# Relative Measurement Theory[1]

## The unification of experimental and theoretical measurements

by
Ken Krechmer[2]
University of Colorado
Interdisciplinary Telecommunications Program
Boulder, CO 80309 USA
krechmer@isology.com

*Abstract:* *The discontinuous, non-causal and instantaneous changes due to a measurement that appear in quantum mechanics (QM) theory are not consistent with a classical understanding of physical reality, but are completely confirmed by experiments. Relative measurement theory explains why. This paper presents the first formal development of an experimental measurement which includes the uncertainty due to calibration and resolution. The uncertainty due to calibration and resolution, previously considered experimental artifacts, is shown to be equal to the uncertainty that appears in QM theory and experiment. When the calibration to a reference and resolution effects are considered, all the QM measurement discontinuities are consistent with classical explanations.*

*Keywords:* measurement uncertainty, measurement disturbance, wave function collapse, entanglement, calibration, reference.

[1] A precursor paper, Relational Measurements and Uncertainty, was published in Measurement Volume 93, November 2016, Pages 36–40. Relative Measurement Theory offers an expanded, clearer and more rigorous development than the precursor paper.

[2] Permanent address: Palo Alto, CA, USA

## 1.0 INTRODUCTION

In quantum mechanics (QM) theory "...the discontinuous, non-causal and instantaneously acting experiments or measurements" [1] create: uncertainty – when measurements of an unchanged observable change [2], disturbance – measuring one observable disturbs a different observable [3], collapse – experimental results have a lower entropy than QM theory predicts [4], and entanglement – measurement results transfer faster than the velocity of light [5]. Strangely, experiments completely support these unreasonable results [6]. And the wave function (the basis of QM theory) is a complete success at describing the probabilities of a quantum system. This agreement of extensive experiments and successful QM theory has caused many to believe that quantum mechanics is not consistent with classical mechanics, i.e., QM is not reasonable in terms of human experience.

In 1935, the EPR paper [7] proposed that the wave function must be an incomplete description of physical reality. The belief expressed in the EPR paper is that physical reality has underlying consistency and it is a fundamental task of physics to formalize this consistency [5]. Whether or not quantum and classical mechanics are consistent has been considered and tested extensively starting before 1935, without a clear resolution.

This paper develops the first formal measurement function [8] that includes calibration and resolution (Section 2.0-2.2), converts probabilistic QM measures to experimental measurement results (Section 2.3), explains how relative measurement theory resolves the unreasonable results (Sections 3 - 5), completes the QM description of physical reality (Section 5), and concludes that all mechanics are consistent (Section 6).

L. Euler [9] identifies that any measurement result is only relative to another measurement result. Fig. 1 presents the minimum empirical single axis *relative measurement system* [10] including three entities: observables[3], measuring apparatus with finite intervals and a reference. In Fig. 1, what is accepted in QM theory is the top half and unshaded. What this paper adds is the bottom half and shaded grey.

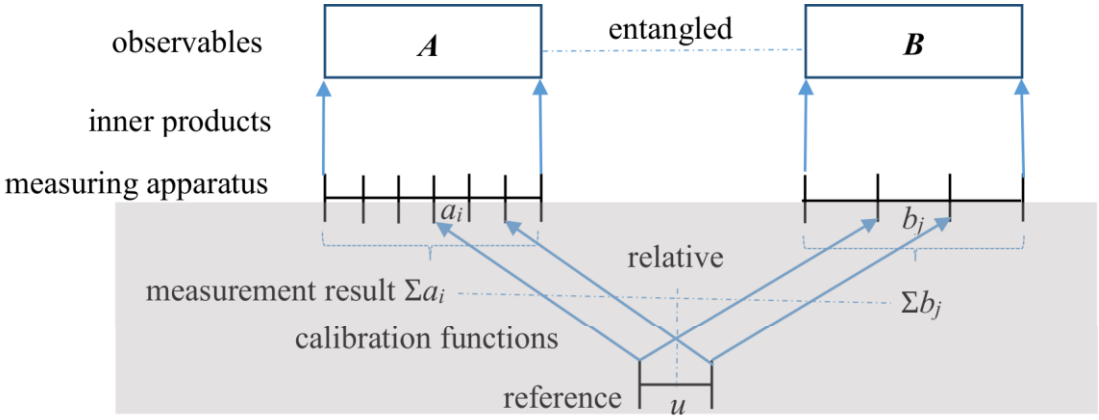


Fig. 1. Relative measurement system

[3] An observable has a magnitude of intervals, each with an interval vector magnitude. In this paper each interval vector magnitude is an independent variable.

