Proof of the Collatz Conjecture Using the Div Sequence

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We define the "Div sequence" that sets up the number of times divided by 2 in the Collatz operation. Using this and the "well-founded induction", we prove the Collatz conjecture.

1. Introduction

1.1 Collatz conjecture

Consider the following operation on an arbitrary positive integer:

- If the number is even, divide it by two.
- If the number is odd, triple it and add one.

Now form a sequence by performing this operation repeatedly, beginning with any positive integer, and taking the result at each step as the input at the next. The Collatz conjecture is: This process will eventually reach the number 1, regardless of which positive integer is chosen initially.

Let's call (3x+1) for odd x and iterate to divide by 2 to be **(one) Collatz operation**.
Let's call the number of Collatz operations **initial value**.
The initial value is also called **Collatz value**.

1.2 Div sequence and Complete Div sequence

Definition 1. *When a Collatz operation is performed continuously with a positive odd number n as the initial value, a sequence formed by arranging the number of times divided by 2 in each operation is called Div sequence of n.*

For example, in the case of 9, the sequence of 3x+1 and /2 is
9,28,14,7,22,11,34,17,52,26,13,40,20,10,5,16,8,4,2,1 (stops when 1 is reached),
the Div sequence of 9 is [2,1,1,2,3,4].

- When the Div sequence becomes a finite sequence, it is the same as reaching 1 in a series of Collatz operation.

- If the Div sequence is an infinite sequence,
  It is the same value as one that does not reach 1 in a series of Collatz operation (whether it enters a loop other than 4-2-1 or whether the Collatz value increases endlessly).

Definition 2. *We name a sequence of multiples of 3 as a Complete Div sequence.*

9[2,1,1,2,3,4] is the Complete Div sequence of 9.

7[1,1,2,3,4] is the Div sequence of 7.

Definition 3. *If there is only one element in the Div sequence of n, the Collatz operation cannot be applied to n.*

Theorem 1. *Performing a Collatz operation on x in a Complete Div sequence (having 2 or more elements),*

yields (some) y and its Div sequence.

Proof: It is obvious from the Collatz operation and the definition of the Div sequence.

Theorem 2. *Performing a Collatz operation on y in a Div sequence (having 2 or more elements),*

yields (some) y' and its Div sequence.

Proof: It is obvious from the Collatz operation and the definition of the Div sequence.

1.3 We can check only odd numbers of multiples of 3

We do not need to check even numbers By dividing every even number by 2, we reach either odd number. Therefore, we can check only "whether all odd numbers will reach 1 with Collatz operation". We can check only odd numbers of multiples of 3 For number x which is not divisible by 3, think backwards by Collatz reverse operation. The remainder obtained by dividing x by 9 is one of 1, 2, 4, 5, 7, and 8, this

$$\begin{aligned} 1*2^6 &\equiv 1 \\ 2*2^5 &\equiv 1 \\ 4*2^4 &\equiv 1 \\ 5*2^1 &\equiv 1 \\ 7*2^2 &\equiv 1 \\ 8*2^3 &\equiv 1 \pmod 9 \end{aligned}$$

like, By multiplying 2 by an appropriate number of times, dividing by 9 makes it possible to make it even one more surplus. If you subtract 1 from this and divide by 3 it will be an odd multiple that is a multiple of 3. Tracing back one Collatz operation from x, it is an odd number of multiples of 3. If an odd number of multiples of 3 arrives at 1, x which operated odd numbers of multiples of 3 once in Collatz operation 1. Therefore,

Theorem 3. *All you have to do is investigate "whether an odd number that is a multiple of 3 reaches 1 by Collatz operation".*

