Direct Sum Decomposition of a Linear Vector Space

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Abstract

The direct sum decomposition of a vector space has been explored to bring out a conflicting feature in the theory. It has been proved that a subspace cannot have dimension less than a third of the dimension of the parent vector space.

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Introduction

The concept of linear vector spaces[1] is fundamental to the edifice of physics and mathematics. Nevertheless this theory is not free from conflicts. The direct sum decomposition of a vector space has been explored to bring out a conflicting feature in the theory. It has been proved that a subspace cannot have dimension than a third of the dimension of the parent vector space

Basic Considerations and Calculations

We consider the direct sum[3] decomposition of a vector space V in subspaces $A$and B .

$$V = A \oplus B \quad (1)$$

By the definition of direct sum

$$A \cap B = \{0\}$$

The dimensions of the three spaces have been stated below.

$$Dim(V) = n, Dim(A) = k \Rightarrow Dim(B) = n - k$$

If possible let there be a subspace B' ,distinct from B such that

$$V = A \oplus B' \quad (2)$$

By the definition of direct sum

$$A \cap B' = \{0\}$$

Since $Dim(V) = n, Dim(A) = k$,we have , $Dim(B') = n - k$

We denote by m, the dimension of the intersection, $B \cap B'$

Now vectors in B-B' cannot span vectors in B'-B

Therefore $n - k - m + m + n - k - m \leq n$

$$\Rightarrow n \leq 2k + m \quad (3)$$

We denote by $n_l(X)$ the number of linearly independent vectors from X[chosen in a certain manner.

Option 1:Let

$n_l(A) + n_l(B - B') + n_l(B \cap B') + n_l(B' - B) > n \quad (4)$

$$k + (n - k - m) + m + (n - k - m) > n$$

$$k + 2n - 2k - m > n$$

$$n - k - m > 0$$

$$n > k + m > 0 \ (5)$$

From (3) and (5) n will be greater than the greater of $2k + m$ and $k + m$

Since

$2k + m \geq k + m$ we have

$$k + m < n \leq 2k + m$$

$$n \leq 2k + k - x; 0 \leq x \leq k$$

$$n \leq 3k - x$$

$$3k \geq n + x \Rightarrow k \geq \frac{n}{3} + \frac{x}{3} \Rightarrow k \geq \frac{n}{3}$$

$$k \geq \frac{n}{3} (6)$$

Option 2.

We consider the alternative to (4). If it materialize It is expected to incorporate the case $k < \frac{n}{3}$

$n_l(A) + n_l(B - B') + n_l(B \cap B') + n_l(B' - B) \leq n$ (7)

We shall show that te alternative to (4) that is, equation (7) will not materialize (7)

$$k + (n - k - m) + (n - k - m) + m \leq n \quad (10)$$

In relation to relation (3)(5) we have to take note of the fact that $A \cap B = A \cap B' = A \cap B \cap B' = \{0\}$

Equation (3) implies

$$m \geq n - k$$

It is not possible to have m (=the dimension of $B \cap B'$) greater than $n - k$, the dimension of Bor of B'

Therefore,

$$m = n - k \ or \ 2n < 3k \ (11)$$

$$\Rightarrow B = B' \ (12)$$

A Pair of Theorems

We consider a basis of $V = \{e_1, e_2, e_3, \ldots\ldots e_k, e_{k+1}, e_{k+2} \ldots . e_n\}$

We further assume $\{e_1, e_2, e_3, \ldots\ldots e_k\}$is a basis of A and that $\{e_{k+1}, e_{k+2} \ldots . e_n\} \equiv \{e_{k+j}\}$forms a basis of B

We have relation (1): $V = A \oplus B$

Now we consider a set of n vectors $\{e_1, e_2, e_3, \ldots\ldots e_k, e'_{k+1}, e'_{k+2} \ldots . e_n{}'\}$ where $e'_{k+j} = e_{k+j} + \alpha_j$ and $\alpha_j \in A; j = 1,2, \ldots n - k; \alpha_j \neq 0[e_{k+j} \in B$,defined earlier in this section.

Theorem 1:We prove that

$$e'_{k+j} \in \overline{A \cup B} \qquad or, e'_{k+j} = e_{k+j} + \alpha_j \in \overline{A \cup B}$$

Assume that $e'_{k+j} \in A$

Now,

$$e_{k+j} = e'_{k+j} - \alpha_j$$

On the left side $e_{k+j} \in B$

On the right side both $e'_{k+j}$and $\alpha_j$belong to $A \Rightarrow e'_{k+j} - \alpha_j \in A; j = 1,2, \dots n-k$. This not possible taking note of the fact that $A \cap B = \{0\}$ and that all the vectors involved are non zero vectors.

