Definitive Proof of the Near-Square Prime Conjecture, Landau's Fourth Problem

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Abstract

The Near-Square Prime conjecture, states that there are an infinite number of prime numbers of the form $x^2 + 1$. In this paper, a function was derived that determines the number of prime numbers of the form $x^2 + 1$ that are less than $n^2 + 1$ for large values of n. Then by mathematical induction, it is proven that as the value of n goes to infinity, the function goes to infinity, thus proving the Near-Square Prime conjecture.

Introduction

The Near-Square Prime conjecture, first proposed by Euler in 1760, states that there are an infinite number of prime numbers of the form $x^2 + 1$. In this paper, a function was derived that determines the number of prime numbers of the form $x^2 + 1$ that are less than $n^2 + 1$ for large values of n. Then by mathematical induction, it is proven that as the value of n goes to infinity, the function goes to infinity, thus proving the Near-Square Prime conjecture.

Functions

Let the function l(x) be the largest prime number of the form 4i+1 that is less than x. For example, l(10.5) = 5, l(20) = 17, l(17) = 13.

Let the function π(n) represent the number of primes of the form $x^2 + 1$ that are less than or equal to $n^2 + 1$.

Let the set 𝕂 equal the set of odd integers of the form $x^2 + 1$.

Let π(n) represent the number of prime numbers in 𝕂 that are less than $n^2 + 1$.

Methodology

We will look only at cases where n is an even number because if n is odd, then n^2+1 will be an even number and thus not prime.

The set of odd integers of the form (x^2+1) less than or equal to $n^2 + 1$ is as follows:

𝕂 = {5,17,37,65,101,145,197,257,325,401,485,…,n^2+1}

These numbers are in the form $4x^2 + 8x + 5$, where x is an integer greater than or equal to 0.

There are n/2 numbers in the set. Notice that not all these numbers are prime.

To identify the numbers that are prime, we will eliminate the values divisible by primes of the form 4i+1 since primes of other forms do not evenly divide numbers of the form x^2+1. This is a known theorem of quadradic residues.

Primes of the form 4i + 1 are

{5,13,17,29,37,41,53,61,73,89,97,101,109,113,137,…}

Note that the minimum gap between primes of the form 4i+1 is 4, and there are no consecutive gaps of 4. This is because for the sequence 5,9,13,17,21,25,29,33…, every 3rd number is divisible by 3. According to Dirichlet's Theorem, there are an infinite number of prime numbers of the form 4i + 1.

We start by eliminating all number of the form $x^2 + 1$ from the set $\mathbb{K}$ that are divisible by the prime number 5.

$\mathbb{K}$ = {5,17,37,65,101,145,197,257,325,401,485, 577,677,785,901,1025,1157,1297,1445,1601, 1765,1937…,n^2+1}

Notice that every 5^{th} number after 5, 2 of them are divisible by 5. This is a property of quadradic equations.

The equation $y = 4x^2 + 8x + 5$ can be written as $y = x(4x+8) + 5$. Values of x=5k or 5k+3 where k is an integer, will result in a value of y that is evenly divisible by 5. Plugging 5k for x gives 5k(4x+8) which is divisible by 5, plugging 5k+3 for x gives x(4(5k+3)+8) = x(20k+20) which is also divisible by 5.

Thus, as n-> ∞, about 2/5ths of the numbers of the form $x^2 + 1$ are evenly divisible by 5.

# of values divisible by 5 limit n-> ∞ = (n/2)(2/5)

Next, we eliminate values divisible by 13, the next higher prime of the form 4i+1, from the set $\mathbb{K}$.

$\mathbb{K}$ = {5,17,37,65,101,145,197,257,325,401,485, 577,677,785,901,1025,1157,1297,1445,1601, 1765,1937…,n^2+1}

Notice that every 13 numbers, 2 are divisible by 13.