2. Star map

Star map is defined. For the complete div sequence, we define the star map. From a complete div sequence of length n to a complete div sequence of length n + 1 (shown later). Too much dividing the Collatz value x by 9,

```
x ≡ 3 mod 9
   The map a finite or infinite sequence [a_1, a_2, a_3, ...]
  into a sequence [6, a_1 -4, a_2, a_3, ...] is written as A[6,-4].
   The map a finite or infinite sequence [a_1, a_2, a_3, ...]
  into a sequence [1, a_1 -2, a_2, a_3, ...] is written as B[1,-2].
x ≡ 6 mod 9
   The map a finite or infinite sequence [a_1, a_2, a_3, ...]
  into a sequence [4, a_1 -4, a_2, a_3, ...] is written as C[4,-4].
   The map a finite or infinite sequence [a_1, a_2, a_3, ...]
  into a sequence [3, a_1 -2, a_2, a_3, ...] is written as D[3,-2].
x ≡ 0 mod 9
   The map a finite or infinite sequence [a_1, a_2, a_3, ...]
  into a sequence [2, a_1 -4, a_2, a_3, ...] is written as E[2,-4].
   The map a finite or infinite sequence [a_1, a_2, a_3, ...]
  into a sequence [5, a_1 -2, a_2, a_3, ...] is written as F[5,-2].
anytime
   The map a finite or infinite sequence [a_1, a_2, a_3, ...]
  into a sequence [a_1 +6, a_2, a_3, ...] is written as G[+6].
```

If the first term of the original sequence is negative, G[+6] is performed beforehand.

example

$117 \equiv 0 (\text{mod } 9)$ 117[5,1,2,3,4]

At this time, E[2,-4] → 9[2,5-4,1,2,3,4] and F[5,-2] → 309[5,5-2,1,2,3,4] can be mapped.

x ≡ 3 mod 9 A[6,-4] y=4x/3-7 B[1,-2] y=x/6-1/2
x ≡ 6 mod 9 C[4,-4] y=x/3-2 D[3,-2] y=2x/3-1
x ≡ 0 mod 9 E[2,-4] y=x/12-3/4 F[5,-2] y=8x/3-3
anytime G[+6] y=64x+21
The function represents the change in Collatz value.

2.1 all complete div sequences are obtained by performing star map on a complete div sequence

Let's see how the Collatz value changes with each star map.

```
3 mod 9
The first term of the div sequence is less than 4 because it is A[6,-4].
A[6,-4]   y=4x/3-7  3+9t  Excludes when t is odd because x is even
    3+18t  (3(3+18t)+1)/2=5+27t  except when t is even
    21+36t  (3(21+36t)+1)/4=16+27t  except when t is odd
    21+72t  (3(21+72t)+1)/8=8+27t  except when t is odd
    21+144t  (3(21+144t)+1)/16=4+27t  except when t is odd
A maps 21+288t to 【21+384t】. 21+288t

The first term of the div sequence is less than 2 because it is B[1,-2].
B[1,-2]   y=x/6-1/2  3+9t  Excludes when t is odd because x is even
    3+18t  (3(3+18t)+1)/2=5+27t  except when t is even
    21+36t  (3(21+36t)+1)/4=16+27t  except when t is odd
B maps 21+72t to 【3+12t】. 21+72t

6 mod 9
The first term of the div sequence is less than 4 because it is C[4,-4].
C[4,-4]   y=x/3-2  6+9t  Excludes when t is even because x is even
    15+18t  (3(15+18t)+1)/2=23+27t  except when t is even
    33+36t  (3(33+36t)+1)/4=25+27t  except when t is even
    69+72t  (3(69+72t)+1)/8=26+27t  except when t is odd
    69+144t  (3(69+144t)+1)/16=13+27t  except when t is even
C maps 213+288t 【69+96t】. 213+288t
```

The first term of the div sequence is less than 2 because it is D[3,-2].
D[3,-2]   y=2x/3-1 6+9t Excludes when t is even because x is even
    15+18t (3(15+18t)+1)/2=23+27t except when t is even
    33+36t (3(33+36t)+1)/4=25+27t except when t is even
D maps 69+72t 【45+48t】. 69+72t

0 mod 9
The first term of the div sequence is less than 4 because it is E[2,-4].
E[2,-4]   y=x/12-3/4 9t Excludes when t is even because x is even
    9+18t (3(9+18t)+1)/2=14+27t except when t is odd
    9+36t (3(9+36t)+1)/4=7+27t except when t is even
    45+72t (3(45+72t)+1)/8=17+27t except when t is even
    117+144t (3(117+144t)+1)/16=22+27t except when t is odd
E maps 117+288t to 【9+24t】. 117+288t

