Espil's high power partial fraction decomposition theorem

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Abstract: the problem of fraction decomposition it's easy to solve by using the cover up method, when there are no repeated linear factors in the denominator. Nevertheless it could turn into a hard work if these factors are raised to a high power, where the cover up method doesn't work. This technique shows how to calculate these coefficients without solving large systems of equations.

1 hypothesis:

Let $P(x)$ be a polynomial of degree less than $m+n$, which it's not divisible by $(x-a)^n$ or $(x-b)^m$.

$$\frac{P(x)}{(x-a)^n(x-b)^m} = \sum_{k=n}^{1} \frac{\alpha_k}{(x-a)^k} + \sum_{j=m}^{1} \frac{\beta_j}{(x-b)^j}$$

$$A(x) = \frac{P(x)}{(x-b)^m}\ ,\ B(x) = \frac{P(x)}{(x-a)^n}$$

2 thesis

$$\alpha_k = \frac{1}{(n-k)!} D^{n-k} A(a)$$

$$\beta_j = \frac{1}{(m-j)!} D^{m-j} B(b)$$

$$\frac{P(x)}{(x-a)^n(x-b)^m} = \sum_{k=n}^{1} \frac{1}{(n-k)!} \frac{D^{n-k}A(a)}{(x-a)^k} + \sum_{j=m}^{1} \frac{1}{(m-j)!} \frac{D^{m-j}B(b)}{(x-b)^j}$$

3 demonstration

3.1 α_k derivation

$$\frac{P(x)}{(x-a)^n(x-b)^m} = \sum_{k=n}^{1}\frac{\alpha_k}{(x-a)^k} + \sum_{j=m}^{1}\frac{\beta_j}{(x-b)^j}$$

$$\frac{P(x)}{(x-b)^m} = \sum_{k=n}^{1}\frac{\alpha_k(x-a)^n}{(x-a)^k} + \sum_{j=m}^{1}\frac{\beta_j(x-a)^n}{(x-b)^j}$$

$$\mathrm{A(x)} = \sum_{k=n}^{1}\frac{\alpha_k(x-a)^n}{(x-a)^k} + \sum_{j=m}^{1}\frac{\beta_j(x-a)^n}{(x-b)^j}$$

$$A(x) = \sum_{k=n}^{1}\alpha_k(x-a)^{n-k} + \sum_{j=m}^{1}\beta_j\frac{(x-a)^n}{(x-b)^j}$$

$$\mathrm{A(x)} = \alpha_n + \alpha_{n-1}(x-a) + \alpha_{n-2}(x-a)^2 + \alpha_{n-3}(x-a)^3 + \cdots + \alpha_1(x-a)^{n-1} + \sum_{j=m}^{1}\frac{\beta_j(x-a)^n}{(x-b)^j}$$

$$\mathrm{DA(x)} = \alpha_{n-1} + 2\alpha_{n-2}(x-a) + 3\alpha_{n-3}(x-a)^2 + \cdots + (n-1)\alpha_1(x-a)^{n-2} + \sum_{j=m}^{1}\frac{\beta_j\mathrm{n}(x-a)^{n-1}}{(x-b)^j}$$

$$D^2\mathrm{A(x)} = 2\alpha_{n-2} + 3.2.\alpha_{n-3}(x-a) + \cdots + (n-1)(n-2)\alpha_1(x-a)^{n-3} + \sum_{j=m}^{1}\frac{\beta_j\mathrm{n}(n-1)(x-a)^{n-2}}{(x-b)^j}$$

$$D^3\mathrm{A(x)} = 3.2.\alpha_{n-3} + \cdots + (n-1)(n-2)(n-3)\alpha_1(x-a)^{n-4} + \sum_{j=m}^{1}\frac{\beta_j\mathrm{n}(n-1)(n-2)(x-a)^{n-3}}{(x-b)^j}$$

$D^rA(a) = r!\,.\,\alpha_{n-r}$, substitution r=n-k ,

$$\alpha_k = \frac{1}{(n-k)!}D^{n-k}A(a)$$

3.2 $\boldsymbol{\beta_j}$ derivation

$$\frac{P(x)}{(x-a)^n(x-b)^m} = \sum_{k=n}^{1} \frac{\alpha_k}{(x-a)^k} + \sum_{j=m}^{1} \frac{\beta_j}{(x-b)^j}$$

$$\frac{P(x)}{(x-a)^n} = \sum_{k=n}^{1} \frac{\alpha_k (x-b)^m}{(x-a)^k} + \sum_{j=m}^{1} \frac{\beta_j (x-b)^m}{(x-b)^j}$$

