Ramanujan summation of the Ln(n) series

Jesús Sánchez

Independent Researcher, Bilbao, Spain
Email: jesus.sanchez.bilbao@gmail.com
https://www.researchgate.net/profile/Jesus_Sanchez64
ORCID 0000-0002-5631-8195

Abstract

In this paper it will calculated that the Ramanujan summation of the Ln(n) series is:

$$\lim_{n\to\infty}(Ln(1)+Ln(2)+Ln(3)+\cdots Ln(n)) = Ln(-\gamma) = Ln(\gamma)+(2k+1)\pi i$$

Being γ the Euler-Mascheroni constant 0.577215... The solution is valid for every integer number k (it has infinite solutions). The series are divergent because Ln(n) tends to infinity as n tends to infinity. But, as in other divergent series, a summation value can be associated to it, using different methods (Cesàro, Abel or Ramanujan).

If we take the logarithm of the absolute value (this is, we take only the real part of the solution), the value corresponds to the smooth continuation to the y axis of the curve that calculates the partial sums at every point, as we will see in the paper.

$$\lim_{n\to\infty}(Ln|1|+Ln|2|+Ln|3|+\cdots Ln|n|) = Ln|-\gamma| = Ln|\gamma|$$

Keywords

Divergent series, natural logarithm, Ramanujan summation, gamma function.

1. Introduction

In this paper it will calculated that the Ramanujan summation [1] of the Ln(n) series is:

$$\lim_{n\to\infty}(Ln(1)+Ln(2)+Ln(3)+\cdots Ln(n)) = Ln(-\gamma) = Ln(\gamma)+(2k+1)\pi i \qquad (1)$$

Being γ the Euler-Mascheroni [2] constant 0.577215... The solution is valid for every integer number k (it has infinite solutions). The series are divergent because Ln(n) tends to infinity as n tends to infinity. But, as in other divergent series, a summation value can be associated to it, using different methods (Cesàro, Abel or Ramanujan [1]).

If we take the logarithm of the absolute value (this is, we take only the real part of the solution), the result would be:

$$\lim_{n\to\infty}(Ln|1| + Ln|2| + Ln|3| + \cdots Ln|n|) = Ln|-\gamma| = Ln|\gamma| \qquad (2)$$

To be able to perform the calculation, first, the gamma function [3] will be extended to have a solution at non-positive numbers (normally considered as divergent or not existing).

Then, it will be shown that the Ramanujan summation corresponds to the value where the curve that approximates the sum, hits the y axis, this is, the value of the curve for x=0.

With these data, the Ramanujan summation of Ln(n) will be calculated. And in consequence, the value for other related series will be shown.

2. Extension of the gamma function to non-positive integers

The gamma function [3] is an extension of the factorial function [4]. The gamma function is not defined at non-positive integers as we can see in the graph:

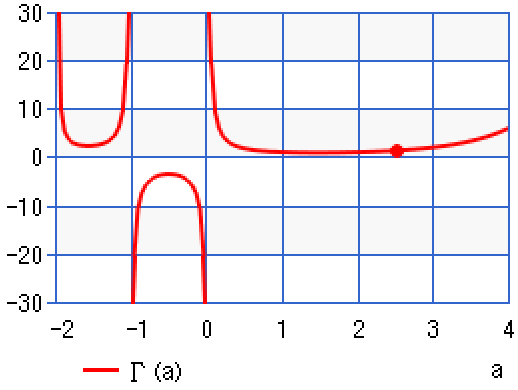

Fig 1. [5]

Anyhow, what we can do is to obtain the Cauchy principal value [6] at these points. For example, we can obtain the Cauchy principal value of Γ(0) as:

$$\Gamma(0) = \lim_{\varepsilon\to 0}\frac{1}{2}(\Gamma(0+\varepsilon) + \Gamma(0-\varepsilon)) \qquad (3)$$

And, in fact, using the Keisan online calculator [5] we get:

$$\frac{1}{2}\big(\Gamma(0.001) + \Gamma(-0.001)\big) = \frac{1}{2}(999.423772 - 1000.578206) = -0.577217 \qquad (4)$$

If we approximate more, we have:

$$\frac{1}{2}\left(\Gamma(0.0001) + \Gamma(-0.0001)\right) = \frac{1}{2}(999.422883 - 1000.577315) = -0.577216 \quad (5)$$

We can check that the solution tends to the negative of the Euler-Mascheroni constant [2]:

$$-\gamma = -0.577215664901532\ldots \quad (6)$$

A formal demonstration of this issue is given in [7]. In fact, in paper [7] a general solution for the gamma function for all the negative integer numbers is given as:

$$\Gamma(-r) = -\frac{(-1)^r}{r!} + \frac{(-1)^r}{r!}\sum_{i=1}^{r}\frac{1}{i} \quad (7)$$

For the special case $\Gamma(0)$ it is calculated in the same paper [7] as:

$$\Gamma(0) = -\gamma \quad (8)$$

3. Ramanujan summation

Using different techniques, Ramanujan obtained the sum of divergent series [1]. But, in general, can be simplified using a graphical representation. The way to do it is the following. First it is represented the result of the function graphically. Then a curve approximates this result. Afterwards, it is obtained the value of this curve in the y axis, this means, when x=0. This value is considered as the Ramanujan sum of the divergent series.