The first term of the div sequence is less than 2 because it is F[5,-2].
F[5,-2]   y=8x/3-3 9t Excludes when t is even because x is even
    9+18t (3(9+18t)+1)/2=14+27t except when t is odd
    9+36t (3(9+36t)+1)/4=7+27t except when t is even
F maps 45+72t to 【117+192t】. 45+72t

G[+6]   y=64x+21 3+6t
G maps 3+6t to 【213+384t】. 3+6t

We can see that every maps is from 3+6t to 3+6t.

2.2 composition

G【213+384t】┐ ─ [21+192t] ──┐ ─ [21+96t] ──┐ ─ [21+48t] ──┐ ─ [21+24t]
A【21+384t】┘ F【117+192t】┘ C【69+96t】┘ D【45+48t】┘ E【9+24t】
──── [9+12t] ──┐ ─ [3+6t] ☆☆☆
B【3+12t】┘

Therefore,

Theorem 4. *All complete div sequences are obtained by performing star map on a complete div sequence.*

If all the complete div sequences are of finite length, the Collatz conjecture is also true.

3. Extended Star map and Extended Complete Div sequence

Extended Star map is defined.

Definition 4. *If the Collatz value x is not 3,9, for a complete div sequence of x, Consider the following map in which the star map is applied multiple times. We call this the extended star map.*

```
No.  case  Extended star map  after
    0 mod 9
1  9  none
2  72t+45  E[2,-4] y=x/12-3/4  6t+3
3  216t+81  DE[3,0,-4] y=x/18-3/2  12t+3
4  216t+153  AE[6,-2,-4] y=x/9-8  24t+9
5  216t+225  FE[5,0,-4] y=2x/9-5  48t+45
6  108t+27  CF[4,1,-2] y=8x/9-3  96t+21
7  108t+63  BF[1,3,-2] y=4x/9-1  48t+27
8  108t+99  EF[2,1,-2] y=2x/9-1  24t+21
    6 mod 9
9  18t+15  C[4,-4] y=x/3-2  6t+3
    3 mod 9
10  3  none
11  36t+21  B[1,-2] y=x/6-1/2  6t+3
12  108t+39  DB[3,-1,-2] y=x/9-4/3  12t+3
13  108t+75  AB[6,-3,-2] y=2x/9-23/3  24t+9
14  108t+111  FB[5,-1,-2] y=4x/9-13/3  48t+45
The extended star map copies the initial value from 6t+3 to 6t'+3.
```

Definition 5. *The div sequence obtained by applying the extended star map is named an extended complete div sequence. The elements of the extended complete div sequence may contain 0 or a negative value.*

Definition 6. *Collatz value and its div sequence series (n, [a, b, ...]) to* $((3n+1)/2^a$*, [b, ...]) map, We call it Extended Collatz Operation.*

Definition 7. *After extended Collatz map, collatz value* $(Oddnumbernotamultipleof3)/2^r$ *(r ≠ 0) The div sequence when it becomes is called the extended div sequence.*

Theorem 5. *If the extended star map of x other than 3,9 is applied to the extended star map, the extended complete div sequence of x'is obtained.*

Proof: It is trivial from the definition of the extended star map.