Therefore $e'_{k+j} \notin A$

Next let $e'_{k+j} \in B$

Now,

$$e'_{k+j} - e_{k+j} = \alpha_j$$

On the left side $e'_{k+j} - e_{k+j}$belongs to B since each $e'_{k+j}$and e_{k+j}belong to B.On the right side of the above $\alpha_j \in A$.This is not possible taking note of the fact that $A \cap B = \{0\}$ and that all the vectors involved are non zero vectors.

Therefore $e'_{k+j} \notin B$

Therefore as claimed we have,

$$e'_{k+j} = e_{k+j} + \alpha_j \in \overline{A \cup B}$$

Theorem 2:The set $\{e_1, e_2, e_3, \dots .. e_k, e'_{k+1}, e'_{k+2} \dots . e_n{}'\}$ form as basis with respect to the space V.

We consider the equation

$$\sum_{i=1}^{k} C_i e_i + \sum_{j=1}^{n-k} C_{k+j} e'{}_{k+j} = 0 \quad (13)$$

Now,

$$e'{}_{k+j} = e_{k+j} + \alpha_j$$

Therefore,

$$\sum_{i=1}^{k} C_i e_i + \sum_{j=1}^{n-k} C_{k+j} \left(e_{k+j} + \alpha_j\right) = 0$$

$$\sum_{i=1}^{k} C_i e_i + \sum_{j=1}^{n-k} C_{k+j} e_{k+j} + \sum_{j=1}^{n-k} C_{k+j} \alpha_j = 0$$

$$\alpha_j = \sum_{l=1}^{k} D_{jl} e_l = \sum_{i=1}^{k} D_{ji} e_i$$

Therefore,

$$\sum_{i=1}^{k} C_i e_i + \sum_{j=1}^{n-k} C_{k+j} e_{k+j} + \sum_{j=1}^{n-k} \sum_{i=1}^{k} C_{k+j} D_{ji} e_i = 0$$

$$\sum_{i=1}^{k} \left(C_i + \sum_{j=1}^{n-k} C_{k+j} D_{ji} \right) e_i + \sum_{j=1}^{n-k} C_{k+j} e_{k+j} = 0$$

Since

$$\{e_1, e_2, e_3, \dots .. e_k, e_{k+1}, e_{k+2} \dots . e_n\}$$

forms a linearly independent set in that they are the basic vectors for V,we have,

$$C_{k+j} = 0; j = 1,2,3 \dots n - k \ (14.1)$$

$$C_i + \sum_{j=1}^{n-k} C_{k+j} D_{ji} = 0; i = 1,2,3 .. k; i = 1,2,3 \dots k \ \ (14.2)$$

Since from (14.1)

$$C_{k+j} = 0; j = 1,2,3 \dots n - k$$

we have from (14.2)

$$C_i = 0; i = 1,2,3 \dots k$$

Therefore,

$$\{e_1, e_2, e_3, \dots .. e_k, e'_{k+1}, e'_{k+2} \dots . e_n{}'\}$$

comprise a linearly independent set of vectors . Since there are ‘n’ such vectors, n being the dimension of V, they span V.

Therefore the above set is a basis for V.

The Conflict

Let us consider B’ spanned by

$$\{e'_{k+1}, e'_{k+2} \dots . e_n{}'\}$$

Since

$$\{e_1, e_2, e_3, \dots .. e_k, e'{}_{k+1}, e'{}_{k+2} \dots . e'{}_n\}$$

spans V, we have the direct sum decomposition,

$$V = A \oplus B'$$

We also do have from(5)

$$B' = B$$

Now from theorem I, we have, $e'_{k+j} \in \overline{A \cup B} \Rightarrow e'_{k+j} \in \overline{A \cup B'} \Rightarrow e'_{k+j} \notin B'$

But e'_{k+j} is a basic vector of B'

Thus we have arrived at a contradiction. Equation (7) will not materialize. Thus equation (4)represents the valid choice. Therefore we have equation (6): $k \geq \frac{n}{3}$. The relation

$$k < \frac{n}{3} \ (15)$$

going with (7) will not materialize in any circumstance. But $k < \frac{n}{3}$is too restrictive

Conclusion

A conflict in the theory of vector spaces can have serious consequences in the areas of mathematics and physics opening up gateways to fundamental research

References

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