Each measuring apparatus is projected (vertical arrows) on each observable $\boldsymbol{A}$ and $\boldsymbol{B}$, establishing the $A$ and the $B$ vector magnitudes in intervals of $a_i$ and $b_j$. The reference $u$ is tightly correlated (relative) by c*alibration* (diagonal arrows) to each experimental measuring apparatus interval (MAI). Calibration defines the interval vector magnitude of each $a_i$ or $b_j$. Fig. 1 does not include resolution effects.

An experimental measurement result of an observable is the sum of each MAI magnitude (e.g., a centimeter ± uncertainty). Often measurement results are assumed to be the product of the vector magnitude (e.g., $A$) of the intervals times the mean ($\langle . \rangle$) interval magnitude which is $A\langle a_i \rangle$. Or measurement results may be assumed to be $Au$. When each $a_i$ is not exactly equal to $u$, or when all $a_i$ are not equal, or when the distribution of $a_i$ is not symmetrical about $u$, i.e., $u \neq \langle a_i \rangle$, or when $\langle a_i \rangle \neq \langle \text{MAI} \rangle$, then:

$$Au \neq A\langle a_i \rangle \neq \Sigma a_i \neq \Sigma \text{MAI} \qquad (1)$$

The unequalities in (1) occur when these distributions are not symmetrical to each other. None of these unequalities are recognized in existing measure theory [11] as they are assumed to be related to the experiment. Only $\Sigma$MAI in (1) describes experimental results. The uncertainty caused by these unequalities can be significant.

When two calibration functions occur (which correlate each MAI of each measuring apparatus to one reference), measurement result $\Sigma a_i$ and measurement result $\Sigma b_j$ and the reference become relative to each other (thin dash-dot line) and can be compared via a common factor of $u$ (*common reference*).

*Relative* describes the now corrolated relation of the measurement result's intervals , e.g., $2a_i = u = b_j$, as well as the now correlated relation of the measurement result's relative magnitudes ($A$=6 and $B$=3) in $u_i$, i.e., $6a_i = 3b_j$. Fig. 1 does not include the uncertainty of the MAI. In QM theory the observables are termed entangled (thin dash-dot line) when the measurement results of two separated observables remain relative to each other. Entanglement is more formally developed in Section 2.1, below.

In metrology (the science of experimental measurement), calibration to a reference establishes the correlation between measurement results and decreases the uncertainty of distributions of measurement results. QM theory is based upon a measure theory [11] which does not consider calibration in a *reference space*[4]. A relative measurement theory (RMT) is needed.

## 2.0 FORMAL MEASUREMENT

In an experimental measurement, the MAI and their coordinate axes are defined by the SI (International System of Units) [12]. The SI is the *experimental reference space*. This reference space must be applied in a measurement theory which describes experimental results. An MAI is correlated to the appropriate SI standard(s) using metrology. But each MAI is not exactly

[4] Reference space describes a vector space that also stipulates the discrete intervals applied for measures or measurements.

equal to the others from the same measuring apparatus or exactly equal to the appropriate SI standard(s).

From Fig. 1, a measurement includes the inner product function and calibration function required to establish a comparable measurement result. In this paper, the magnitudes of observables and their intervals are formalized without consideration for interactions with the measuring apparatus (i.e., observer effects) or any external effects such as noise. In carefully designed inner product and calibration experiments, these observer and external effects may be minimized or canceled and are not considered inherent.

## 2.1. INNER PRODUCT FUNCTION

An observable (e.g., $\boldsymbol{A}$ or $\boldsymbol{B}$ ) exists prior to any relative relation. Therefore the observable is a norm or unity. Norms (***bold***) are self-relative and represent all the magnitude possibilities. In QM theory an observable is a superposition of complex amplitudes which represents all the magnitude possibilities.

The inner product function converts an observable's norm to a magnitude of interval norms. The measuring apparatus's intervals, before the calibration function, are norms, $\boldsymbol{u}_i$. A *measure* (observable's magnitude in $\boldsymbol{u}_i$) is calculated when each interval of the measuring apparatus ($\boldsymbol{u}_i$) which projects on the observable is counted. In Fig. 2 each projection is indicated by upward arrows.

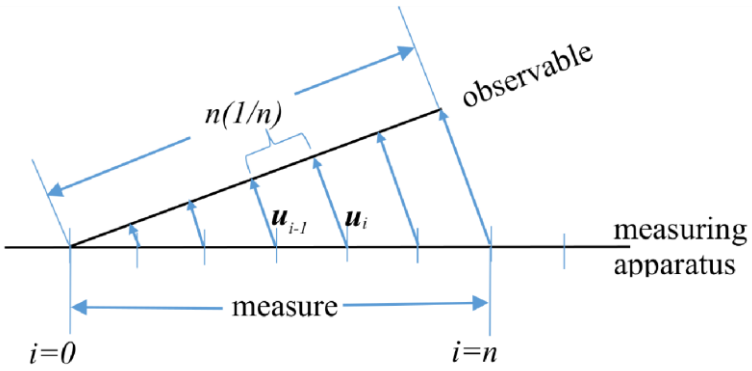


Fig. 2. Inner product function

Equation (2) formalizes Fig. 2, as a sum of inner products $\langle .,. \rangle$ where $i \in \{1,2,\ldots,n\}$ [13].

