

Interval Sieve Algorithm

Creating a Countable Set of Real Numbers from a Closed Interval

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I. Abstract

The Interval Sieve Algorithm is a method for generating a list of real numbers on any closed interval of real numbers $[r_i, r_j]$ where $r_i < r_j$. Georg Cantor in his 1891 paper demonstrated a constructive proof that the real numbers are uncountable. Cantor developed a method for showing how a particular objective cannot be accomplished; in this case, establishing a one to one correspondence between the natural numbers and the real numbers, using his diagonal argument.

What Cantor didn't show is that there are no ways of demonstrating a one to one correspondence between the natural numbers and the real numbers. This is important because even if one can demonstrate one or more ways that something cannot be done, it is only necessary to develop one way that shows how it can be done to invalidate the ways that show it cannot.

The interval sieve algorithm partitions a closed interval of real numbers $[r_i, r_j]$ where $r_i < r_j$ to create a complete list, L , of numbers in the interval. We will prove that the list L is complete, and derive the bijective function $f: \mathbb{N} \rightarrow [r_1, r_2]$.

II. Definitions

1. The **lower bound** of an interval is the leftmost member of the interval. In the interval $[r_1, r_2]$, r_1 is the lower bound of the interval.
2. The **upper bound** of an interval is the rightmost member of the interval. In the interval $[r_1, r_2]$, r_2 is the upper bound of the interval.
3. Given the set, $S = \{1, 2, 3\}$. We define a **closed interval** of the set as $I_S = [1, 3]$ where both upper and lower bounds are included in the interval.
4. Given the set, $S = \{1, 2, 3\}$. We define an **open interval** of the set as $I_S = (1, 3)$ where the upper and lower bounds are excluded from the interval.
5. Given the set, $S = \{1, 2, 3\}$. We define a **lower open interval** of the set as $I_S = (1, 3]$ where the lower bound is excluded from the interval and the upper bound is included in the interval.
6. Given the set, $S = \{1, 2, 3\}$. We define an **upper open interval** of the set as $I_S = [1, 2)$ where the lower bound is included in the interval and the upper bound is excluded from the interval.
7. A **conjoined interval pair** is a pair of intervals where the upper bound of one and the lower bound of the other are the same member. $[r_i, [r_k, r_j]$ is an example of a conjoined interval pair where r_k is both the upper bound of $[r_i, r_k]$, the lower bound of $[r_k, r_j]$ and $r_i < r_k < r_j$.

8. A **relative bound** is the number that is common to both intervals in a conjoined interval pair. In the conjoined interval pair $[r_1, [r_3, r_2]$, r_3 is the relative bound in both intervals $[r_1, r_3]$ and $[r_3, r_2]$.

9. An interval of a set may be **partitioned** by creating a conjoined interval pair per definition 7 and then splitting the conjoined interval pair into sub-intervals with the relative bound being the upper bound of one sub-interval and the lower bound of the other sub-interval.

Example:

$$S = \{1, 2, 3\}$$

$$I_S = [1, 3] \quad (I_S \longrightarrow \text{the interval } I \text{ on set } S)$$

