Extending an Irrationality Proof of Sondow: From e to $\zeta(n)$

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Introduction

Jonathan Sondow's geometric proof that e is irrational [1] uses nested closed intervals and the Bolzano-Weierstrass theorem. It's a trap: the endpoints of the intervals are systematically excluded as possible values for e. They are collectively all possible rational values, so e is proven to be irrational.

Here we re-frame Sondow's idea replacing his intervals with concentric circles with classes from natural number moduli on them. We call such sets of points a circular moduli lattice (CML). This idea leads to a general criterion for irrationality of a series.

We explore some applications of the CML idea by by giving Sondow's original proof for the irrationality of e using the CML associated with it. Next, we see how Sondow's proof doesn't generalize to show

$$z_n = \sum_{k=2}^{\infty} \frac{1}{k^n}$$

is irrational. Finally, we give a proofs for the irrationality of e and z_2 using the criterion given earlier in the article.

Circular moduli lattice

Let's suppose the circle in Figure 1 has a radius of $1/\sqrt{\pi}$. Then its area is 1. We've placed equally spaced moduli classes for modulus 5 around the circle. Now sector areas correspond to fractions with numerators given by classes and denominators with the value of the modulus. The area associated with the radial in the figure is $3/5$. Clearly, for any rational number m/n, $0 < m < n$, this procedure can be done.

Definition 1. *We will designate the set of such points in this arrangement with CK_n, where n is the modulus used and refer to such sets as clocks.*

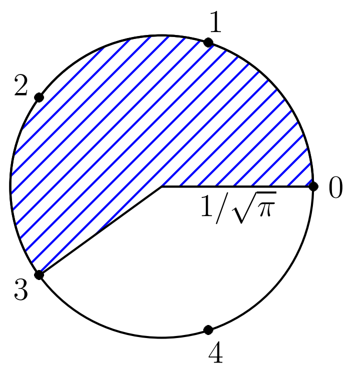
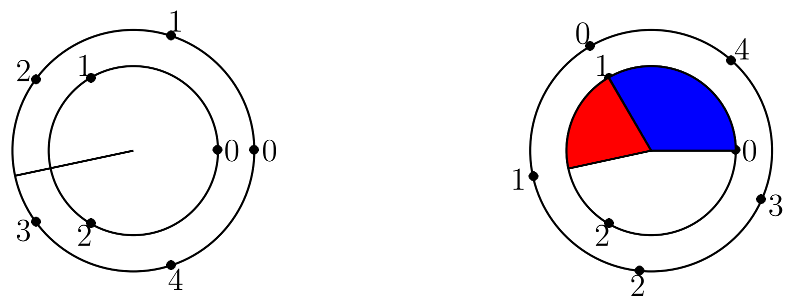


Figure 1: The shaded area is given by a modulo class.

Additional clocks can be added. In order to make them all sweep the same areas we use radii of $\sqrt{n/\pi}$. For example, in Figure 2(a) there are a 3-clock and a 5-clock. The radial given in this figure sweeps the same area in the inner circle and the annulus. In this way the clocks can be used as a crude measurement device. We can infer from Figure 2(b) that the area associated with the sector given by the radial shown in Figure 2(a) measures neither thirds or fifths of the inner circle's area.

The circles can also be used to construct areas corresponding to the addition of fractions. In Figure 2(b) an addition method is given. It is similar to the head to tail method of vector addition. The 5-clock is rotated so as to place its 0 position at the 1 position of the 3-clock. The new 1 position of the 5-clock corresponds, gives the area $1/3+1/5$. The radial generated is the same as that in Figure 2(a). Thus we can infer that $1/3 + 1/5$ is not in the set $\{1/3, 2/3, 1/5, 2/5, 3/5, 4/5\}$ or any un-reduced form of these fractions. The clocks give both a way to measure and construct addition of fractions.

In Figure 3, the first few terms for z_2 are given by clocks and their addition method. We formalize the idea with a definition.



(a) (b)

Figure 2: Circles as measuring device (a) and construction device (b).

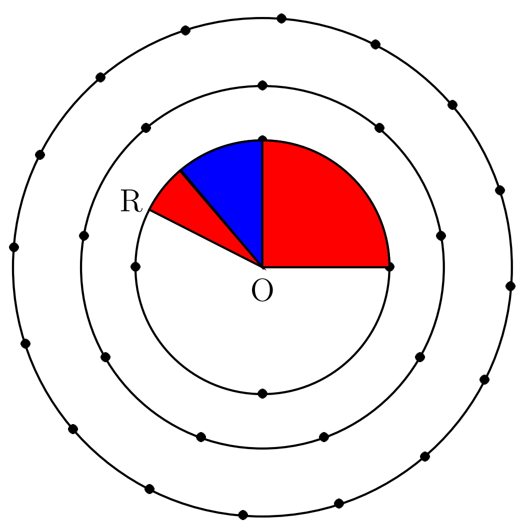


Figure 3: A partial sum for z_2 is constructed using $CML\{k^2\}$.