If we subtract 65 from both sides of $y = 4x^2 + 8x + 5$, we get $y-65 = 4x^2 + 8x – 60$ which can be written as (4x – 12)(4x + 20). Values of x = 13k + 3 or 13k + 8 will result in an integer value of y/13. If we plug x = 13k+3 in the left side, we get 52k(4x+20) which is divisible by 13. If we plug 13k + 8 in the right side we get (4x-12)(52k+52) which is divisible by 13.

Thus, as n-> ∞, about 2/13ths of the values are eliminated. However, notice that 65 and 325 are also divisible by 5. About 2/5ths of the numbers divisible by 13 are also divisible by 5. So to avoid double counting, we must multiply the number divisible by 13 by 3/5.

# of values divisible by 13 and not 5 limit n-> ∞ = (n/2)(3/5)(2/13)

Next, we eliminate values divisible by 17, the next higher prime of the form 4i+1, from the set $\mathbb{K}$.

$\mathbb{K}$ = {5,17,37,65,101,145,197,257,325,401,485, 577,677,785,901,1025,1157,1297,1445,1601, 1765,1937…,n^2+1}

Notice that every 17 numbers after 17, 2 are divisible by 17.

If we subtract 17 from both sides of $y = 4x^2 + 8x + 5$, we get $y – 17 = 4x^2 + 8x + 5 – 17$ which can be written as (4x – 4)(4x + 12). Values of x = 17k + 1 or 17k + 14 will result in an integer value of y/17. Thus, there will always be at least 2 values of x every 17 numbers.

# of values divisible by 17 and not 5 or 13 limit n-> ∞ = (n/2)(3/5)(11/13)(2/17)

The fact that $y = 4x^2 + 8x + 5$ is quadratic, for every p numbers, there will always be 2 values of x that will result in a y that is evenly divisible by p.

The general formula for number of values in the set 𝕂 that are divisible by p where p is a prime number of the form 4i+1 is:

# of values evenly divisible by only p limit n-> ∞ = (n/2)(3/5)(11/13)(15/17)…(2/p)

This can be written as

# of values evenly divisible by only p limit n-> ∞ = (n/2)(2/p)$\prod_{q=5}^{p}(q-2)/q$

where the product is over prime numbers of the form 4i+1.

We only need to go up to l(n) since prime numbers greater than l(n) will not evenly divide any odd number less than n^2+1 that is not already divisible by a lower prime.

Summing up all these gives us the total number of composite numbers in set 𝕂 that are less than or equal to $n^2 + 1$.

# of composite numbers in 𝕂 limit n-> ∞

= (n/2)(2/5) + (n/2)(3/5)(2/13) + (n/2)(3/5)(11/13)(2/17) + … + (n/2)(2/l(n))$\prod_{q=5}^{l(l(n))}(q-2)/q$

= (n/2)[(2/5) + (3/5)(2/13) + (3/5)(11/13)(2/17) + … + (2/l(n))$\prod_{q=5}^{l(l(n))}(q-2)/q$]

= (n/2)[$\sum_{p=5}^{l(n)}(\frac{2}{p})\prod_{q=5}^{l(p)}(q-2)/q$]

where the sum and products are over prime numbers of the form 4i+1.

If we define the function W(x) as follows

$W(x) = \sum_{p=5}^{x}(\frac{2}{p}) \prod_{q=5}^{p}(q-2)/q$

where x is a prime number and the sum and products are over prime numbers of the form 4i+1,

Examples of values of W(x) are:

W(5) = 2/5

W(13) = (2/5) + (3/5)(2/13)

W(17) = (2/5) + (3/5)(2/13) + (3/5)(11/13)(2/17)

W(29) = (2/5) + (3/5)(2/13) + (3/5)(11/13)(2/17) + (3/5)(11/13)(15/17)(2/29)

Etc

The equation for the total number of composite values in set $\mathbb{K}$ is:

# of composite numbers in $\mathbb{K}$ limit n-> ∞ = (n/2)(W(l(n))

The number of primes of the form x^2+1 in $\mathbb{K}$ that are less than n^2+1 limit n-> ∞ equals the total number of values in $\mathbb{K}$, which is (n/2), minus the # of composite values in $\mathbb{K}$.

