ and $r_B = r_C = 2r$ and the identity still holds.

**Thales' theorem**

We consider a triangle $BAC$ with $A(-1,0), C(1,0), \angle BOC = \theta; O(0,0)$ on the unit circle. Then, the gradients of the lines $AB$ and $CB$ are given by

$$\frac{\sin\theta}{\cos\theta + 1}$$

and

$$\frac{\sin\theta}{\cos\theta - 1},$$

respectively. We see that for $\theta = \pi$ and $\theta = 0$, they are zero, respectively.

For a triangle OAB ($O(0,0), A(1,0), B(0,\tan\theta), \theta = \angle OAB$), we consider the escribed circle

$$(x + r)^2 + y^2 = r^2$$

of A. Then, from

$$\frac{r}{r+1} = \tan\frac{\theta}{2},$$

we have

$$r = \frac{\tan\frac{\theta}{2}}{1 - \tan\frac{\theta}{2}}.$$

For $\theta = \pi/2$, we have the reasonable result

$$r = \frac{1}{1-1} = 0.$$

On the point $(p,q)(0 \le p, q \le 1)$ on the unit circle, we consider the tangential line $L_{p,q}$ of the unit circle. Then, the common points of the line $L_{p,q}$ with $x$-axis and $y$-axis are $(1/p, 0)$ and $(0, 1/q)$, respectively. Then, the area $S_p$ of the triangle formed by three points $(0,0)$, $(1/p,0)$ and $(0,1/q)$ is given by

$$S_p = \frac{1}{2pq}.$$

Then,

$$p \longrightarrow 0; \quad S_p \longrightarrow +\infty,$$

however,

$$S_0 = 0$$

(H. Michiwaki: 2015.12.5.). We denote the point on the unit circle on the $(x, y)$ plane with $(\cos\theta, \sin\theta)$ for the angle $\theta$ with the positive real line. Then, the tangential line of the unit circle at the point meets at the point $(R_\theta, 0)$ for $R_\theta = [\cos\theta]^{-1}$ with the $x$-axis for the case $\theta \neq \pi/2$. Then,

$$\theta\left(\theta < \frac{\pi}{2}\right) \to \frac{\pi}{2} \Longrightarrow R_\theta \to +\infty,$$

$$\theta\left(\theta > \frac{\pi}{2}\right) \to \frac{\pi}{2} \Longrightarrow R_\theta \to -\infty,$$

however,

$$R_{\pi/2} = \left[\cos\left(\frac{\pi}{2}\right)\right]^{-1} = 0,$$

by the division by zero. We can see the strong discontinuity of the point $(R_\theta, 0)$ at $\theta = \pi/2$ (H. Michiwaki: 2015.12.5.).

The line through the points $(0, 1)$ and $(\cos\theta, \sin\theta)$ meets the $x$ axis with the point $(R_\theta, 0)$ for the case $\theta \neq \pi/2$ by

$$R_\theta = \frac{\cos\theta}{1 - \sin\theta}.$$

Then,

$$\theta\left(\theta < \frac{\pi}{2}\right) \to \frac{\pi}{2} \Longrightarrow R_\theta \to +\infty,$$

$$\theta\left(\theta > \frac{\pi}{2}\right) \to \frac{\pi}{2} \Longrightarrow R_\theta \to -\infty,$$

however,

$$R_{\pi/2} = 0,$$

by the division by zero. We can see the strong discontinuity of the point $(R_\theta, 0)$ at $\theta = \pi/2$.

Note also that

$$\left[1 - \sin\left(\frac{\pi}{2}\right)\right]^{-1} = 0.$$

In a triangle ABC, let X be the leg of the perpendicular line from A to the line BC and let Y be the common point of the bisector line of A and the line BC. Let P and Q be the tangential points on the line BC with the incircle of the triangle and the inscribed circle in the sector with the angle A, respectively. Then, we know that

$$\frac{XP}{PY} = \frac{XQ}{QY}.$$

If $AB = AC$, then, of course, X=Y=P=Q. Then, we have

$$\frac{0}{0} = \frac{0}{0} = 0.$$

Let X, Y, Q be the common points with a line and three lines AC, BC and AB, respectively. Let P be the common point with the line AB and the line through the point C and the common point of the lines AY and BX. Then, we know the identity

$$\frac{AP}{AQ} = \frac{BP}{BQ}.$$

If two lines XY and AB are parallel, then the point Q may be considered as the point at infinity. Then, by the interpretation $AQ = BQ = 0$, the identity is valid as

$$\frac{AP}{0} = \frac{BP}{0} = 0.$$

We write lines by

$$L_k : a_k x + b_k y + c_k = 0, k = 1, 2, 3.$$

The area $S$ of the triangle surrounded by these lines is given by

$$S = \pm\frac{1}{2} \cdot \frac{\triangle^2}{D_1 D_2 D_3},$$

where $\triangle$ is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

and $D_k$ is the co-factor of $\triangle$ with respect to $c_k$. $D_k = 0$ if and only if the corresponding lines are parallel. $\triangle = 0$ if and only if the three lines are parallel or they have a common point. We can see that the degeneracy (broken) of the triangle may be stated by $S = 0$ beautifully, by the division by zero.

Similarly we write lines by

$$M_k : a_{k1}x + a_{k2}y + a_{3k} = 0, k = 1, 2, 3.$$

The area $S$ of the triangle surrounded by these lines is given by

$$S = \frac{1}{A_{11}A_{22}A_{33}} \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}$$

where $A_{kj}$ is the co-factor of $a_{kj}$ with respect to the matrix $[a_{kj}]$. We can see that the degeneracy (broken) of the triangle may be stated by $S = 0$ beautifully, by the division by zero.

**Acknowledgements**

The author expresses his thanks to Professors Masakazu Nihei and Hiroshi Okumura and Mr. Hiroshi Michiwaki and V. V. Puha for their kind sendings of their results. Further, the author wishes to express his deep thanks to Professor Hiroshi Okumura and Mr. José Manuel Rodríguez Caballero for their very exciting informations.

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