$$\sum_{i=1}^{i=n} \left\langle {}^{1}\!/_{n}, \boldsymbol{u}_i \right\rangle = \text{ magnitude (e.g., } A \text{ of } \boldsymbol{A} \text{ ) in } \boldsymbol{u}_i \quad (2)$$

Equation (2) may also be formalized in bra-ket notation [14]. Since $\boldsymbol{u}_i - \boldsymbol{u}_{i-1} = {}^{1}\!/_{n}$ and $N$ is a vector magnitude expressing the sum of $n$ equal intervals of both the observable and the measuring apparatus, then Fig. 2 provides a derivation of the Born Rule [15]. The Born Rule identifies that the inner product of the bra and the ket in (3) is the probability amplitude of the magnitude of a measure in $\boldsymbol{u}_i$.

$$\sum_{i=1}^{i=n}\left\langle \tfrac{1}{n}, \boldsymbol{u}_i \right\rangle = \langle N \,|\, N \rangle \qquad (3)$$

From Fig. 2, the inner product (2) makes the observable and the measuring apparatus in (3) entangled. That is, when they are separated, their relative absolute magnitude relationship, $|N| = |N|$, continues. Fig. 1 shows that two observables, each entangled with different, but relative, measuring apparatus, are also entangled with each other. Equation (3) is an indication that Fig. 2 is valid.

An example of entanglement in classical mechanics: A blindfolded carpenter takes two dowels, $\boldsymbol{A}$ and $\boldsymbol{B}$, aligns by feel one end of the two dowels and makes one saw cut across both dowels. The alignment (zero setting) and saw cut (projection) is an inner product which entangles the dowels. Each dowels' length magnitude, before it is entangled is "one dowel", a norm or all the possible dowel lengths. The entangled dowels, when separated to anywhere in space-time, have an equal, previously unknown, length magnitude, $|A| = |B|$, relative to a stipulated $u$.

It is not the purpose of this paper to describe how QM observables become entangled, only to explain that when the entangled observables are separated in space-time, the continued correlation of their measurement magnitude is not "spooky action at a distance" [5].

The inner product measure (2) is in a reference (Hilbert) space. The linear result of an inner product is a scalar measure which is not relative to a coordinate axis. A non-linear calibration function which changes the reference space to an SI reference space is required to transform a scalar measure into a measurement result.

## 2.2 CALIBRATION FUNCTION

Equations (2), (3), and Fig. 2 present each $\boldsymbol{u}_i$ as normalized and equal. Then multiple measures of a fixed observable appear as a probability distribution, not a distribution of experimental measurement results. Exactly equal $u_i$ are not possible in any experimental measuring apparatus [16]. Therefore a measurement result is the $\Sigma u_i$ in theory and $\Sigma$MAI in experiments, not the sum of $\Sigma \boldsymbol{u}_i$ interval norms. To convert Fig. 2 to an SI measurement result, a separate calibration function correlates $u$ to $\boldsymbol{u}_i$, creating the *applied references* $u_i \cong$ MAI, where $\cong$ indicates equal with an uncertainty, as an exact (in theory) measurement result of each MAI is not possible (experimentally) due to the measuring apparatus resolution (Fig. 3).

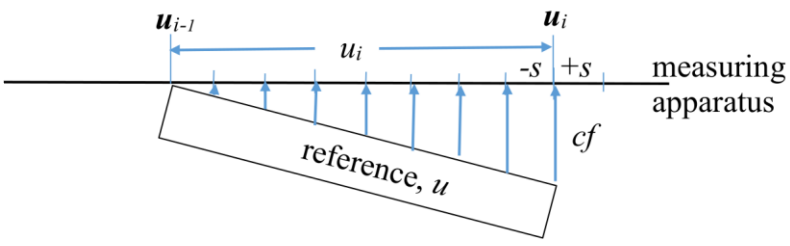


Fig. 3. Calibration function $\boldsymbol{u}_i \rightarrow u_i$

From Fig. 3, the calibration function (*cf*) projects each $u$ to transform the $i^{th}$ $\boldsymbol{u}_i$ to the $i^{th}$ $u_i$ with an inherent uncertainty $\Delta u_i + \Delta s$ :

$$\boldsymbol{u}_i \rightarrow u_i = u + \Delta u_i + \Delta s \quad (4)$$

By stipulation, $u$ may be a primary standard, e.g., a meter, kilogram or second, or correlated to a primary standard, or $u$ may be a common reference, where $u$ is referenced by two or more calibration functions to establish relative relations. From Fig. 3 the $\Delta u_i$ difference between $u$ and $u_i$ is calculated from: $\sqrt{1-\left(cf/u\right)^2} = u_i$.

