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The Proof of The ABC Conjecture - Part I: The Case $c = a + 1$

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Abstract In this paper, we consider the *abc* conjecture in the case $c = a + 1$. Firstly, we give the proof of the first conjecture that $c < rad^2(ac)$. It is the key of the proof of the *abc* conjecture. Secondly, the proof of the *abc* conjecture is given for $\epsilon \geq 1$, then for $\epsilon \in]0, 1[$ for the two cases: $c \leq rad(ac)$ and $c > rad(ac)$. We choose the constant $K(\epsilon)$ as $K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}$. A numerical example is presented.

Keywords Elementary number theory · real functions of one variable.

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To the memory of my Father who taught me arithmetic
To the memory of my colleague and friend Dr.Eng. Chedly Fezzani (1943-2019) for his important work in the field of Geodesy and the promotion of the geographic sciences in Africa

1 Introduction and notations

Let a a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as :

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{1}$$

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We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{2}$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) ([1]). It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

Conjecture 1 (***abc* Conjecture**): Let a, b, c positive integers relatively prime with $c = a + b$, then for each $\epsilon > 0$, there exists a constant $K(\epsilon)$ such that :

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{3}$$

$K(\epsilon)$ depending only of ϵ.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ ([1]). A conjecture was proposed that $c < rad^2(abc)$ ([2]). Here we will give the proof of it in the case $c = a + 1$.

Conjecture 2 Let a, b, c positive integers relatively prime with $c = a + b$, then:

$$c < rad^2(abc) \Longrightarrow \frac{Logc}{Log(rad(abc))} < 2 \tag{4}$$

This result, I think is the key to obtain a proof of the veracity of the *abc* conjecture.

2 A Proof of the conjecture (2), Case : $c = a + 1$

Let a, c positive integers, relatively prime, with $c = a + 1$. If $c < rad(ac)$ then we obtain:

$$c < rad(ac) < rad^2(ac) \tag{5}$$

and the condition (4) is verified.

If $c = rad(ac)$, then a, c are not relatively coprime.

In the following, we suppose that $c > rad^2(ac) \Longrightarrow \mu_a.rad(a) + 1 > rad^2(a).rad^2(c) \Longrightarrow 1 > rad(a).(rad(a)rad^2(c) - \mu_a)$, we obtain :

- if $(rad(a)rad^2(c) - \mu_a) > 0$, as $rad(a) \geq 2 \Longrightarrow 1 < rad(a).(rad(a)rad^2(c) - \mu_a)$, then the contradiction, hence $c < rad^2(ac)$.

- if $rad(a)rad^2(c) - \mu_a = 0 \Longrightarrow$ that a, c are not coprime, then the contradiction, hence $c < rad^2(ac)$.

- if $rad(a)rad^2(c) - \mu_a < 0 \Longrightarrow \mu_a > rad(a)rad^2(c)$. From $c = a + 1$, we obtain $rad(a) = \dfrac{c-1}{\mu_a}$. As it is supposed $c > rad^2(a)rad^2(c) \Longrightarrow c > \dfrac{(c-1)^2}{\mu_a^2}.rad^2(c)$. We obtain that c verifies the inequality:

$$c^2 - c\left(2 + \left(\frac{\mu_a}{rad(c)}\right)^2\right) + 1 < 0 \tag{6}$$

Then, we consider the equation :

$$P(X) = X^2 - X\left(2 + \left(\frac{\mu_a}{rad(c)}\right)^2\right) + 1 = 0 \tag{7}$$

We verify that the discriminant of $P(X)$ is > 0. The roots $X_1 < X_2$ of $P(X)$ are given by:

$$X_1 = \frac{1}{2}\left[2 + \left(\frac{\mu_a}{rad(c)}\right)^2 - \sqrt{4\left(\frac{\mu_a}{rad(c)}\right)^2 + \left(\frac{\mu_a}{rad(c)}\right)^4}\right] > 0$$
$$X_2 = \frac{1}{2}\left[2 + \left(\frac{\mu_a}{rad(c)}\right)^2 + \sqrt{4\left(\frac{\mu_a}{rad(c)}\right)^2 + \left(\frac{\mu_a}{rad(c)}\right)^4}\right] > 0 \tag{8}$$

c verifies (6) $\Longrightarrow c \in]X_1, X_2[$, we obtain:

$$\mu_a\left(1 - \sqrt{1 + 4\frac{rad^2(c)}{\mu_a^2}}\right) < 2rad(a)rad^2(c) < \mu_a\left(1 + \sqrt{1 + 4\frac{rad^2(c)}{\mu_a^2}}\right) \tag{9}$$

From the right member of the above inequality, we have :

$$\mu_a > 2\frac{rad(a)rad^2(c)}{1 + \sqrt{1 + 4\frac{rad^2(c)}{\mu_a^2}}} = t \quad with \quad t < rad(a)rad^2(c) \tag{10}$$

Then the contradiction with $\mu_a > rad(a)rad^2(c)$. We deduce that the condition $c > rad^2(a)rad^2(c)$ is false and $c < rad^2(a)rad^2(c)$.

We announce the theorem:

Theorem 1 ***(Abdelmajid Ben Hadj Salem, 2019)*** *Let a, c positive integers relatively prime with $c = a + 1, a \geq 2$, then $c < rad^2(abc)$.*

3 The Proof of The ABC Conjecture (1) Case: $c = a + 1$

We denote $R = rad(ac)$.