First example. To obtain the value of the divergent series [8]:

$$1 + 1 + 1 + 1 \ldots \quad (9)$$

First, we represent the sum of the function (y axis) as a function of the number of the element we are summing (x axis). This is represented in grey (the ladder type function).

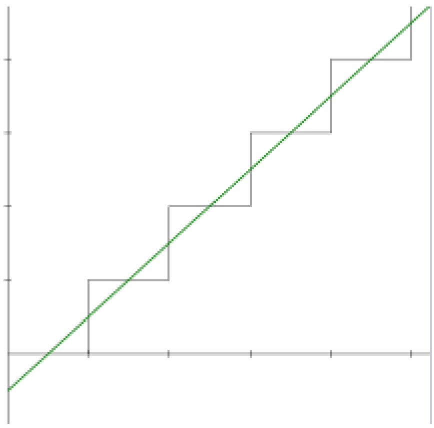

Fig. 2 [8]

Then, we approximate this discontinuous function by a curve (in this case a straight line) in green. The value of the curve in green when x=0 (when it hits the y axis) is the Ramanujan summation. In this case, we obtain [8]:

$$1 + 1 + 1 + 1 \ldots = -\frac{1}{2} \qquad (10)$$

Second example. We want to obtain the Ramanujan summation of the divergent series [9]:

$$1 + 2 + 3 + 4 \ldots \qquad (11)$$

First, we represent the sum of the function (y axis) as a function of the number of the element we are summing (x axis). This is represented in grey (the ladder type function).

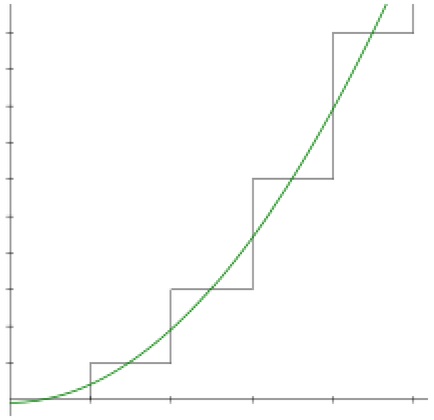

Fig.3 [9]

Then, we approximate this discontinuous function by the curve in green. The value of the curve in green when x=0 (when it hits the y axis) is the Ramanujan summation. In this case, we obtain [9]:

$$1+2+3+4\ldots=-\frac{1}{12} \qquad (12)$$

4. Ramanujan summation of the L(n) series

Our target is to get the Ramanujan summation of the following series:

$$\lim_{n\to\infty}(Ln(1)+Ln(2)+Ln(3)+\cdots Ln(n)) \qquad (13)$$

In the previous examples, we have got a curve that is the extension of the result for the real values (not only integers). In this case, we can calculate the partial sums and draw the function as follows:

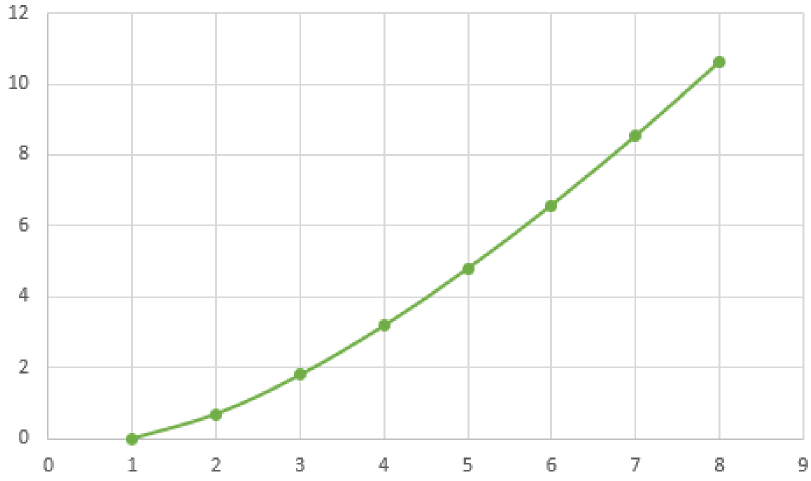

Fig. 4

But we do not know how to continue the curve to get to x=0 (as we have seen in the previous examples). The first thing, we can do is to convert the sum of logarithms into logarithm of a multiplication [10]:

$$\lim_{n\to\infty}(Ln(1)+Ln(2)+Ln(3)+\cdots Ln(n))=\lim_{n\to\infty}\big(Ln(1\cdot 2\cdot 3\cdot\ldots\cdot n)\big) \qquad (14)$$