π(n) = (n/2) - (n/2)(W(l(n))

π(n) = (n/2)(1-W(l(n)) Equation 1

To verify that I derived equation 1 properly, I plotted the number of primes of the form $x^2 + 1$ that are less than or equal to $n^2 + 1$ (blue line) and $\pi(n)$ (orange line) for values of n up to 1000 and as can be seen, the lines correspond very nicely.

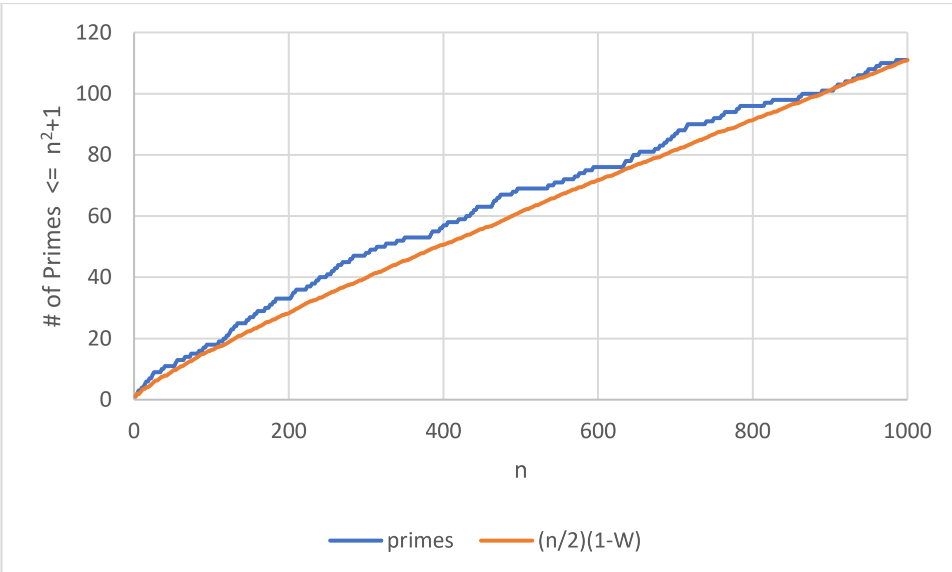

Figure 1. Number of primes of the form $x^2 + 1$ that are less than or equal to $n^2 + 1$.

Since I will be using mathematical induction to prove the Near-Square Prime conjecture, I need to define $1 - W(p_{i+1})$ in terms of $W(p_i)$. Below are the values of $1 - W(p_i)$.

1 - W(5) = 1 – (2/5) = 3/5

1 - W(13) = 1 - (2/5) - (3/5)(2/13) = (3/5)(11/13)

1 - W(17) = 1 - (2/5) - (3/5)(2/13) - (3/5)(11/13)(2/17) = (3/5)(11/13)(15/17)

1 - W(29) = 1 - (2/5) - (3/5)(2/13) - (3/5)(11/13)(2/17) - (3/5)(11/13)(15/17)(2/29) = (3/5)(11/13)(15/17)(27/29)

Notice the value of 1 - $W(p_{i+1})$ is equal to $((p_{i+1} - 2)/p_{i+1})$ times the previous value of 1 - $W(p_i)$. This gives us the following equation:

$1-W(p_{i+1}) = ((p_{i+1} - 2)/p_{i+1})(1-W(p_i))$ Equation 2

Let $l(n) = p_i$ and let's approximate $n = p_i$. Since n is an even integer, n is at least $p_i + 1$ so this approximation errs on the side of caution. Plugging p_i for l(n) and n into equation 1 gives the following:

$\pi(p_i) = (p_i/2)(1-W(p_i))$

$\pi(p_{i+1}) = (p_{i+1}/2)(1-W(p_{i+1}))$

$\pi(p_{i+1}) = (p_{i+1}/2)\ (p_{i+1}-2/p_{i+1})(1-W(p_i))$ Using equation 2

$\pi(p_{i+1}) = ((p_{i+1}-2)/2)(1-W(p_i))$

Taking the ratio of $\pi(p_{i+1})/\ \pi(p_i)$ gives:

$\pi(p_{i+1})/\pi(p_i) = ((p_{i+1} - 2)/2)(1-W(p_i))\ /\ (p_i/2)(1-W(p_i))$

$\pi(p_{i+1})/\pi(p_i) = (p_{i+1} - 2)/p_i > 1$

Since p_{i+1} is at least $p_i + 4$, this proves that $\pi(p_{i+1})$ will always be bigger than $\pi(p_i)$. However, plugging in $p_i + 4$ for p_{i+1} gives $(p_i + 4 - 2)/p_i = (p_i + 2)/p_i$ which approaches 1 as p_i goes to infinity. This could mean that $\pi(p_i)$ approaches a constant.

To prove that $\pi(p_i)$ goes to infinity as p_i goes to infinity, I will prove that $\pi(p_i)^2$ goes to infinity. This is done because it is easier to prove that $\pi(p_i)^2$ goes to infinity.

$\pi(p_i)^2 = (p_i^2/4)(1-W(p_i))^2$

$\pi(p_{i+1})^2 = ((p_{i+1}-2)^2/4)(1-W(p_i))^2$

Let $\Delta\pi(p_i)$ represent the difference between $\pi(p_{i+1})^2$ and $\pi(p_i)^2$.

$\Delta\pi(p_i) = \pi(p_{i+1}) - \pi(p_i)$

$\Delta\pi(p_i) = ((p_{i+1}-2)^2/4)\ (1-W(p_i))^2 - (p_i^2/4)(1-W(p_i))^2$

$\Delta\pi(p_i) = ((p_{i+1} - 2)^2 - p_i^2)(1-W(p_i))^2/4$

We know that p_{i+1} is at least $p_i + 4$, so to simplify things, let's substitute p_{i+1} with $p_i + 4$. We will call this new function $\Delta\pi^*(p_i)$ which will always be less than or equal to $\Delta\pi(p_i)$.

$\Delta\pi^*(p_i) = ((p_i + 4 - 2)^2 - p_i^2)(1-W(p_i))^2/4$

$\Delta\pi^*(p_i) = ((p_i + 2)^2 - p_i^2)(1-W(p_i))^2/4$

$\Delta\pi^*(p_i) = ((p_i^2 + 4p_i + 4) - p_i^2)(1-W(p_i))^2/4$

$\Delta\pi^*(p_i) = (4p_i + 4)(1-W(p_i))^2/4$

$\Delta\pi^*(p_i) = (p_i + 1)(1-W(p_i))^2$

I will prove $\Delta\pi^*(p_i) > 0$ by mathematical induction.

Base case

$p_0 = 5$

$\Delta\pi^*(5) = (5+1)(1-W(5))^2$

$\Delta\pi^*(5) = (6)(1- 2/5)^2$

$\Delta\pi^*(5) = 6(3/5)^2$

$\Delta\pi^*(5) = 6(9/25)$

$\Delta\pi^*(5) = 72/25 > 1$

Assuming the following

$\Delta\pi^*(p_i) > 0$

prove that

$\Delta\pi^*(p_{i+1}) > 0$

$\Delta\pi^*(p_i) = (p_i + 1)(1-W(p_i))^2$ Assume > 1

$\Delta\pi^*(p_{i+1}) = (p_{i+1} + 1)(1-W(p_{i+1}))^2$

$\Delta\pi^*(p_{i+1}) = (p_{i+1} + 1)[((p_{i+1} - 2)/p_{i+1})(1-W(p_i))]^2$

$\Delta\pi^*(p_{i+1}) = (p_{i+1} + 1)((p_{i+1} - 2)^2/p_{i+1}^2)(1-W(p_i))^2$

$\Delta\pi^*(p_{i+1})/ \Delta\pi^*(p_i) = (p_{i+1} + 1)((p_{i+1} - 2)^2/p_{i+1}^2)(1-W(p_i))^2 / (p_i + 1)(1-W(p_i))^2$