3.1 Case: $\epsilon \geq 1$

Using the result of the theorem above, we have $\forall \epsilon \geq 1$:

$$c < R^2 \leq R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, \ \epsilon \geq 1 \tag{11}$$

We verify easily that $K(\epsilon) > 1$ for $\epsilon \geq 1$ and it is a decreasing function from e the base of the neperian logarithm to 1.

3.2 Case: $\epsilon < 1$

3.2.1 Case: $c \leq R$

In this case, we can write :

$$c \leq R < R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, \ \epsilon < 1 \tag{12}$$

here also $K(\epsilon) > 1$ for $\epsilon < 1$ and the *abc* conjecture is true.

3.2.2 Case: $c > R$

In this case, we confirm that :

$$c < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, 0 < \epsilon < 1 \tag{13}$$

If not, then $\exists \epsilon_0 \in]0, 1[$, so that the triplets (a, c) checking $c > R$ and:

$$c \geq R^{1+\epsilon_0}.K(\epsilon_0) \tag{14}$$

are in finite number. We have:

$$\begin{aligned} c \geq R^{1+\epsilon_0}.K(\epsilon_0) \Longrightarrow R^{1-\epsilon_0}.c \geq R^{1-\epsilon_0}.R^{1+\epsilon_0}.K(\epsilon_0) \Longrightarrow \\ R^{1-\epsilon_0}.c \geq R^2.K(\epsilon_0) > c.K(\epsilon_0) \Longrightarrow R^{1-\epsilon_0} > K(\epsilon_0) \end{aligned} \tag{15}$$

As $c > R$, we obtain:

$$c^{1-\epsilon_0} > R^{1-\epsilon_0} > K(\epsilon_0) \Longrightarrow$$

$$c^{1-\epsilon_0} > K(\epsilon_0) \Longrightarrow c > K(\epsilon_0)^{\left(\frac{1}{1-\epsilon_0}\right)} \tag{16}$$

We deduce that it exists an infinity of triples $(a, 1, c)$ verifying (14), hence the contradiction. Then the proof of the *abc* conjecture in the case $c = a + 1$ is finished. We obtain that $\forall \epsilon > 0$, $c = a + 1$ with a, c relatively coprime, $2 \leq a < c$:

$$c < K(\epsilon).rad(ac)^{1+\epsilon} \quad with \quad K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)} \tag{17}$$

Q.E.D

4 Examples

In this section, we are going to verify some cases of one numerical example. The example is given by:

$$1 + 5 \times 127 \times (2 \times 3 \times 7)^3 = 19^6 \quad (18)$$

$a = 5 \times 127 \times (2 \times 3 \times 7)^3 = 47\,045\,880 \Rightarrow \mu_a = 2 \times 3 \times 7 = 42$ and $rad(a) = 2 \times 3 \times 5 \times 7 \times 127$,
$b = 1 \Rightarrow \mu_b = 1$ and $rad(b) = 1$,
$c = 19^6 = 47\,045\,880 \Rightarrow rad(c) = 19$. Then $rad(abc) = rad(ac) = 2 \times 3 \times 5 \times 7 \times 19 \times 127 = 506\,730$..

We have $c > rad(ac)$ but $rad^2(ac) = 506\,730^2 = 256\,775\,292\,900 > c = 47\,045\,880$.

4.0.1 Case $\epsilon = 0.01$

$c < K(\epsilon).rad(ac)^{1+\epsilon} \Longrightarrow 47\,045\,880 \overset{?}{<} e^{10000}.506\,730^{1.01}$. The expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{\frac{1}{0.0001}} = e^{10000} = 8,7477777149120053120152473488653e{+}4342 \quad (19)$$

We deduce that $c \ll K(0.01).506\,730^{1.01}$ and the equation (17) is verified.

4.0.2 Case $\epsilon = 0.1$

$K(0.1) = e^{\frac{1}{0.01}} = e^{100} = 2,6879363309671754205917012128876e + 43 \Longrightarrow c < K(0.1) \times 506\,730^{1.01}$. And the equation (17) is verified.

4.0.3 Case $\epsilon = 1$

$K(1) = e \Longrightarrow c = 47\,045\,880 < e.rad^2(ac) = 697\,987\,143\,184,212$. and the equation (17) is verified.

4.0.4 Case $\epsilon = 100$

$$K(100) = e^{0.0001} \Longrightarrow c = 47\,045\,880 \overset{?}{<} e^{0.0001}.506\,730^{101} = 1,5222350248607608781853142687284e + 576$$

and the equation (17) is verified.

5 Conclusion

This is an elementary proof of the *abc* conjecture in the case $c = a + 1$. We can announce the important theorem:

Theorem 2 *(David Masser, Joseph Œsterlé & Abdelmajid Ben Hadj Salem; 2019) Let* a, c *positive integers relatively prime with* $c = a + 1$, $a \geq$ *then for each* $\epsilon > 0$, *there exists* $K(\epsilon)$ *such that :*

$$c < K(\epsilon).rad(ac)^{1+\epsilon} \tag{20}$$

where $K(\epsilon)$ *is a constant depending of* ϵ *equal to* $e^{\left(\frac{1}{\epsilon^2}\right)}$.

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