Partition I_S as follows -

$$I_S = [1, 3]$$

$$= [1, [2], 3]$$

$$= [1, 2], [2, 3]$$

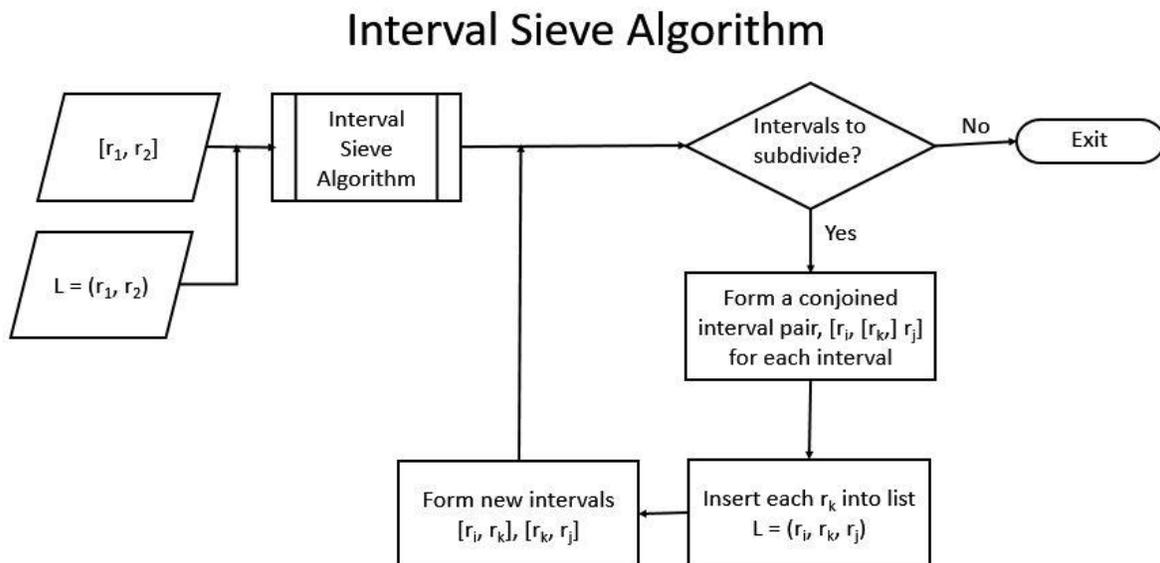
10. When no sub-intervals can be further subdivided then the interval is called **fully partitioned**.

11. The **immediate predecessor** of a number λ is a number β such that there exists no number δ where $\beta < \delta < \lambda$.

12. The **immediate successor** of a number λ is a number β such that there exists no number δ where $\lambda < \delta < \beta$.

13. For any 2 real numbers λ and β in $[r_1, r_2]$, we can always find another real number, δ , such that if $\lambda > \beta$ then $\beta < \delta < \lambda$ and if $\lambda < \beta$ then $\lambda < \delta < \beta$. Therefore from definitions 11 and 12 we know that there are no immediate predecessors or successors of any of the elements of $[r_1, r_2]$; that is, $[r_1, r_2]$ is a **continuum**.

III. The Interval Sieve Algorithm



Procedure:

0. We begin the procedure given the interval $[r_1, r_2]$ where $r_1 < r_2$ and r_1, r_2 are real numbers and the list $L = (r_1, r_2)$.

1. If there are intervals that can be sub-divided next step else stop..

2. Sub-divide each interval $[r_i, r_j]$ by selecting a number r_k such that $r_i < r_k < r_j$ to get a conjoined interval pair $[r_i, [r_k, r_j]$

3. Insert the relative bound number, r_k , into the list L to get $L = (r_i, r_k, r_j)$.

4. Form new sub-intervals $[r_i, r_k], [r_k, r_j]$.

5. Return to step 1.

IV. Partitioning Intervals Using the Algorithm

We will use the partitioning of a set interval according to the algorithm as a method for creating a list of the elements of the interval. Partitioning allows us to construct a list whereby all the elements of the interval will be included in the list.

Below are two examples of partitioning set intervals.

Example 1 -

Let:

$$S = \{1, 2, 3, 4, 5\}$$

$$I_s = [1, 5]$$

First we note that in defining the interval $[1, 5]$ we have specified the first two elements of the list L . That is $L = (1, 5)$. To continue populating L , partition $[1, 5]$, adding each relative bound to L as it is created, and continue the process until no intervals are left that can be partitioned.

$$I_s = [1, 5]$$

$$L = (1, 5)$$

$$= [1, [3], 5]$$

$$L = (1, 3, 5)$$

$$= [1, 3], [3, 5]$$

$$= [1, [2], 3], [3, [4], 5]$$

$$L = (1, 2, 3, 4, 5)$$

$$= [1, 2], [2, 3], [3, 4], [4, 5]$$

Since no interval in I_s can be further subdivided the interval is fully partitioned and L is complete.