Calibration is relative to a reference in space or time. When a reference in time is applied, the calibration function is the Lorentz transformation in special relativity [17], where the velocity of light $=c= u$, $v = cf$ and $n=$ a count of MAI (e.g., light years). Then the Lorentz transformation: $\frac{1}{\sqrt{1-\left(v/c\right)^2}} = \frac{1}{u_i}$ and $\sum_{i=1}^{i=n} \text{MAI}/u_i =$ a measurement result in MAI calibrated in time (relativistic) rather than space. This is a strong indication that Fig. 3 is valid for all measurements.

One part of the calibration function (4) describes the smallest magnitude ($s$, a sample, in Fig. 3) that the perfect measuring apparatus[5] can reliably detect. In QM theory, the sample may be a Planck, $\hbar$ . In a perfect measuring apparatus (which cannot create perfect measurement results), when $s$ of the measuring apparatus projects on the observable, it is $s$. When $s$ does not project on the observable, it is $0$. When the magnitude of the measuring apparatus's projection is $>0$ and $< s$, that $s$ will be uncertain, $0$ or $s$. Each of the two possible $s$ increments of $u_i$ shown in Fig. 3 may be uncertain due to zero setting or projection. This is *resolution uncertainty*.

The resolution uncertainty ( $\Delta s$ ) has three possible magnitudes: $+s$, $0$ (i.e., $<|s|$), $-s$. The zero magnitude is not a state (does not occur in theory), but is a transition between the $+s$ and $-s$ states, which has a low experimental probability,[6] and is usually treated as an experimental error (e.g., see Appendix A, below). The standard deviation of such a three magnitude resolution distribution is developed in Section 4.

## 2.3 RELATIVE MEASUREMENT FUNCTION

Replacing $\boldsymbol{u}_i$ in (2) with $u_i$ from (4) formalizes the theoretical *relative measurement function* in space (5):

$$\Sigma u_i = \sum_{i=1}^{i=n} \left\langle 1/n, \left(u + \Delta u_i + \Delta s\right) \right\rangle \quad (5)$$

Equation (5) includes both *accuracy* (6), relative to a reference, and *precision* (7), self-relative, due to resolution:

$$\sum_{i=1}^{i=n} \Delta u_i = \text{accuracy} \quad (6), \qquad \sum_{i=1}^{i=n} \Delta s = \text{ precision} \quad (7)$$

[5] In QM theory a perfect measuring apparatus is represented by a Kronecker delta.

[6] The Stern-Gerlach experiments [5] present a two state resolution uncertainty, without the zero magnitude.

Multiple measurements of a fixed observable using the same measuring apparatus will exhibit uncertainty relative to a reference. $\Delta s$ and $\Delta u_i$ are shown to be responsible for the inherent uncertainty of measurement results from observables that appear continuous.

Expanding on (1), different experimental measurements or theories cause different terms to dominate or be assumed: usually $s \ll u_i$ ; $n$ may be large; (6) or (7) may be assumed to cancel (possibly invalid for $\Delta u_i$ and likely valid for $\Delta s$ ). Measurement results without significant uncertainty are termed weak measurements [18] in QM. They occur when $\langle \text{MAI} \rangle >>$ (6) or (7). The examples in Section 3 explain other effects.

## 3.0 EXPERIMENTAL UNCERTAINTY

Consider a four digit voltmeter where *00.01* is the voltmeter's display of an MAI. Measuring a fixed voltage source[7] multiple times produces a distribution of measurement results. Practically, the uncertainty of this distribution is specified by the manufacturer (i.e., it has been calibrated previously), for all voltmeters of this model, to be ±MAI.

A theoretical measurement calculation example: maintaining $\pm u_i = 0.01$ with uncertainty requires $\leq \pm 0.005$ volt ($v$) resolution ($s$) based upon sampling theory [19], i.e., each $u_i$ is between *0.005* and *0.015 v*. The smallest resolution (magnitude of one sample) of a measuring apparatus is one cause of the uncertainty of measurement results, and is never zero. This resolution uncertainty also has three possible magnitudes: *+0.005*, *0.000* (<|*0.005*|), *-0.005*. A *1.000 ±0.001 v* fixed voltage source is applied. *±0.001 v* is the independently determined maximum uncertainty ( $\pm|\Delta u_i|$ ) of the *1.000 v* fixed source relative to a primary standard. Applying (4), there are nine possible magnitudes of $u_i \geq 0.001$ when $u = 0.010$:

| $\pm\Delta u_i - s$ | $u_i$ | $\pm\Delta u_i + s$ | $u_i$ | $\pm\Delta u_i + s$ | $u_i$ |
|---|---|---|---|---|---|
| *-0.001 -0.005* | *0.004* | *-0.001 +0.000* | *0.009* | *-0.001 +0.005* | *0.014* |
| *+0.000 -0.005* | *0.005* | *+0.000 +0.000* | *0.010* | *+0.000 +0.005* | *0.015* |
| *+0.001 -0.005* | *0.006* | *+0.001 +0.000* | *0.011* | *+0.001+0.005* | *0.016* |