$\Delta\pi^*(p_{i+1})/ \Delta\pi^*(p_i) = (p_{i+1} + 1)((p_{i+1} - 2)^2/p_{i+1}^2) / (p_i + 1)$

$\Delta\pi^*(p_{i+1})/ \Delta\pi^*(p_i) = (p_{i+1} + 1)(p_{i+1} - 2)^2/(p_{i+1}^2 (p_i + 1))$

$\Delta\pi^*(p_{i+1})/ \Delta\pi^*(p_i) = (p_{i+1} + 1)(p_{i+1}^2 - 4p_{i+1} + 4)/(p_{i+1}^2 p_i + p_{i+1}^2)$

$\Delta\pi^*(p_{i+1})/ \Delta\pi^*(p_i) = (p_{i+1}^3 - 4p_{i+1}^2 + 4 p_{i+1} + p_{i+1}^2 - 4p_{i+1} + 4)/(p_{i+1}^2 p_i + p_{i+1}^2)$

$\Delta\pi^*(p_{i+1})/ \Delta\pi^*(p_i) = (p_{i+1}^3 - 3p_{i+1}^2 + 4)/(p_{i+1}^2 p_i + p_{i+1}^2)$

The minimum p_{i+1} can be is $p_i + 4$. Substituting p_i with $p_{i+1} – 4$ gives

$\Delta\pi^*(p_{i+1})/ \Delta\pi^*(p_i) = (p_{i+1}^3 - 3p_{i+1}^2 + 4)/(p_{i+1}^2(p_{i+1} – 4) + p_{i+1}^2)$

$\Delta\pi^*(p_{i+1})/ \Delta\pi^*(p_i) = (p_{i+1}^3 - 3p_{i+1}^2 + 4)/(p_{i+1}^3 – 4 p_{i+1}^2 + p_{i+1}^2)$

$\Delta\pi^*(p_{i+1})/ \Delta\pi^*(p_i) = (p_{i+1}^3 - 3p_{i+1}^2 + 4)/(p_{i+1}^3 – 3p_{i+1}^2) > 1$

Since the numerator is always greater than the denominator, the ratio will always be greater than 1, thus proving that $\Delta\pi^*(p_{i+1}) > \Delta\pi^*(p_i)$ for any p_i and p_{i+1}. Since $\Delta\pi^*(p_0) = 72/25$, then $\Delta\pi^*(p_i) > 72/25$ for all p_i.

The first value of $\pi(p_0)^2$ is

$\pi(5)^2 = [(5/2)(1-(2/5))]^2 = [(5/2)(3/5)]^2 = 9/4$.

Since the gap between $\pi(p_i)^2$ and $\pi(p_{i+1})^2$ is always greater than 72/25, then as p_i goes to infinity, $\pi(p_i)^2$ goes to infinity. Therefore, $\pi(p_i)$ also goes to infinity as p_i goes to infinity.

This proves that there are an infinite number of primes of the form $n^2 + 1$ thus proving the near square primes conjecture.

Summary

It has been shown that as n goes to infinity, the number of prime numbers of the form $x^2 + 1$ that are less than or equal to $n^2 + 1$ approaches the following equation:

$$\pi(n) = (n/2)(1\text{-}W(l(n))$$

where W(x) is defined as follows:

$$W(x) = \sum_{p=5}^{x} (\tfrac{2}{p}) \prod_{q=5}^{p} (q-2)/q$$

where x is a prime number and the sum and products are over prime numbers of the form 4i+1.

By mathematical induction, it is proven that $\pi(p_i)^2$ goes to infinity as p_i goes to infinity thus proving that there are an infinite number of prime numbers of the form $x^2 + 1$.

References

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