$$\mathrm{B(x)} = \sum_{k=n}^{1} \frac{\alpha_k (x-b)^m}{(x-a)^k} + \sum_{j=m}^{1} \beta_j (x-b)^{m-j}$$

$$\mathrm{B(x)} = \sum_{k=n}^{1} \frac{\alpha_k (x-b)^m}{(x-a)^k} + \beta_m + \beta_{m-1}(x-b) + \beta_{m-2}(x-b)^2 + \beta_{m-3}(x-b)^3 + \cdots + \beta_1 (x-b)^{m-1}$$

$$\mathrm{D\ B(x)} = \sum_{k=n}^{1} \frac{m\alpha_k (x-b)^{m-1}}{(x-a)^k} + \beta_{m-1} + 2\beta_{m-2}(x-b) + 3.\beta_{m-3}(x-b)^2 + \cdots + (m-1)\beta_1 (x-b)^{m-2}$$

$$D^2\ \mathrm{B(x)} = \sum_{k=n}^{1} \frac{m(m-1)\alpha_k (x-b)^{m-2}}{(x-a)^k} + 2\beta_{m-2} + 3.2.\beta_{m-3}(x-b) + \cdots + (m-1)(m-2)\beta_1 (x-b)^{m-3}$$

$$D^3\ \mathrm{B(x)} = \sum_{k=n}^{1} \frac{m(m-1)(m-2)\alpha_k (x-b)^{m-3}}{(x-a)^k} + 3.2.\beta_{m-3} + \cdots + (m-1)(m-2)(m-3)\beta_1 (x-b)^{m-4}$$

$$D^t B(b) = t!\,.\beta_{m-t} \quad \text{substitution j=m-t}$$

$$\beta_j = \frac{1}{(m-j)!} D^{m-j} B(b)$$

remark: what we should notice about this formula :

$$\frac{P(x)}{(x-a)^n(x-b)^m} = \sum_{k=n}^{1} \frac{1}{(n-k)!}\frac{D^{n-k}A(a)}{(x-a)^k} + \sum_{j=m}^{1} \frac{1}{(m-j)!}\frac{D^{m-j}B(b)}{(x-b)^j}$$

is that we are changing the hard work, instead of solving a (m+n)x(m+n) system of equations, we could just take "simple" derivatives but when when we go back to the definition to see who A(x) and B(x) are, we find that both of them are rational functions and it doesn't sound "so easy" to make high order derivatives of a rational function. The following examples show how to "avoid" this problem with the help of Horner's rule and a clever rearrangement of the expressions A(x) and B(x).

4 Examples

Example 4.1

$$\frac{x^3-2x^2+7x-1}{(x-2)^4(x-3)^3}$$

remark: first of all, we must expand the numerator in terms of the right factor: (x-3) applying Horner's method

	x^3	x^2	x^1	x^0
	1	**-2**	**7**	**-1**
3		**3**	**3**	**30**
	1	**1**	**10**	**29**
3		**3**	**12**	
	1	**4**	**22**	
3		3		
	1	**7**		

$$\frac{1(x-3)^3+7(x-3)^2+22(x-3)+29}{(x-2)^4(x-3)^3}\ \frac{1+7(x-3)^{-1}+22(x-3)^{-2}+29(x-3)^{-3}}{(x-2)^4} = \frac{\mathbf{A(x)}}{(x-2)^4}$$

we could do the same work for the left factor (x-2)

	x^3	x^2	x^1	x^0
	1	**-2**	**7**	**-1**
2		**2**	**0**	**14**
	1	**0**	**7**	**13**
2		**2**	**4**	
	1	**2**	**11**	
2		2		
	1	**4**		

$$\frac{1(x-2)^3+4(x-2)^2+11(x-2)+13}{(x-2)^4(x-3)^3}\ \frac{1(x-2)^{-1}+4(x-2)^{-2}+11(x-2)^{-3}+13(x-2)^{-4}}{(x-3)^3} = \frac{\mathbf{B(x)}}{(x-3)^3}$$

$$\frac{P(x)}{(x-a)^n(x-b)^m}=\sum_{k=n}^{1}\frac{1}{(n-k)!}\frac{D^{n-k}A(a)}{(x-a)^k}+\sum_{j=m}^{1}\frac{1}{(m-j)!}\frac{D^{m-j}B(b)}{(x-b)^j}$$