Table 1. Nine magnitudes of $u_i$

Summing the nine possible magnitudes of $u_i$ over $n = 100$ of the $u_i$ (of the *1.000 v* fixed source) identifies $9^{100}$ combinations of sums which will, as the number of different voltmeter measurements of the same observable increase, converge to a normal distribution as described by the central limit theorem. Therefore the uncertainty of the measurement result distribution without the calibration function (QM theory) is: *100* times *0.004* to *0.016 = 0.400* to *1.600 v*.

An experimental calibration example: The four digit display of the voltmeter is adjusted to *00.00* (zero setting). The fixed voltage source (*1.000v*) is applied and a second voltmeter adjustment (reference setting) changes the display to *1.00*. These calibration adjustments remove most of the inherent uncertainty, $\sum(\Delta u_i + \Delta s)$, and establish $\langle \text{MAI} \rangle = 0.010 \pm 0.006$ *v*. The theoretical measurement is *1.00 ± 0.60 v*. The calibrated measurement result is

[7] In metrology this is a measurand [12], a non-normalized observable.

$100\langle \text{MAI} \rangle = 1.00 \pm 0.006$. The change from calculated distribution to calibrated measurement result distribution decreases the relative measurement system entropy. This is termed a wave function collapse (or jump) in QM and appears in the Compton and Simons experiments [4].

## 4.0 RELATIVE UNCERTAINTY

The following develops the standard deviations of three concepts from metrology: applied reference ($u_i$), accuracy (6), and precision (7) to compare them with concepts in current QM theory.

$$\sigma(\Delta s) = \sqrt{\frac{1}{n}\sum_{i=1}^{i=n}\left(\Delta s - \langle \Delta s \rangle\right)^2} \qquad (8)$$

In the neutron spin experiments [20] discussed in Appendix A, the spin resolution distribution has two states, $+1$ and $-1$, as zero is considered an error in these neutron spin experiments. Then the mean is very close to zero and the standard deviation of this resolution distribution (8) is slightly $< 1$, which is experimentally verified in [20].

When the $\sigma(\Delta s)$ is slightly $< 1$, the $\log_e \sigma(\Delta s)$ which calculates the entropy of the $\Delta s$ Gaussian distribution, is near zero and the entropy due to $\Delta s$ is near constant [21]. Resolution entropy change is usually not experimentally significant.

In the uncertainty relations in QM theory, the standard deviation of the measure magnitude is calculated relative to a mean, as shown in (9). The use of interval norms ($\boldsymbol{u}_i$) and the mean as the reference shown in (9) are not valid in relative measurements. The effect of the calibration function on (9) is shown in (11).

$$\sigma(N\boldsymbol{u}_i) = \sqrt{\frac{1}{n}\sum_{i=1}^{i=n}\left(N - \langle N \rangle\right)^2} \qquad (9)$$

Depending upon the relative measurement system, uncertainty due to $\Delta u_i$ may be non-statistical [22], e.g., when calibration causes a uniform shift from each $i^{th}$ $\boldsymbol{u}_i$ to the $i^{th}$ $u_i$. Such distributions (10) will have a higher entropy than Gaussian distributions. Then, when $n$ is large, the entropy increase due to $\sum_{i=1}^{i=n}\Delta u_i$ is significant. As demonstrated in Section 3 above, experimental calibration to $\langle \text{MAI} \rangle$ will cancel most of this entropy increase making the theoretical measures appear more uncertain than experimental measurement results, an unreasonable result.

As developed in Section 3, rigorous experimental measurement results are calibrated to a reference, $u$ in the SI reference space. This is shown in (10) which calculates the *relative standard deviation* by applying (4), where each $u_i$ is relative to $u$.

$$\sigma(\Delta u_i) = \sqrt{\frac{1}{n}\sum_{i=1}^{i=n}\left(u_i - u\right)^2} \qquad (10)$$

The uncertainty of a measure (9) is transformed into a reference space by summing (8) and (10). Then the uncertainty of a measurement result (11) is:

$$\sigma(N\boldsymbol{u}_i) \rightarrow \sigma(\sum u_i) = \sigma(u_i) = \sigma(\Delta u_i) + \sigma(\Delta s) \quad (11)$$

Equation (11), the *relative uncertainty relation*, presents the inherent uncertainty of one measurement as the sum of the standard deviation of accuracy (6) and precision (7) when $n$ is fixed. Equation (11) is an indication of the validity of (5).