Definition 2. *Given an infinite series with positive, strictly decreasing terms of the form* $1/a_j$, $a_j \in \mathbb{N}$, *let the set of all points on* CK_{a_j} *be called the circular moduli lattice for the series. Designate this set with* $CML\{a_j\}$.

Using sets of clocks associated with an infinite series, we can frame the question of convergence to an irrational point. In Figure 4 the partial sum $1/4 + 1/9 + 1/16$ for z_2 is depicted using the original, un-rotated clocks. The

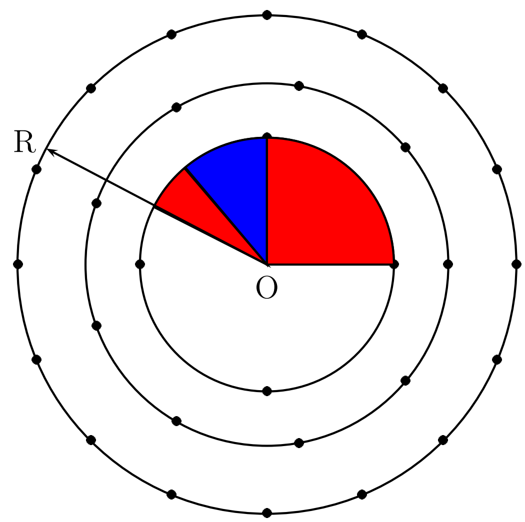


Figure 4: The radial for the partial doesn't intersect lattice points.

radial OR generates a sector of this sums area and it doesn't intersect any of the points on the three circles. This means $1/4 + 1/9 + 1/16$ doesn't have a reduced form associated with CK_4, CK_9, or CK_{16}. If this is always true, i.e., if the radial for z_2, the infinite series, doesn't go through a lattice point and all the lattice points give all the possible rational areas, then z_2 is irrational. We formalize the notion of all possible rational areas with a definition.

Definition 3. *For a given series with terms $1/a_j$, if there exists for every m/n, with $0 < m < n$, CK_r and modulus class s such that $s/r = m/n$ then the CML associated with the series, $CML\{a_j\}$ is said to be expressive (complete?).*

We can give a necessary and sufficient condition for a series to converge to an irrational number.

Theorem 1. *Given an expressive $CML\{a_k\}$ associated with a series if the partial sums for the series are such that*

$$\sum_{k=2}^{n} \frac{1}{a_k} \in \bigcup_{k=2}^{\infty} CK_{a_k} \setminus \bigcup_{k=2}^{\varphi(n)} CK_{a_k}, \tag{1}$$

where $\varphi(n)$ is a natural number, strictly increasing function, then the series converges to an irrational number.

Proof. Using (1),

$$\lim_{n\to\infty} \bigcup_{k=2}^{\infty} CK_{a_k} \setminus \bigcup_{k=2}^{\varphi(n)} CK_{a_k} = \emptyset,$$

so the the limit of the partials is not in $CML\{a_k\}$ and must be irrational. $\square$

Sondow's proof

Here's Sondow's proof that e is irrational. Figure 5 has a radial that sweeps an arc giving a sector of area e. The second sector is used to delimit where the sum radial line can exist. So the first 1/4 area is in white (its passed), but the second in red, indicating its boundary radial lines give limits to the convergence radial. As the series rate of convergence is fast and terms are multiplies of each other, factorial values, the sectors perpetually nest (neatly) with $CK_{k!}$ points being excluded.

Here's z_2

It doesn't work. They don't nest. It's a mess.

The technique given in section 2 replaces interval endpoint exclusions with general, in effect, denominator exclusions.

Examples

It's easy. We need to move away from the $\epsilon - \delta$ world of point set topology and analysis and use just sets. To show what we mean by this consider the task of proving the limit of $1-1/n$ is not of the form m/n, $0 < m < n$. That is we want to show the limit is an integer and not a fraction, of one class and not another. Now