$$\frac{x^3-2x^2+7x-1}{(x-2)^4(x-3)^3}=\sum_{k=4}^{1}\frac{1}{(4-k)!}\frac{D^{4-k}A(2)}{(x-2)^k}+\sum_{j=3}^{1}\frac{1}{(3-j)!}\frac{D^{3-j}B(3)}{(x-3)^j}$$

$$\frac{A(2)}{(x-2)^4}+\frac{D^1A(2)}{(x-2)^3}+\frac{1}{2!}\frac{D^2A(2)}{(x-2)^2}+\frac{1}{3!}\frac{D^3A(2)}{(x-2)}+\frac{B(3)}{(x-3)^3}+\frac{D^1B(3)}{(x-3)^2}+\frac{1}{2!}\frac{D^2B(3)}{(x-3)}$$

$A(x)$	$1+7(x-3)^{-1}+22(x-3)^{-2}+29(x-3)^{-3}$	$A(2)=-13$
$D^1A(x)$	$-7(x-3)^{-2}-44(x-3)^{-3}-87(x-3)^{-4}$	$D^1A(2)=-50$
$D^2A(x)$	$14(x-3)^{-3}+132(x-3)^{-4}+348(x-3)^{-5}$	$D^2A(2)=-230$
$D^3A(x)$	$-42(x-3)^{-4}-528(x-3)^{-5}-1740(x-3)^{-6}$	$D^3A(2)=-1254$

$B(x)$	$1(x-2)^{-1}+4(x-2)^{-2}+11(x-2)^{-3}+13(x-2)^{-4}$	$B(3)=29$
$D^1B(x)$	$-1(x-2)^{-2}-8(x-2)^{-3}-33(x-2)^{-4}-52(x-2)^{-5}$	$D^1B(3)=-94$
$D^2B(x)$	$2(x-2)^{-3}+24(x-2)^{-4}+132(x-2)^{-5}+260(x-2)^{-6}$	$D^2B(3)=418$

$$\frac{x^3-2x^2+7x-1}{(x-2)^4(x-3)^3}=\frac{A(2)}{(x-2)^4}+\frac{D^1A(2)}{(x-2)^3}+\frac{1}{2!}\frac{D^2A(2)}{(x-2)^2}+\frac{1}{3!}\frac{D^3A(2)}{(x-2)}+\frac{B(3)}{(x-3)^3}+\frac{D^1B(3)}{(x-3)^2}+\frac{1}{2!}\frac{D^2B(3)}{(x-3)}$$

$$\frac{x^3-2x^2+7x-1}{(x-2)^4(x-3)^3}=\frac{\mathbf{-13}}{(x-2)^4}-\frac{50}{(x-2)^3}-\frac{115}{(x-2)^2}-\frac{209}{(x-2)}+\frac{29}{(x-3)^3}-\frac{94}{(x-3)^2}+\frac{209}{(x-3)}$$

Example 4.2

$$\frac{1}{(x-1)^4(x-2)^2}$$

$$\frac{1}{(x-1)^4(x-2)^2}=\frac{A(1)}{(x-1)^4}+\frac{D^1A(1)}{(x-1)^3}+\frac{1}{2!}\frac{D^2A(1)}{(x-1)^2}+\frac{1}{3!}\frac{D^3A(1)}{(x-1)}+\frac{B(2)}{(x-2)^2}+\frac{D^1B(2)}{(x-2)}$$

$$A(x) = (x-2)^{-2}\,, B(x) = (x-1)^{-4}$$

$A(x)$	$(x-2)^{-2}$	$A(1)$ =1
$D^1A(x)$	$-2(x-2)^{-3}$	$D^1A(1)$ =2
$D^2A(x)$	$6(x-2)^{-4}$	$D^2A(1)$ =6
$D^3A(x)$	$-24(x-2)^{-5}$	$D^3A(1)$ =24

$B(x)$	$(x-1)^{-4}$	$B(2)$ =1
$D^1B(x)$	$-4(x-1)^{-5}$	$D^1B(2)$ = - 4

$$\frac{1}{(x-1)^4(x-2)^2} = \frac{1}{(x-1)^4} + \frac{2}{(x-1)^3} + \frac{3}{(x-1)^2} + \frac{4}{(x-1)} + \frac{1}{(x-2)^2} - \frac{4}{(x-2)}$$

5 Conclusion: this method is helpful to turn rational functions into a partial fraction decomposition, avoiding any kind of large system of equations, and if we apply it with the clever rearrengement of the numerator, we could just be able to face higher powers of the denomiator, because this kind of numerator rearrengment allows allways to find an easy pattern for the n-derivatives which involves of course factorials.