The element inside the logarithm is in fact the factorial of n. And the factorial of n is related to the gamma function in the following way [3]:

$$\lim_{n\to\infty}\left(Ln(1\cdot 2\cdot 3\cdot \ldots \cdot n)\right) = \lim_{n\to\infty}\left(Ln(n!)\right) = \lim_{n\to\infty}\left(Ln(\Gamma(n+1))\right) \quad (15)$$

This means, we need the value of the gamma function for n+1 when n tends to infinity. This clearly diverges. But, we can use the Ramanujan summation the following way.

The strict factorial function is a discontinuous function that only applies to natural numbers. The gamma function is the continuous extension of the factorial function for the real numbers (this is, the green curve in the previous examples).

So, if we obtain the point where the green curve (in this case the gamma function) hits the y axis (the value when x=0), we will get the Ramanujan value of n! when n tends to infinity.

If we check the gamma function graph again, we see that the curve does not hit the y axis (it diverges at the point where x=0).

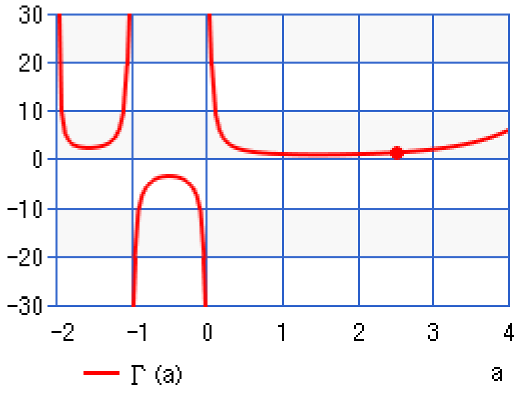

Fig 1. [5]

But, as we have commented, and indicated in paper [7], the gamma function in y axis (when x=0) is defined to be:

$$\Gamma(0) = -\gamma \qquad (8)$$

This means the Ramanujan of n! when n tends to infinity is:

$$\lim_{n\to\infty}(1\cdot 2\cdot 3\cdot 4\cdot \ldots \cdot n) = \lim_{n\to\infty}(n!) = \lim_{n\to\infty}(\Gamma(n+1)) = \Gamma(x=0) = -\gamma \quad (16)$$

So, from (14),(15) and(16) we get:

$$\lim_{n\to\infty}(Ln(1)+Ln(2)+Ln(3)+\cdots Ln(n)) = \lim_{n\to\infty}\big(Ln(n!)\big) = Ln(-\gamma) \qquad (17)$$

Being γ the Euler Mascheroni constant [2]. Using the properties of the natural logarithm [10] in the complex plane we get:

$$\lim_{n\to\infty}(Ln(1)+Ln(2)+\cdots Ln(n)) = Ln(-\gamma) = Ln(\gamma e^{\pi i}) = Ln(\gamma)+(2k+1)\pi i \quad (18)$$

If we take the logarithm of the absolute value (this is, we take only the real part of the solution), the value is the smooth continuation of the curve as we can see clearly in the graph.

$$\lim_{n\to\infty}(Ln|1|+Ln|2|+Ln|3|+\cdots Ln|n|) = Ln|-\gamma| = Ln|\gamma| \qquad (19)$$

Putting in the curve the value Ln|γ| for x=0, we get the smooth continuation as expected:

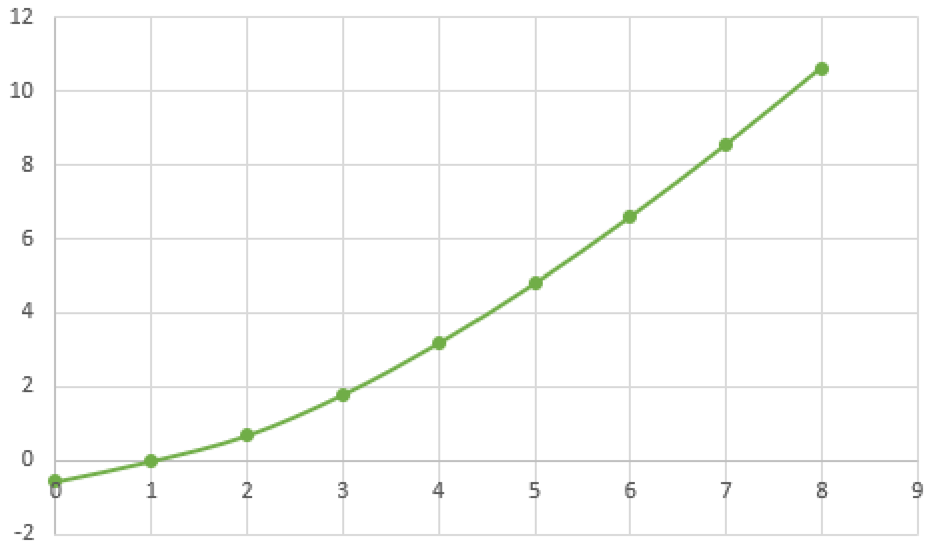

Fig. 5

5. Other sums of interest

Now, that we have obtained this sum, we can relatively easy obtain related sums. For example:

$$\lim_{n\to\infty}\left(Ln(1)+Ln\left(\frac{1}{2}\right)+Ln\left(\frac{1}{3}\right)+\cdots Ln(\tfrac{1}{n})\right) = \lim_{n\to\infty}\left(Ln\left(1\cdot\frac{1}{2}\cdot\frac{1}{3}\cdot\ldots\cdot\frac{1}{4}\right)\right) = \lim_{n\to\infty}\left(Ln\left(\frac{1}{n!}\right)\right) = Ln\left(-\frac{1}{\gamma}\right) = Ln\left(\frac{1}{\gamma}\right)+(2k+1)\pi i = -Ln(\gamma)+(2k+1)\pi i \qquad (20)$$

For the above case, again, if we take the logarithm of the absolute values (we take only the real part of the solution), the result would be just the real value -Ln|γ|.

And in general, being s any complex number we get:

$$\lim_{n\to\infty}(Ln(1^s)+Ln(2^s)+\cdots Ln(n^s)) = \lim_{n\to\infty}\left(Ln(1^s\cdot 2^s\cdot 3^s\cdot \ldots \cdot n^s)\right) = \lim_{n\to\infty}(Ln((1\cdot 2\cdot 3\cdot \ldots \cdot n)^s)) = \lim_{n\to\infty}(Ln((n!)^s)) = s(Ln(-\gamma)) = s(Ln(\gamma)+(2k+1)\pi i) \quad (21)$$

It is to be noted that the Riemann zeta function [11] has these series of logarithms as exponents of it. But regretfully, this summation cannot be used as the series is in the exponents, and not as a direct sum:

$$\zeta(s) = \frac{1}{1^s}+\frac{1}{2^s}+\cdots\frac{1}{n^s}+\cdots = e^{-sLn(1)}+e^{-sLn(2)}+\cdots+e^{-sLn(n)}+\cdots \qquad (22)$$

6. Conclusions

In this paper, it has been calculated the Ramanujan summation [1] of the Ln(n) series as:

$$\lim_{n\to\infty}(Ln(1)+Ln(1)+Ln(2)+\cdots Ln(n)) = Ln(-\gamma) = Ln(\gamma)+(2k+1)\pi i \quad (1)$$

Being γ the Euler-Mascheroni constant [2] and k any integer number. To get to that point an extension of the gamma function [3] to non-positive values has been shown [7] and an explanation of the Ramanujan summation [1] graphically.

Taking only the real part of the solution (this is, taking the logarithms of the absolute values), the result is the following. And it corresponds with the smooth continuation to the y axis, of the curve that calculates the partial sums at every point:

$$\lim_{n\to\infty}(Ln|1|+Ln|2|+Ln|3|+\cdots Ln|n|) = Ln|-\gamma| = Ln|\gamma| \qquad (2)$$

Other similar summations (even one related to Riemann zeta function [11]) have been calculated).

Bilbao, 20th January 2019 (viXra-v1).

12. Acknowledgements

To my family and friends.

13. References

[1] https://en.wikipedia.org/wiki/Ramanujan_summation

[2] https://en.wikipedia.org/wiki/Euler%E2%80%93Mascheroni_constant

[3] https://en.wikipedia.org/wiki/Gamma_function

[4] https://en.wikipedia.org/wiki/Factorial

[5]https://keisan.casio.com/exec/system/1180573444?lang=en&charset=utf-8&var_a=0.0001&ketasu=22

[6] https://en.wikipedia.org/wiki/Cauchy_principal_value

[7] http://www.naturalspublishing.com/files/published/t4uv81624e25vq.pdf

[8] https://en.wikipedia.org/wiki/1_%2B_1_%2B_1_%2B_1_%2B_%E2%8B%AF

[9] https://en.wikipedia.org/wiki/1_%2B_2_%2B_3_%2B_4_%2B_%E2%8B%AF

[10] https://en.wikipedia.org/wiki/Natural_logarithm

[11] https://en.wikipedia.org/wiki/Riemann_zeta_function