Example 2 -

Let:

$$S = \{1, 2, 3, \dots \omega\}$$

where ω is the first infinite ordinal number.

$$I_s = [1, \omega)$$

Partition the interval $[1, \omega)$ to create a list L of natural numbers.

$$\begin{array}{ll} I_s = [1, \omega) & L = (1) \\ = [1, [2], \omega) & L = (1, 2) \\ = [1, 2], [2, \omega) & \\ = [1, 2], [2, [3], \omega) & L = (1, 2, 3) \\ = [1, 2], [2, 3], [3, \omega) & \\ \dots & \dots \end{array}$$

Taking the procedure to its limit will create a list of all the natural numbers, $L = (1, 2, 3, \dots)$. Since I_s is defined as an upper open interval, ω is not included in L.

V. Creating L over $[r_1, r_2]$ where r_1, r_2 are Real Numbers

Let:

$$S = \{\mathbb{R}\}$$

$$I_s = [r_1, r_2] \text{ where } r_1 < r_2$$

Partition I_s to create the list L of real numbers between r_1 and r_2 .

$$\begin{array}{ll} I_s = [r_1, r_2] & L = (r_1, r_2) \\ = [r_1, [r_3], r_2] & L = (r_1, r_3, r_2) \\ = [r_1, r_3], [r_3, r_2] & \\ = [r_1, [r_4], r_3], [r_3, [r_5], r_2] & L = (r_1, r_4, r_3, r_5, r_2) \\ = [r_1, r_4], [r_4, r_3], [r_3, r_5], [r_5, r_2] & \\ = [r_1, [r_6], r_4], [r_4, [r_7], r_3], [r_3, [r_8], r_5], [r_5, [r_9], r_2] & L = (r_1, r_6, r_4, r_7, r_3, r_8, r_5, r_9, r_2) \\ = [r_1, r_6], [r_6, r_4], [r_4, r_7], [r_7, r_3], [r_3, r_8], [r_8, r_5], [r_5, r_9], [r_9, r_2] & \\ \dots & \dots \end{array}$$

At the limit of the process L will appear as follows: $L = (r_1, \dots, r_6, \dots, r_4, \dots, r_7, \dots, r_3, \dots, r_8, \dots, r_5, \dots, r_9, \dots, r_2)$.

By definition 13 there are no immediate predecessors or successors in $[r_1, r_2]$. It follows that the partitioning of sub-intervals of $[r_1, r_2]$ can go on indefinitely. Also, except for r_1 and r_2 , every number in the original interval must, at some point during the process, become a relative bound and only then added to L. And because no number will be a relative bound more than once, there will be no duplicates in L.

As can be seen in the examples above, each relative bound becomes the lower bound of one sub-interval and the upper bound of another sub-interval. This means that every number in the each

sub-interval will be approached from left below and right its value and the interval lengths will become infinitesimally small.

Example 3, using numbers:

Let:

$$S = \{\mathbb{R}\}$$

$$I_s = [1, 4]$$

Partition I_s to create the list L of real numbers between 1 and 4.

$$\begin{aligned}
 I_s &= [1, 4] & L &= (1, 4) \\
 &= [1, [\pi], 4] & L &= (1, \pi, 4) \\
 &= [1, \pi], [\pi, 4] \\
 &= [1, [e], \pi], [\pi, [3.2], 4] & L &= (1, e, \pi, 3.2, 4) \\
 &= [1, e], [e, \pi], [\pi, 3.2], [3.2, 4] \\
 &= [1, [\sqrt{2}], e], [e, [3], \pi], [\pi, [3.15], 3.2], [3.2, [3.3], 4] & L &= (1, \sqrt{2}, e, 3, \pi, 3.15, 3.2, 3.3, 4) \\
 &= [1, \sqrt{2}], [\sqrt{2}, e], [e, 3], [3, \pi], [\pi, 3.15], [3.15, 3.2], [3.2, 3.3], [3.3, 4] \\
 &\dots
 \end{aligned}$$

At the limit of the process $L = (1, \dots, \sqrt{2}, \dots, e, \dots, 3, \dots, \pi, \dots, 3.15, \dots, 3.2, \dots, 3.3, \dots, 4)$.