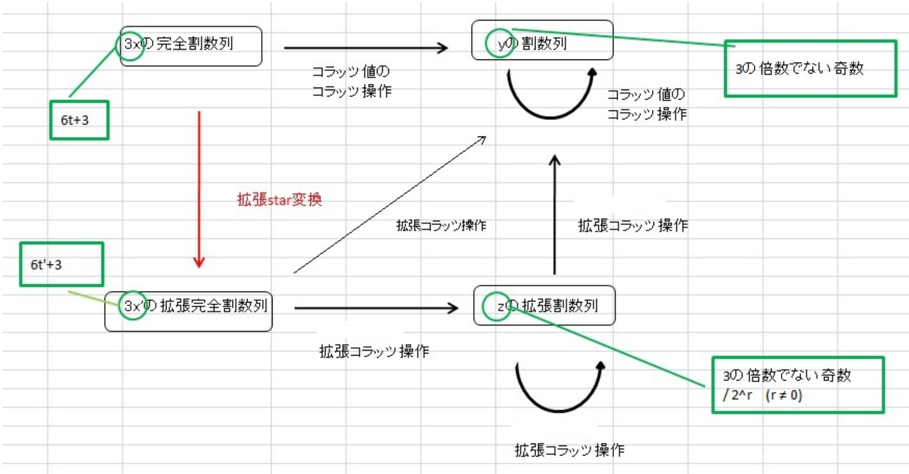
Theorem 6. *If you apply the extended Collatz operation to the extended complete div sequence of x, you will get the extended div sequence of z or the div sequence of y.*

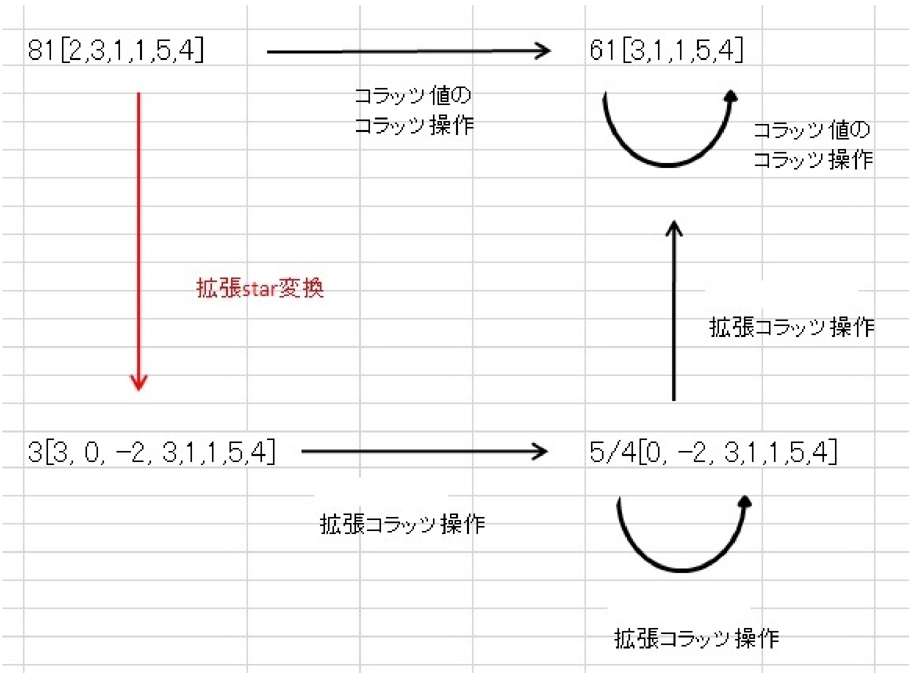
Proof: If the Collatz value after the operation is a natural number y, the div sequence of y is obtained. Otherwise, an extended div sequence of z is obtained.

Theorem 7. *When the extended Collatz operation is performed on the extended div sequence of z, the expanded div sequence of z'or the div sequence of y is obtained.*

Proof: If the Collatz value after the operation is a natural number y, the div sequence of y is obtained. Otherwise, an extended div sequence of z'is obtained.

The following diagram can be drawn by summarizing Theorem 1-1 to Theorem 1-2 and Theorem 3-1 to Theorem 3-3.





Theorem 8. *From a certain Collatz value (y for the Collatz operation once), If you perform expanded Collatz operation multiple times by converting the expanded star once, it will return to y.*

Proved with Egison.