$$1 - 1/n \in \bigcup_{k=1}^{\infty} CK_k \setminus \bigcup_{k=2}^{n-1} CK_k$$

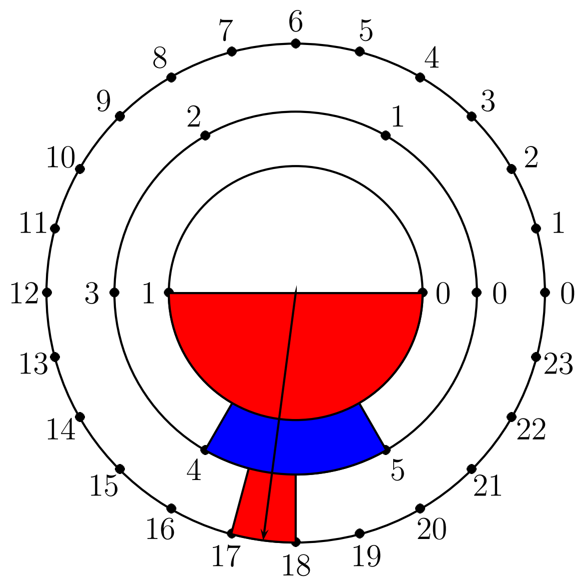


Figure 5: Sondow's proof that e is irrational using $CML\{k!\}$.

and

$$\lim_{n\to\infty}\bigcup_{k=1}^{\infty} CK_k \setminus \bigcup_{k=2}^{n-1} CK_k = CK_1.$$

Recall that $(n-1, n) = 1$, so $(n-1)/n$ is a reduced fraction.

Consider next a less trivial example. Suppose we want to show that $.\overline{1}$ base 4 converges to a denominator that is relatively prime to 4.

$$\lim_{n\to\infty}\bigcup_{k=2}^{\infty} CK_k \setminus \bigcup_{k=1}^{n-1} CK_{4^k} \neq \emptyset. \tag{2}$$

We can deduce from this example that a necessary condition for an infinite decimal to be rational is the limit formed using its partials, as in (2), is not empty. If, instead of $.\overline{1}$ base 4, we used $\sqrt{2}$ given as an infinite, non-repeating decimal, we would still get a non-empty set, so the test is not sufficient to show rationality. However, if

$$\lim_{n\to\infty}\bigcup_{k=2}^{\infty} CK_{b_k} \setminus \bigcup_{k=1}^{n} CK_{\text{for partials}} = \emptyset$$

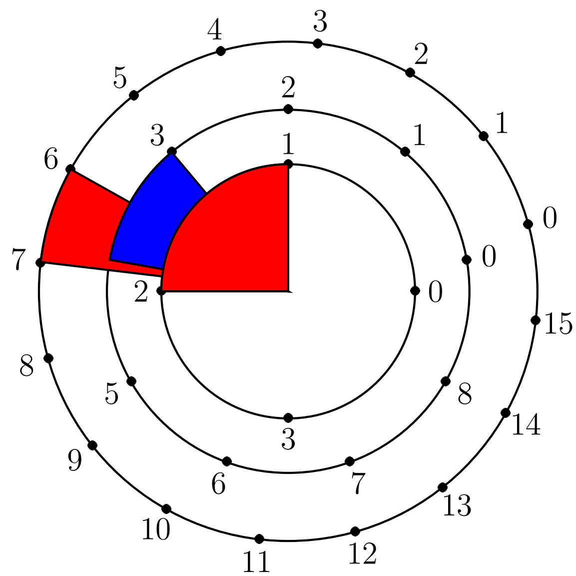


Figure 6: Sondow's interval technique fails for proving z_2 is irrational.

and $\cup CK_{b_k}$ contains all candidate rationals for the series in question, the series converges to an irrational number.

Irrationality proofs

The irrationality of e:

$$\lim_{n\to\infty} \bigcup_{k=2}^{\infty} CK_{k!} \setminus \bigcup_{k=2}^{n} CK_{k!} = \emptyset$$

and all candidate rational numbers, p/q, can be written as $p(q-1)!/q$. As the partials for e, $\sum_2^n 1/k!$, require $k!$ denominators, the result follows.

The irrationality of z_2:

$$\lim_{n\to\infty} \bigcup_{k=2}^{\infty} CK_{k^2} \setminus \bigcup_{k=2}^{n} CK_{k^2} = \emptyset$$

and all candidate rational numbers, p/q, can be written as pq/q^2. We need to show that the partials for z_2, $\sum_2^n 1/k^2$, require greater than n^2 denominators, for example. At prime p partials will be given by

$$\frac{a}{b} + \frac{1}{p^2} = \frac{p^2 a + b}{p^2 b},$$

where $p \nmid b$. This implies that partials for upper limit a prime require at least the upper limit squared in their denominators.

Conclusion

Is our proof of the irrationality of z_2 a geometric proof? The radial is perpetually offset from all possible rational convergence points, so its limit is not rational. Look at the picture and remember that $(n, n+1) = 1$.

References

[1] J. Sondow, A geometric proof that e is irrational and a new measure of its irrationality, *Amer. Math. Mon.* 113 (2006), 637-641.