## 5.0 RELATIVE MEASUREMENT IN QM

From Fig. 1, the measuring apparatus with $a_i$ intervals is projected upon the observable $\boldsymbol{A}$. The measuring apparatus with $b_j$ intervals is projected upon the observable $\boldsymbol{B}$. The vector magnitude of the observables $\boldsymbol{A}$ and $\boldsymbol{B}$ in $\boldsymbol{u}_i$ is determined to be $A$ and $B$ respectively. Transforming $\boldsymbol{u}_i$ to $u_i$ (4) introduces relative measurements into QM theory: the measurement results from observables $\boldsymbol{A}$ and $\boldsymbol{B}$ are $\Sigma a_i$ and $\Sigma b_j$. These measurement results are relative to $u$ as shown in (5). Then, applying the three left hand terms of (11), the Kennard form of Heisenberg uncertainty (also termed disturbance) eq. 3 in [3] transforms to:

$$\sigma(\boldsymbol{Q})\sigma(\boldsymbol{P}) \rightarrow \sigma(q_i)\sigma(p_j) \geq \hbar/2 \quad (12)$$

$q_i$ are the $u_i$ of position and $p_j$ are the $u_i$ of momentum. These $u_i$ are related by $p_j = m(dq_i/dt_i)$, where $m$ = a fixed mass. At the instant the $q_i$ are calibrated to $q$ (the reference), both measurement results of the observables $\boldsymbol{Q}$ and $\boldsymbol{P}$ will change, since $q_i$ and $p_j$ are different functions of $q_i$. The Kennard or Heisenberg uncertainty shows the effect of the calibration function on two variables related to $q$.

The Robertson form of the uncertainty relation (where $\boldsymbol{A}$ and $\boldsymbol{B}$ are not relative to one reference) from eq. 4 in [3] is:

$$\sigma(\boldsymbol{A})\sigma(\boldsymbol{B}) \geq \frac{1}{2}\left|\langle\psi,[AB\text{-}BA]\psi\rangle\right| \quad (13)$$

In (13) $\langle\psi,[AB\text{-}BA]\psi\rangle$ identifies that the inner product of the state vector $\psi$ represents two magnitudes ($A$ and $B$) that do not commute. This appears experimentally when observable $\boldsymbol{A}$ is measured first, then the observable $\boldsymbol{B}$ and the reverse in the second term. The magnitude $AB$ is almost always different from $BA$, suggesting the first measurement changes a second differently each time. Since $A$ and $B$ in (13) are not changed, this does not represent reasonable physical reality. When the vector magnitudes $A$ and $B$ in (13) are replaced by $\Sigma a_i$ and $\Sigma b_j$ respectively, expressing both the fixed magnitude and $u_i$, the now reasonable difference between $AB - BA$ follows directly from the $\Delta s$ term in (11).

Since QM theory lacks the concept of measurements relative to a reference, the uncertainty of a measurement result appears as the product of the uncertainty of the measurement results of each of two observables $\boldsymbol{A}$ and $\boldsymbol{B}$ that are in the same reference space (comparable). Equation (14) is the product of (11) for each of the two observables:

$$\sigma(\Sigma a_i)\sigma(\Sigma b_j) = \left[\sigma(\Delta a_i) + \sigma(\Delta s_a)\right]\left[\sigma(\Delta b_j) + \sigma(\Delta s_b)\right] \quad (14)$$

$\Delta s_a$ and $\Delta s_b$ are the minimum sample magnitude of the measuring apparatus projected on $\boldsymbol{A}$ and $\boldsymbol{B}$ respectively. $i$ and $j$ are intervals of the respective measuring apparatus. The product of the right side of (14):

$$\sigma(\Delta a_i)\sigma(\Delta b_j)+\sigma(\Delta a_i)\sigma(\Delta s_b)+\sigma(\Delta s_a)\sigma(\Delta b_j)+\sigma(\Delta s_a)\sigma(\Delta s_b) \quad (15)$$

In the Appendix, (15) is shown to be equal to the universal uncertainty relation (A.1) formally developed by Ozawa in 2003 [3] and experimentally verified [20]. This is an independent development of (11) and therefore a proof of the relative measurement function (5).