4. divSeq and allDivSeq

4.1 divSeq

divSeq t is a function that returns a complete divisor sequence of 6t + 3 with 6t + 3 as the Collatz value. By using CoList, both finite length and infinite length can be obtained.

```
divSeq : Nat -> CoList Integer
divSeq n = divSeq' (S (S (S (n+n+n+n+n+n)))) (S (S (S (n+n+n+n+n+n)))) where
  divSeq' : Nat -> Nat -> CoList Integer
  divSeq' n     Z     = []
  divSeq' Z     (S k) = []
  divSeq' (S n) (S k) with (parity n)
    -- We don't actually come here
    divSeq' (S (S (j + j))) (S k) | Odd  = divSeq' (S j) k
    divSeq' (S (j + j))     (S k) | Even =
      map toIntegerNat
        (unfoldr
          (\b => if b <= 1 then Nothing
                                else Just (countEven (b*3+1) (b*3+1) 0) ) (S (j + j)))
```

4.2 allDivSeq

allDivSeq t is 6t+3 as a Collatz value, 6t+3 is a complete div sequence, and This is a function that returns all extended complete div sequences of 6t+3.

5. well-founded

For the proof, we use the foundation induction library wfInd, step.

- S is a function that adds +1 to a natural number. Z is 0.
- (★ 1) wfInd is the original function of foundational induction. Pass step to this.
- (★ 2) step is a function implemented by the user. You can use the function rs whose type is ((y: Nat) → LT 'y x → P y).
- (★ 3) LT 'y x means y < x.
- (★ 4) In each case, Pass the proof of LT 'y x to the function rs to get P y. Apply firstToAll and IsFirstLimited ** there to get P x.

6. Final proof of theorem

Theorem 9. *makeLimitedDivSeq : (n : Nat) → ((z : Nat) → ((FirstLimited . allDivSeq) z → (AllLimited . allDivSeq) z)) → (FirstLimited . allDivSeq) n*

Proof by using well-founded induction and case classification. Proof by using the foundation and induction. For the number of cases

(1) use rs to reduce the number,

(2) Change the predicate to AllLimited using firstToAll,

(3) Undo predicates and numbers using IsFirstLimitedxx.

(6t+3 in the table below is n in the source code)

```
No.  case  Extended star map  after  become smaller
  0 mod 9
1 9 none
2 72t+45 E[2,-4] y=x/12-3/4 6t+3 72t+45 > 6t+3
3 216t+81 DE[3,0,-4] y=x/18-3/2 12t+3 216t+81 > 12t+3
4 216t+153 AE[6,-2,-4] y=x/9-8 24t+9 216t+153 > 24t+9
5 216t+225 FE[5,0,-4] y=2x/9-5 48t+45 216t+225 > 48t+45
6 108t+27 CF[4,1,-2] y=8x/9-3 96t+21 108t+27 > 96t+21
7 108t+63 BF[1,3,-2] y=4x/9-1 48t+27 108t+63 > 48t+27
8 108t+99 EF[2,1,-2] y=2x/9-1 24t+21 108t+99 > 24t+21
  6 mod 9
9 18t+15 C[4,-4] y=x/3-2 6t+3 18t+15 > 6t+3
  3 mod 9
10 3 none
11 36t+21 B[1,-2] y=x/6-1/2 6t+3 36t+21 > 6t+3
12 108t+39 DB[3,-1,-2] y=x/9-4/3 12t+3 108t+39 > 12t+3
13 108t+75 AB[6,-3,-2] y=2x/9-23/3 24t+9 108t+75 > 24t+9
14 108t+111 FB[5,-1,-2] y=4x/9-13/3 48t+45 108t+111 > 48t+45
```

Theorem 10. *limitedDivSeq : (n : Nat) → (FirstLimited . ProofColDivSeqBase.allDivSeq) n*

Prove the sufficient condition of makeLimitedDivSeq.