VI. Proving the List L is Complete

The question remains as to whether or not the list L will contain all real numbers in $[r_1, r_2]$. We will prove that: All the real numbers in $[r_1, r_2]$ are contained in the list L .

Proof by Construction/Contradiction: Create a number X such that $r_1 < X < r_2$ and assume that $X \notin L$. Demonstrate that ' $r_1 < X < r_2$ and that $X \notin L$ ' leads to a contradiction.

To create X we employ Cantor's diagonal method:

Row	$[r_1, r_2]$							
1	.d ₁	d ₂	d ₃	d ₄	d ₅	d ₆	d ₇	...
2	.d ₁	d ₂	d ₃	d ₄	d ₅	d ₆	d ₇	...
3	.d ₁	d ₂	d ₃	d ₄	d ₅	d ₆	d ₇	...
4	.d ₁	d ₂	d ₃	d ₄	d ₅	d ₆	d ₇	...
5	.d ₁	d ₂	d ₃	d ₄	d ₅	d ₆	d ₇	...
6	.d ₁	d ₂	d ₃	d ₄	d ₅	d ₆	d ₇	...
7	.d ₁	d ₂	d ₃	d ₄	d ₅	d ₆	d ₇	...
...			...					
	X =	b ₁	b ₂	b ₃	b ₄	b ₅	b ₆	b ₇ ...

Each b_i of X differs from each number in the list at d_i . Using the diagonal method we have created a number X such that $r_1 < X < r_2$ and $X \notin L$ (Cantor's proof).

1. Since X is an element of $[r_1, r_2]$ we can rewrite $[r_1, r_2]$ as $[r_1, \dots, X, \dots, r_2]$.
2. We can now select X as the first cut point in the partitioning of $[r_1, r_2]$ and form the relative bound of the conjoined interval pair $[r_1, [X], r_2]$.
3. Once X is designated a relative bound it will be inserted into L per the algorithm and we'll have $L = (r_1, X, r_2)$.
4. Having shown that $X \in L$, we can say that the original assertion, $X \notin L$, leads to a contradiction and must be false.
5. Since the list of $[r_1, r_2]$ created using the diagonal method was lacking X , and we have shown that by partitioning $[r_1, r_2]$ using the interval sieve includes X as a member of the list of $[r_1, r_2]$ we can assert that, at the limit, L will be complete and this ends the proof.

VII. Derivation of $f: \mathbb{N} \rightarrow [r_1, r_2]$

We will now demonstrate that there exists a bijective function from \mathbb{N} to $[r_1, r_2]$, $f: \mathbb{N} \rightarrow [r_1, r_2]$.

We have used the Interval Sieve Algorithm to create: $L = (r_1, \dots, r_6, \dots, r_4, \dots, r_7, \dots, r_3, \dots, r_8, \dots, r_5, \dots, r_9, \dots, r_2)$

and have proved L is complete. It is readily apparent that for every r in the list there is an associated natural number subscript. Since L is complete, containing all numbers in $[r_1, r_2]$ and each number in L is associated with a single unique natural number we can assert that $f: \mathbb{N} \rightarrow [r_1, r_2]$ exists.

The existence of $f: \mathbb{N} \rightarrow [r_1, r_2]$ confirms a one to one correspondence between the natural numbers and any closed interval of real numbers.

VIII. Final Thoughts

The fact that we have shown the existence of $f: \mathbb{N} \rightarrow [r_1, r_2]$ implies that Cantor's continuum hypothesis is true for closed intervals of real numbers. Interestingly, it's not that there are no infinite sets with cardinality between \aleph_0 and \aleph_1 , rather \mathbb{N} and $[r_1, r_2]$ turn out to be the same size.