## 6.0 CONCLUSION

Measurement uncertainty of observables with unchanged magnitude appears because the variation of the experimental measuring apparatus intervals is not treated. Measurement disturbance appears when one calibration function changes two relative measurement results. Wave function collapse occurs because representational measures do not include the calibration function. Measurement entanglement appears unreasonable only because a measurement result is not recognized as relative to another measurement result. These explanations of the unreasonable results from QM theory and experiments are the strongest arguments for Relative Measurement Theory.

In 1935, the EPR paper recognized that the wave function, based upon representational measures, was an incomplete description of physical reality. RMT adds the missing description: calibration in a stipulated reference space.

Representational measure theory should be expanded into a measurement theory which includes RMT. Then both finite experimental and theoretical measurements are understood as inherently uncertain, are formalized the same, and produce equal (within an equal uncertainty) distributions of measurement results of the same observable. This unifies experimental and theoretical measurements.

**ACKNOWLEDGEMENTS**

The author acknowledges Chris Fields, the *Measurement* reviewers and Elaine Baskin for their assistance in the development of this paper.

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

# Appendix

This Appendix compares Ozawa's formal development [3] of the universal uncertainty relation (A.1) (equation (266) in [3]) to (15) showing that they are equal. It also presents the verification of [3] using neutron spin experiments [20] and identifies where the calibration function appears.

$$\varepsilon(A,\rho)\eta(B,\rho)+\varepsilon(A,\rho)\sigma(B,\rho)+\sigma(A,\rho)\eta(B,\rho)\geq\frac{1}{2}\left|Tr\left([A,B]\rho\right)\right| \qquad \text{(A.1)}$$

Equation (A.1) maintains the notation used in [3]. The four terms applied in the SI reference space (15) and the equivalent terms from (A.1) in Hilbert space are shown in Table A.1. $\rho$ represents a pure state in Hilbert space.

| (15) | (A.1) |
|---|---|
| $\sigma(\Delta a_i)$ | $\varepsilon(A,\rho)$ |
| $\sigma(\Delta b_j)$ | $\eta(B,\rho)$ |
| $\sigma(\Delta s_a)$ | $\sigma(A,\rho)$ |
| $\sigma(\Delta s_b)$ | $\sigma(B,\rho)$ |

Table A.1. Equal terms

When comparing the left side of (A.1) to (15), (A.1) must be slightly modified. In (A.1) $\varepsilon(A,\rho)$ is the root-mean-square magnitude relative to $a$ ($a$ is termed a true value in [3] page 17 bottom of the right column). In RMT this $a$ is a reference, $u$.

Notice that an additional fourth term in (A.1), $\sigma(A,\rho)\sigma(B,\rho)$, appears mathematically consistent. The same suggestion is also presented in (56) of [3]. Modifying (A.1) to include this fourth term does not change the (A.1) inequality. This modification is supported by: $A$ and $B$ are correlated to a common reference (time) in the neutron spin experiments. This correlation is also expressed in (235) in [3]. Therefore the product of the pre-measurement uncertainties, $\sigma(A,\rho)\sigma(B,\rho)$, page 25 section C in [3], should be included in (A.1). And, the experimental results in [20] identify that both resolution uncertainties are slightly less than one, therefore their product is close to one, not zero.

Discussing the above in relation to RMT: The point of modifying (A.1) to include this fourth term is to factor the modified (A.1) into (15). The experimental results can ignore this fourth term, because the resolution entropy is $n$ times a constant plus the log of close to one (which is close to zero), so the fourth term has little effect when $n$ is small (2 spin states) on the experimental results [21]. But it does affect the understanding of measurement uncertainty.

EXPERIMENTAL VERIFICATION

Experimental verification of (A.1) is provided by neutron spin axis position experiments on a stream of neutrons [20]. The notation below is from [20].

When $\phi_{OA}$ is zero, $O_A$=$A$. The resolution $s$ of this experiment is $\sim 1.5°$, (page 8 in [20]), which is small relative to each $u_i$, $\left(2\pi/8\right)$. This is good experimental practice. The standard deviations, $\sigma(A, \rho)$ and $\sigma(B, \rho)$ represent the distribution of the observable magnitudes ($+1$, $0$, $-1$). In the neuron spin experiments, $n=2$ ($+1$, $-1$) and $0$ is an experimental artifact caused by the finite resolution.

These experiments detune between $0$ and $2\pi$. Changing detune to de-calibrate in [20] (all places) then relates [20] to the calibration function.

# References


[1] J. von Neumann, *Mathematical Foundations of Quantum Mechanics*, Princeton University Press, Princeton NJ, USA, 1955. This quote is on page 349.

[2] W. Heisenberg, The Physical Content of Quantum Kinematics and Mechanics, *Quantum Theory and Measurement*, eds.: J. A. Wheeler and W. H. Zurek, Princeton University Press, Princeton, NJ, 1983, page 64.