Acknowledgment

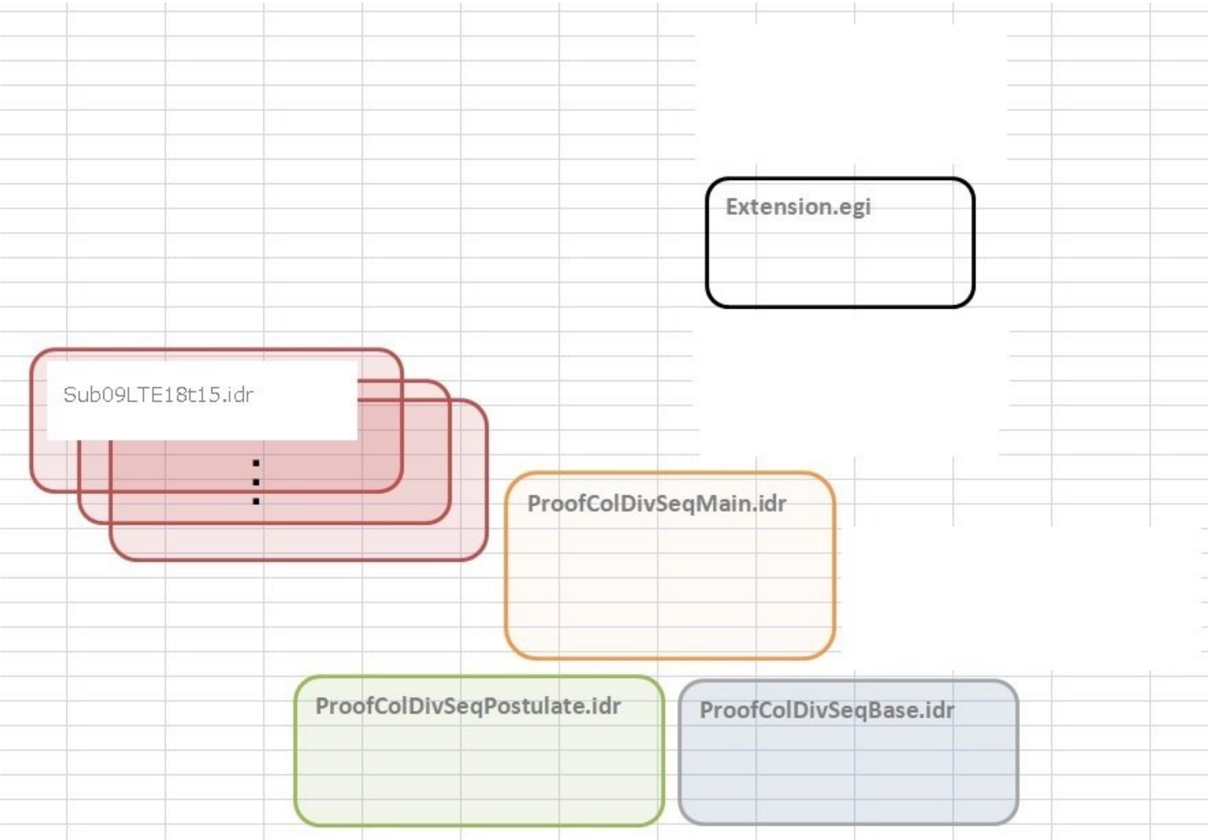
I thank everyone who read this paper.

In 5ch "コラッツ予想がとけたらいいな その2", I gave a meaningful opinion.

In particular, I am grateful to "前786 ◆ 5A/gU5yzeU" san.

Appendix: Description of each file

A.1 image



A.2 other

A.2.1 Extension.egi

Theorem 4-1

From a certain Collatz value (α that was operated once Collatz)

・ Once Extended Star map has been performed multiple times, expanded Collatz operation will return to α

Is proof of. We're using Egison.

A.3 Idris

A.3.1 ProofColDivSeqBase.idr

We are making the base part. divSeq, etc.

A.3.2 ProofColDivSeqMain.idr

This is the main processing. Using well-founded, 14 patterns of cases are classified and proved.

A.3.3 ProofColDivSeqPostulate.idr

We put a postulate proposition, but since We proved everything, it is now an empty file.

A.3.4 Sub02... Sub14....idr

The case of 14 patterns of the main function is divided and processed for each file. The proposition is of the form $a \leq b$. For these reasons, these files are dirty for technical reasons.

References

1) Lagarias, Jeffrey C., ed. (2010). The Ultimate Challenge: The 3x+1 Problem, American Mathematical Society, ISBN 978-0-8218-4940-8, Zbl 1253.11003.