[3] M. Ozawa, Uncertainty Relations for Noise and Disturbance in Generalized Quantum Measurements, *Ann. Phys.* 311, 350-416 (2004). http://arxiv.org/abs/quant-ph/0307057

[4] J. von Neumann, *Mathematical Foundations of Quantum Mechanics*, pages 212-218.

[5] J. S. Bell, *Speakable and unspeakable in quantum mechanics*, Cambridge University Press, Cambridge, GB, 1989, Chapter 16, Bertlmann's socks and the nature of reality, page 139. This provides detailed discussions of entanglement including the Stern-Gerlach experiments, the major proposals to explain entanglement and a reference to Einstein's quote, "spooky action at a distance".

[6] J. von Neumann, *Mathematical Foundations of Quantum Mechanics*.

[7] A. Einstein, B. Podolsky, N. Rosen, Can Quantum-Mechanical Description of Physical Reality Be Considered Complete, *Physical Review*, Vol. 47, May 15, 1935. This paper is often referred to as the EPR paper.

[8] The following works do not offer a formal experimental measurement development:

D. J. Hand, *Measurement Theory and Practice*, Oxford University Press, London, England, 2004.

B. Ellis, *Basic Concepts of Measurement*, Cambridge University Press, Great Britain, 1966.

C. West Churchman, P. Ratoosh, eds., *Measurement: Definitions and Theories*, John Wiley & Sons, Inc., 1959.

H. von Helmholtz, *Counting and Measuring*, D. van Nostrand Co. Inc. New York, NY, USA, 1930.

A.E. Fridman, *The Quality of Measurements,* Springer, New York, NY, USA, 2012, page 44 equation (2.31) is a formal development of measurement errors. This supports the RMT development.

[9] L. Euler, *Elements of Algebra*, Chapter I, Article I, #3. Third edition, Longman, Hurst, Rees, Orme and Co., London England, 1822. "It is not possible to determine or measure one quantity other than by assuming

that another quantity of the same type is known and determining the ratio between the quantity being measured and that quantity".

[10] L. Mari and S. Sartori, A relative theory of measurement: Traceability as a solution to the non-transitivity of measurement results, *Measurement* 40 (2007), page 233-242. Their development recognizes calibration and proposes relative measurements as an extension to representational measures.

[11] D. H. Krantz, R. D. Luce, P. Suppes, A. Tversky, *Foundations of Measurement*, Academic Press, New York, 1971, Vol. 1, page 32, Section 1.5 Roles of Theories of Measurement in the Sciences. This three volume work is the foundational text on representational measure. "The construction and calibration of measuring devices is a major activity, but it lies rather far from the sorts of qualitative theories we examine here."

[12] *International Vocabulary of Metrology* (VIM) 3rd edition, BIPM JCGM 200:2012, http://www.bipm.org/en/publications/guides/vim.html

[13] J. von Neumann, *Mathematical Foundations of Quantum Mechanics*, page 74. Theorem 10, an inner product projection, defines (2).

[14] P.A.M. Dirac, *The Principles of Quantum Mechanics*, 4th edition, Oxford University Press, London, England 1962, Chapter III Representations.

[15] M. Born, On the Quantum Mechanics of Collisions, *Quantum Theory and Measurement*, eds.: J. A. Wheeler and W. H. Zurek, Princeton University Press, Princeton, NJ, 1983, pages 52-55.

[16] A.E. Fridman, *The Quality of Measurements,* Springer, page 23. "It is in principle impossible to find the true value of a quantity."

[17] W. Pauli, *Theory of Relativity,* Dover Publications, Mineola, NY, 1981, Section 24, Four-dimensional representation of the Lorentz transformation.

[18] B. Tamir, E. Cohen, Introduction to Weak Measurements and Weak Values, *Quanta*, Vol. 2, Issue 1 pages 7-17, May 2013.

[19] C.E. Shannon, Communications in the Presence of Noise, *Collected Papers*, eds., N. J. A. Sloane and A. D. Wyner, IEEE Press, New York, NY, 1993, page 161, The Sampling Theorem.

[20] G. Sulyok, S. Sponar, J. Erhart, G. Badurek, M. Ozawa and Y. Hasegawa, Violation of Heisenberg's error-disturbance uncertainty relation in neutron-spin measurements, *Physical Review A* 88, 022110 (2013). http://arxiv.org/abs/1305.7251

[21] C. E. Shannon, *The Mathematical Theory of Communications*, University of Illinois Press, Urbana, IL, 1963, page 89, provides the derivation of the entropy of a single coordinate axis Gaussian distribution.

[22] A.E. Fridman, *The Quality of Measurements,* page 59.

# Figure Captions

Fig. 1. Relative measurement system

Fig. 2. Inner product function

Fig. 3. Calibration function $\boldsymbol{u}_i \rightarrow u_i$